NON-ARISTOTELIAN LOGIC

BY

HENRY BRADFORD SMITH

Assistant Professor of Philosophy in the University of Pennsylvania

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CHAPTER I

§ 1. The problem of a science is to define the elements, of which it treats, by an enumeration of their formal properties. These properties are to be found within the system, of which these elements are the parts. The task of logic is, then, to develop completely its own system, by constructing all the true and all the untrue propositions, into which its elements enter exclusively.

§ 2. The propositions, which are recognized by the logician, are:

(1) The *Categorical* forms, made up of *terms* (represented in the proposition $x(ab)$ by the *subject* a and the *predicate* b) and *relationships* (an adjective of quantity, *all*, *some*, *no*, and the copula, *is*), viz.,

$$\alpha(ab) = \textit{All } a \textit{ is all } b,$$
$$\beta(ab) = \textit{Some } a \textit{ is some } b,$$
$$\gamma(ab) = \textit{All } a \textit{ is some } b,$$
$$\epsilon(ab) = \textit{No } a \textit{ is } b,$$

the word *some*, explicit in β and γ, being interpreted to mean *some at least*, *not all*. This meaning of the word will be unambiguously determined by the propositions, which we say shall be true or untrue in our science. Whenever we wish to designate some one of the four forms but desire to leave unsettled which one is meant, we shall employ the notation, $x(ab)$, $y(ab)$, $z(ab)$, etc.

(2) The *Hypothetical* forms,

$$x(ab) \angle y(ab) = x(ab) \textit{ implies } y(ab) \textit{ is true},$$
$$\{x(ab) \angle y(ab)\}' = x(ab) \textit{ implies } y(ab) \textit{ is untrue}.$$

(3) The *Conjunctive* form,

$$x(ab) \cdot y(ab) = x(ab) \textit{ and } y(ab) \textit{ are true},$$

(4) The *Disjunctive* form,

$$x(ab) + y(ab) = x(ab) \textit{ or } y(ab) \textit{ is true}.$$

$x(ab)$ *is untrue* will be represented by $x'(ab)$, $x'(ab)$ *is untrue* by $x''(ab)$, etc. Since in the proposition $x(ab)$ the terms are *subject*, a, and *predicate*, b, the *term-order* is the order subject-predicate. Whenever we wish to leave the term-order unsettled we shall place a comma between the terms. Thus, $x(a, b)$ stands for $x(ab)$ or for $x(ba)$.

§ 3. A principle, which is altogether fundamental and which will be taken for granted at each step of our progress, is this: *If a proposition is true in general, it is because it remains true for all specific meanings of the terms that enter into it*, although an untrue proposition may well enough become true in the same circumstances. Thus $\gamma(ab) \angle \gamma(ba)$ is untrue in general, but it becomes true for the case, in which a and b are identical, viz., $\gamma(aa) \angle \gamma(aa)$. Accordingly, when we write $\{x(a, b) \angle y(a, b)\}'$, we only assert that there is at least one value of a and one value of b, which will render $x(a, b) \angle y(a, b)$ a false proposition. If it has been established that $x(aa) \angle y(aa)$ is untrue, then we may at once infer that the more general implication, $x(a, b) \angle y(a, b)$, is untrue as well. *In order to establish the untruth of a given proposition, it will be enough to point to a special instance of its being untrue.*

§ 4. In presenting the materials of our subject-matter we shall have to deal with two types of proposition. The truth of

$$(x \angle y)(y \angle z) \angle (x \angle z)$$

is independent of x, y and z no matter for what *propositions* x, y and z may stand. Such a general truth will be termed a *principle*. The truth of

$$\gamma(ba)\gamma(cb) \angle \gamma(ca)$$

is independent of a, b and c no matter for what *classes* a, b and c may stand. If such a general truth has to be taken for granted, it will be termed a *postulate*.

Principles are, accordingly, independent of *forms*; postulates are independent of *terms*.

§ 5. We begin by setting down four postulates, the truth of which may be verified at once empirically by the familiar device of Euler's circular diagrams,

(i) $\alpha(ba)\alpha(bc) \angle \alpha(ca)$,

(ii) $\alpha(ba)\beta(bc) \angle \beta(ca)$,

(iii) $\alpha(ba)\gamma(cb) \angle \gamma(ca)$,

(iv) $\alpha(ba)\,\epsilon(bc) \angle \epsilon(ca)$,

and we shall add to these,

(v) $\alpha'(aa) \angle \alpha(aa)$.

This last assumption illustrates an extension of the common meaning of implication and is forced upon us, if we are to allow $\alpha(aa)$ to stand for a true proposition. The uses, to which such an extension of meaning may be put, will become clear in the sequel. It will be enough to state that a proposition, which is true for all meanings of the terms, will be implied by the proposition whose symbol is i, and behaves like a unit multiplier in this algebra.

Only a small number of the principles, which we shall introduce as necessity requires, are independent but it will not concern our purpose to point out the manner of their inter-connection.

From the principle,

$$(x' \angle x) \angle (y \angle x),$$

we obtain, by (v), the theorem,

$$i \angle \alpha(aa).$$

By (i), for $a = b$, $\alpha(aa)\alpha(ac) \angle \alpha(ca)$, and

$$\{\alpha(aa)\alpha(ac) \angle \alpha(ca)\}\{i \angle \alpha(aa)\} \angle \{\alpha(ac) \angle \alpha(ca)\},$$

by $(xy \angle z)(w \angle x) \angle (wy \angle z)$, omitting the unit multiplier, i, as it is the custom to do.

Similarly, we obtain

$$\beta(ac) \angle \beta(ca), \text{ from (ii)},$$
$$\gamma(ca) \angle \gamma(ca), \text{ from (iii)},$$
$$\epsilon(ac) \angle \epsilon(ca), \text{ from (iv)}.$$

Again,

$$\{\alpha(ab) \angle \alpha(ba)\}\{\alpha(ba) \angle \alpha(ab)\} \angle \{\alpha(ab) \angle \alpha(ab)\},$$

by $(x \angle y)(y \angle z) \angle (x \angle z)$.

Accordingly,

$$\alpha(ab) \angle \alpha(ab),$$
$$\beta(ab) \angle \beta(ab),$$
$$\gamma(ab) \angle \gamma(ab),$$
$$\epsilon(ab) \angle \epsilon(ab).$$

A complete induction of the members of this set and an application of the principle,

$$(x \angle y) \angle (y' \angle x'),$$

yields the general result,

$$k'(ab) \angle k'(ab), \qquad \text{I}$$

$k(ab)$ being understood as representing any one of the unprimed letters, α, β, γ, ϵ.

§ 6. Each one of the propositions so far derived, being true, is implied by the unit multiplier, i. The *contradictory* (as it is called) of i will be any proposition, which is untrue for all meanings of the terms that enter into it. It will be represented by the symbol, o, and will be defined by

$$o \angle i, \qquad (i \angle o)',$$

wherein it will be seen that i stands for o'. For a verbal interpretation of o and i we may read:

o = *No proposition is true*,
i = *One proposition is true*.

§ 7. The utility of the concept of *zero*, o (the *null*-proposition), and of *one*, i (the *one*-proposition), will be illustrated in part by the derivations, which follow. We have

$$\{\alpha(ab) \angle \alpha(ab)\} \angle \{\alpha(ab)\alpha'(ab) \angle o\},$$

by $(x \angle y) \angle (xy' \angle o)$;

$$\{\alpha(ab)\alpha'(ab) \angle o\} | \angle \{\alpha(ab) \angle \alpha''(ab)\},$$

by $(xy \angle o) \angle (x \angle y')$. Thus we should obtain

$$\alpha(ab) \angle \alpha''(ab),$$
$$\beta(ab) \angle \beta''(ab),$$
$$\gamma(ab) \angle \gamma''(ab),$$
$$\epsilon(ab) \angle \epsilon''(ab),$$

and, if we add to these the following assumptions,

$$\alpha''(ab) \angle \alpha(ab),$$
$$\beta''(ab) \angle \beta(ab),$$
$$\gamma''(ab) \angle \gamma(ab),$$
$$\epsilon''(ab) \angle \epsilon(ab),$$

a general result will be obvious, viz.,

$$k(ab) \angle k''(ab), \qquad k''(ab) \angle k(ab), \qquad \text{II}$$

wherein the same restriction is imposed upon $k(ab)$ as before.

§ 8. The principle, that the truth of any one of the four categorical forms implies the falsity of each one of the others, a generalization, which will now be established, is characteristic of the logic, which we are constructing. We shall begin by setting down the three characteristic postulates,

(vi) $$\beta(aa) \angle \beta'(aa),$$
(vii) $$\gamma(aa) \angle \gamma'(aa),$$
(viii) $$\epsilon(aa) \angle \epsilon'(aa).$$

Then, by the principle,

$$(x \angle x') \angle (x \angle y),$$

we may establish at once,

$$\beta(aa) \angle o,$$
$$\gamma(aa) \angle o,$$
$$\epsilon(aa) \angle o.$$

Postulate (ii) above yields, for $a = c$, $\alpha(ba)\beta(ba) \angle \beta(aa)$;

$\{\alpha(ba)\beta(ba) \angle \beta(aa)\}\{\beta(aa) \angle o\} \angle \{\alpha(ba)\beta(ba) \angle o\}$,

by $(x \angle y)(y \angle z) \angle (x \angle z)$;

$$\{\alpha(ba)\beta(ba) \angle o\} \angle \{\alpha(ba) \angle \beta'(ba)\},$$

by $(xy \angle o) \angle (x \angle y')$.

Similarly, by (iii), $\alpha(ba) \angle \gamma'(ab)$;

$$\{\alpha(ab) \angle \alpha(ba)\}\{\alpha(ba) \angle \gamma'(ab)\} \angle \{\alpha(ab) \angle \gamma'(ab)\},$$

by $(x \angle y)(y \angle z) \angle (x \angle z)$; and, by (iv), we obtain $\alpha(ab) \angle \epsilon'(ab)$.

If now we postulate,

(ix) $$\gamma(ba)\gamma(cb) \angle \gamma(ca),$$

(x) $$\epsilon(ba)\gamma(cb) \angle \epsilon(ca),$$

there result as before $\epsilon(ab) \angle \gamma'(ab)$ and $\gamma(ab) \angle \gamma'(ba)$, and if

(xi) $$\beta(ab) \angle \gamma'(ab),$$

(xii) $$\beta(ab) \angle \epsilon'(ab),$$

all of the remaining implications of this form, viz.,

$$\epsilon(ab) \angle \alpha'(ab), \qquad \gamma(ab) \angle \alpha'(ab),$$
$$\epsilon(ab) \angle \beta'(ab), \qquad \gamma(ab) \angle \beta'(ab),$$
$$\epsilon(ab) \angle \gamma'(ab), \qquad \beta(ab) \angle \alpha'(ab),$$

are obtained at once from those that have just been established by

$$(x \angle y') \angle (y \angle x').$$

If, now, $k(ab)$ and $w(ab)$ can not represent the same categorical form, $\gamma(ab)$ and $\gamma(ba)$ being considered distinct, and if further $k(ab)$ and $w(ab)$ can stand only for the unprimed letters, α, β, γ, ϵ,

$$k(ab) \angle w'(ab). \qquad \text{III}$$

This generalization is one of the characteristics, which marks an equivalence of the logic, whose foundations are here set down, with the classical logic of Aristotle. If we were to follow an accepted modern tradition, which regards $\epsilon(aa')$ as a true proposition, ($a' = non$-a), not all implications of this type will hold, for $\gamma(ab) \angle \epsilon'(ab)$ and $\epsilon(ab) \angle \gamma'(ab)$ become $\epsilon(oi) \angle \gamma'(oi)$ and $\gamma(oi) \angle \epsilon'(oi)$, for $a = o, b = i$ (see § 13 below). In such a logic, which is, indeed, an alternative, or *Non-Aristotelian* system, but which gives up the advantage gained by our symmetry, we should have to write $\{\gamma(ab) \angle \epsilon'(ab)\}'$ and $\{\epsilon(ab) \angle \gamma'(ab)\}'$. Moreover, postulate (x), from which these results are derived becomes itself untrue and the same remark applies to $\epsilon(ab)\gamma(cb) \angle \epsilon(ca)$ and $\gamma(ab)\epsilon(c, b) \angle \epsilon(ca)$, implications, which will later be made to depend on postulate (x).

§ 9. We shall now establish the untruth of certain forms of implication, making them ultimately depend upon the invalidity of $i \angle o$, whose untruth is set down as a matter of definition.

Suppose $\alpha(aa) \angle \alpha'(aa)$ were true.

$$\{i \angle \alpha(aa)\} \angle \{\alpha'(aa) \angle o\},$$

by $(x' \angle y) \angle (y' \angle x)$;

$$\{i \angle \alpha(aa)\}\{\alpha(aa) \angle \alpha'(aa)\} \angle \{i \angle \alpha'(aa)\},$$

by $(x \angle y)(y \angle z) \angle (x \angle z)$;

$$\{i \angle \alpha'(aa)\}\{\alpha'(aa) \angle o\} \angle \{i \angle o\},$$

by the same principle.

But $i \angle o$ is untrue.

$$\therefore \quad \alpha(aa) \angle \alpha'(aa) \text{ is untrue.}$$

Again,

$$\{\alpha(aa) \angle \alpha'(aa)\}'\{\beta(aa) \angle \alpha'(aa)\} \angle \{\alpha(aa) \angle \beta(aa)\}',$$

by $(x \angle z)'(y \angle z) \angle (x \angle y)'$;

$$\{\alpha(aa) \angle \beta'(aa)\}\{\alpha(aa) \angle \beta(aa)\}' \angle \{\beta'(aa) \angle \beta(aa)\}',$$

by $(x \angle y)(x \angle z)' \angle (y \angle z)'$.

Accordingly, we have

$$\{\alpha (aa) \angle \alpha'(aa)\}',$$
$$\{\beta'(aa) \angle \beta (aa)\}',$$
$$\{\gamma'(aa) \angle \gamma (aa)\}',$$
$$\{\epsilon'(aa) \angle \epsilon (aa)\}'.$$

and, since the untruth of any proposition is implied, whenever we can point to a special instance of its being untrue, it follows that

$$\{\alpha\,(ab) \angle \alpha'(ab)\}',$$
$$\{\beta'(ab) \angle \beta(ab)\}',$$
$$\{\gamma'(ab) \angle \gamma(ab)\}',$$
$$\{\epsilon'(ab) \angle \epsilon\,(ab)\}',$$

The first and third members of the set

$$\{\alpha'(ab) \angle \alpha\,(ab)\}',$$
$$\{\beta\,(ab) \angle \beta'(ab)\}',$$
$$\{\gamma\,(ab) \angle \gamma'(ab)\}',$$
$$\{\epsilon\,(ab) \angle \epsilon'(ab)\}',$$

will be established on making $a = o$, $b = i$. For the reduction of the second and fourth see exercise (10) at the end of this chapter.

As a result of a complete induction of the members of these sets and upon application of $(x \angle y)' \angle (y' \angle x')'$, it follows that

$$\{k\,(ab) \angle k'\,(ab)\}', \qquad \{k'\,(ab) \angle k\,(ab)\}', \qquad \text{IV}$$
$$\{k'(ab) \angle k''(ab)\}', \qquad \{k''(ab) \angle k'(ab)\}'.$$

§ 10. If $\alpha'(ab) \angle \beta(ab)$ were a true implication, we should have:

$$\{\gamma(ab) \angle \alpha'(ab)\}\{\alpha'(ab) \angle \beta(ab)\} \angle \{\gamma(ab) \angle \beta(ab)\},$$

by $(x \angle y)(y \angle z) \angle (x \angle z)$;

$$\{\gamma(ab) \angle \beta(ab)\}\{\beta(ab) \angle \gamma'(ab)\} \angle \{\gamma(ab) \angle \gamma'(ab)\},$$

by the same principle.

$$\therefore \quad \alpha'(ab) \angle \beta(ab) \text{ is untrue.}$$

Applying the same method of reduction there will result:

$$\{\alpha'(ab) \angle \beta(ab)\}', \qquad \{\beta'(ab) \angle \gamma(ab)\}',$$
$$\{\alpha'(ab) \angle \gamma(ab)\}', \qquad \{\beta'(ab) \angle \epsilon(ab)\}',$$
$$\{\alpha'(ab) \angle \epsilon(ab)\}', \qquad \{\gamma'(ab) \angle \epsilon(ab)\}',$$
$$\{\gamma'(ab) \angle \gamma(ba)\}',$$

and upon application of

$$(x' \angle y)' \angle (y' \angle x)',$$
$$\{\epsilon'(ab) \angle \alpha(ab)\}', \qquad \{\gamma'(ab) \angle \alpha(ab)\}',$$
$$\{\epsilon'(ab) \angle \beta(ab)\}', \qquad \{\gamma'(ab) \angle \beta(ab)\}',$$
$$\{\epsilon'(ab) \angle \gamma(ab)\}', \qquad \{\beta'(ab) \angle \alpha(ab)\}'.$$

We are now prepared to lay down the final generalizations which are given below. From the propositions that have just been enumerated there will follow

$$\{w'(ab) \angle k(ab)\}', \qquad \text{V}$$

from III and V, by

$$(x \angle y) \angle (y' \angle x'),$$
$$(x \angle y)' \angle (y' \angle x')',$$
$$k''(ab) \angle w'(ab), \qquad \{w'(ab) \angle k''(ab)\}', \qquad \text{VI}$$

from III and IV, by

$$(x \angle z)'(y \angle z) \angle (x \angle y)',$$
$$(x \angle y)' \angle (y' \angle x')',$$
$$\{k(ab) \angle w''(ab)\}', \qquad \{w''(ab) \angle k(ab)\}'. \qquad \text{VII}$$

§ 11. In order to classify the categorical forms under the heads, *contradictories*, *contraries*, *subcontraries*, and *subalterns*, let us consider what special meanings of $x(ab)$ and $y(ab)$ render true or untrue,

(1) $$x(ab) \angle y'(ab),$$
(2) $$y'(ab) \angle x(ab).$$

If $x(ab)$ and $y(ab)$ satisfy (1) and (2) together, $x(ab)$ is said to be *contradictory* to $y(ab)$. By I, $k'(ab)$ is contradictory to $k(ab)$ and, by II, $k(ab)$ is contradictory to $k'(ab)$.

If $x(ab)$ and $y(ab)$ satisfy (1) alone, $x(ab)$ is said to be *contrary* to $y(ab)$. By III and V, $k(ab)$ is contrary to $w(ab)$.

If $x(ab)$ and $y(ab)$ satisfy (2) alone $x(ab)$ is said to be *subcontrary* to $y(ab)$. By VI, $k'(ab)$ is subcontrary to $w'(ab)$.

If $x(ab)$ and $y(ab)$ satisfy neither (1) nor (2), $x(ab)$ is said to be *subaltern* to $y(ab)$. By VII, $k(ab)$ is subaltern to $w'(ab)$, and, by IV, $k(ab)$ and $k'(ab)$ are each the subalterns of themselves.

§ 12. In order to classify terms under the heads, *contradictories*, *contraries*, *subcontraries* and *subalterns*, let us consider what special meanings of a and b render true or untrue,

(1) $$\alpha'(ab') \angle \gamma(ab'),$$
(2) $$\alpha'(b'a) \angle \gamma(b'a).$$

The postulate $\alpha'(a'a') \angle \alpha(a'a')$ implies $\alpha'(a'a') \angle \gamma(a'a')$, for

$$\{\alpha'(a'a') \angle \alpha(a'a')\} \angle \{\alpha'(a'a') \angle \alpha''(a'a')\},$$

by $(x \angle y) \angle (y' \angle x')$;

$$\{\alpha'(a'a') \angle \alpha''(a'a')\} \angle \{\alpha'(a'a') \angle \gamma(a'a')\},$$

by $(x \angle x') \angle (x \angle y)$.

If *contradictory* terms be those meanings of a and b that render (1) and (2) true together, then, by $\alpha'(a'a') \angle \gamma(a'a')$, it follows that a and a' are contradictory.

If *contrary* terms be those meanings of a and b that cause (1) alone to become true, then if we assume,

$$\alpha'(oi) \angle \gamma(oi),$$
$$\{\alpha'(io) \angle \gamma(io)\}',$$

where $o' = i$, $i' = o$, we derive in particular the fact that o is the contrary of itself.

If *subcontrary* terms be those meanings of a and b that cause (2) alone to become true, then, from the assumptions just written down, it follows in particular that i is the subcontrary of itself.

If (1) and (2) remain untrue for some special meaning of a and b, then a is said to be the *subaltern* of b. From the two equivalent propositions,

$$\{\alpha'(aa') \angle \gamma(aa')\}', \qquad \{\alpha(a'a) \angle \gamma(a'a)\}',$$

whose untruth may be established on making $a = i$, $a' = o$, in the first, $a' = i$, $a = o$, in the second, and which imply

$$\{\alpha'(ab') \angle \gamma(ab')\}', \qquad \{\alpha'(b'a) \angle \gamma(b'a)\}',$$

it will be seen that a is in general the subaltern of b and of itself.

§ 13. If the meaning of *zero* (o) is unique; that is, if we assume,

$$i \angle \{\alpha'(io) \angle \gamma(io)\}',$$

which is the same as,

$$\{\alpha'(io) \angle \gamma(io)\} \angle o,$$

we should have,

$$\alpha(io) \angle o, \qquad \gamma(io) \angle o,$$

and from

$$\alpha'(oi) \angle \gamma(oi),$$
$$\alpha(oi) \angle o, \qquad \gamma'(oi) \angle o.$$

These last results and others, that have gone before (viz., III), yield:

$$\begin{array}{llll} \alpha'(oo) \angle o, & \alpha\,(oi) \angle o, & \alpha(io) \angle o, & \alpha'(ii) \angle o, \\ \beta\,(oo) \angle o, & \beta\,(oi) \angle o, & \beta(io) \angle o, & \beta\,(ii) \angle o, \\ \gamma\,(oo) \angle o, & \gamma'(oi) \angle o, & \gamma(io) \angle o, & \gamma\,(ii) \angle o, \\ \epsilon\,(oo) \angle o, & \epsilon\,(oi) \angle o, & \epsilon(io) \angle o, & \epsilon\,(ii) \angle o. \end{array}$$

We may now establish

$$\{\alpha'(aa') \angle \alpha(aa')\}', \qquad \{\epsilon'(aa') \angle \epsilon\,(aa')\}',$$
$$\{\gamma'(aa') \angle \gamma(aa')\}', \qquad \{\gamma(aa') \angle \gamma'(aa')\}',$$

and if we postulate

(xiii) $\alpha(aa') \angle \alpha'(aa'),$

(xiv) $\beta(aa') \angle \beta'(aa'),$

(xv) $\{\epsilon\,(aa') \angle \epsilon'(aa')\}',$

(xvi) $\{\gamma'(aa') \angle \gamma(a'a)\}',$

the truth or untruth of every remaining variety of *immediate inference*, $x(a, b) \angle y(a, b)$, may be derived.

§ 14. The operation of *simple conversion* consists in the interchange of subject and predicate. From $\gamma(io) \angle o$ and $\gamma'(oi) \angle o$, which are imposed upon us by the definition of the *null-class* (o)*, it will appear that the inconvertibility of γ is fundamental; for

$$\{i \angle o\}'\{\gamma(io) \angle o\} \angle \{i \angle \gamma(io)\}'$$

* The *null-class* (o) is to be understood as the class that contains no objects, or none of the objects that are in question. The *universe* (i) is the class that contains all of the objects that are in question.

by $(x \angle z)'(y \angle z) \angle (x \angle y)'$, and

$$\{i \angle \gamma(oi)\}\{i \angle \gamma(io)\}' \angle \{\gamma(oi) \angle \gamma(io)\}'$$

by $(x \angle y)(x \angle z)' \angle (y \angle z)'$.

$$\therefore \quad \gamma(ab) \angle \gamma(ba) \text{ is untrue.}$$

EXERCISES

1. The meaning of logical equality is given by

$$(x \angle y)(y \angle x) \angle (x = y),$$
$$(x = y) \angle (x \angle y)(y \angle x).$$

If $k(ab) = k(ab)k(ab)$ and $k(ab) \angle w'(ab)$, show that

$$\alpha(ab) \angle \beta'(ab)\gamma'(ab)\,\epsilon'(ab)\gamma'(ba),$$
$$\beta(ab) \angle \alpha'(ab)\gamma'(ab)\,\epsilon'(ab)\gamma'(ba),$$
$$\gamma(ab) \angle \alpha'(ab)\beta'(ab)\,\epsilon'(ab)\gamma'(ba),$$
$$\epsilon(ab) \angle \alpha'(ab)\beta'(ab)\gamma'(ab)\gamma'(ba),$$

by the aid of

$$(x \angle y)(y \angle z) \angle (x \angle z),$$
$$(x \angle y) \angle (zx \angle zy).$$

2. If

$$\beta'(ab)\gamma'(ab)\,\epsilon'(ab)\gamma'(ba) \angle \alpha(ab),$$
$$\alpha'(ab)\gamma'(ab)\,\epsilon'(ab)\gamma'(ba) \angle \beta(ab),$$
$$\alpha'(ab)\beta'(ab)\,\epsilon'(ab)\gamma'(ba) \angle \gamma(ab),$$
$$\alpha'(ab)\beta'(ab)\gamma'(ab)\gamma'(ba) \angle \epsilon(ab),$$

establish

$$\alpha'(ab) = \beta(ab) + \gamma(ab) + \epsilon(ab) + \gamma(ba),$$
$$\beta'(ab) = \alpha(ab) + \gamma(ab) + \epsilon(ab) + \gamma(ba),$$
$$\gamma'(ab) = \alpha(ab) + \beta(ab) + \epsilon(ab) + \gamma(ba),$$
$$\epsilon'(ab) = \alpha(ab) + \beta(ab) + \gamma(ab) + \gamma(ba),$$

assuming that the contradictory of a product is the sum of the contradictories of the separate factors and assuming the right to substitute $k(ab)$ directly for $k''(ab)$.

3. Assuming $x(ab) = x(ab)x(ab)$, $x(ab) = x(ab) + x(ab)$, show that

$$\alpha'(ab)\beta'(ab) = \gamma(ab) + \epsilon(ab) + \gamma(ba), \text{ etc., etc.,}$$
$$\alpha(ab) + \beta(ab) = \gamma'(ab)\epsilon'(ab)\gamma'(ba), \text{ etc., etc.}$$

4. Establish the general results,

$$k(ab) = k(ab)w'(ab), \qquad k'(ab) = k'(ab) + w(ab),$$
$$k'(ab)w(ab) = o,$$

5. From the principle, $(x \angle z)'(y \angle z) \angle (x \angle y)'$, and the postulate, $\{\alpha(aa) \angle \alpha'(aa)\}'$, derive

$$\{\alpha(aa) \angle \beta(aa)\}', \qquad \{\alpha(aa) \angle \gamma(aa)\}', \qquad \{\alpha(aa) \angle \epsilon(aa)\}'.$$

6. By the aid of the principles,

$$(x \angle y)(y \angle z) \angle (x \angle z), \qquad (x \angle x') \angle (x \angle y),$$

from the postulate, $\alpha'(aa) \angle \alpha(aa)$, and results already established, (viz., III), show that all propositions of the form, $x(aa) \angle y(aa)$, except the three cases in the last example, are true implications, $x(aa)$ and $y(aa)$ representing only the unprimed letters.

7. Show by the method of the last example that $\alpha(aa) \angle \alpha'(aa)$ is the only untrue implication of the form $x(aa) \angle y'(aa)$.

8. Derive seven true implications of the form, $x'(aa) \angle y(aa)$, and nine untrue implications of the same form.

9. Establish the untruth of

$$k(a, b) \angle w(a, b), \qquad k'(a, b) \angle w'(a, b).$$

10. Establish the untruth of $\beta(ab) \angle \beta'(ab)$ and $\epsilon(ab) \angle \epsilon'(ab)$ by making them depend upon the untruth of $\beta(ba)\beta(cb) \angle \beta'(ca)$ and $\epsilon(ba)\epsilon(cb) \angle \epsilon'(ca)$ respectively (see the postulates of the next chapter). Thus,

$$\{\beta(ba)\beta(cb) \angle \beta'(ca)\}' \angle \{\beta(ba)\beta(cb)\beta(ca) \angle o\}',$$

by $(x \angle y')' \angle (xy \angle o)'$;

$$\{\beta(ba)\beta(cb)\beta(ca) \angle o\}'\{\beta(cb)\beta(ca)\cdot o \angle o\}$$
$$\angle \{\beta(ba)\beta(cb)\beta(ca) \angle \beta(cb)\beta(ca)\cdot o\}',$$

by $(xy \angle z)'(w \angle z) \angle (xy \angle w)'$;

$$\{\beta(ba)\beta(cb)\beta(ca) \angle \beta(cb)\beta(ca)\cdot o\}' \angle \{\beta(ba) \angle o\}'$$

by $(xz \angle zy)' \angle (x \angle y)'$;

$$\{\beta(ba) \angle o\}' \angle \{\beta(ba) \angle \beta'(ba)\}',$$

by $(x \angle o)' \angle (x \angle x')'$.

CHAPTER II

§ 15. A *syllogism* is an implication belonging to one of the types,

$$1.\ x(ba)y(cb) \angle z(ca),$$
$$2.\ x(ab)y(cb) \angle z(ca),$$
$$3.\ x(ba)y(bc) \angle z(ca),$$
$$4.\ x(ab)y(bc) \angle z(ca).$$

These differences are known as the first, second, third and fourth *figures* of the syllogism respectively. The two forms conjoined to the left of the implication sign are called the *premises* and the form to the right of the implication sign is called the *conclusion*. The predicate of the conclusion is called the *major term* and points out the *major premise*, which by convention is written first, while the subject of the conclusion is called the *minor term* and points out the *minor premise*. The term, which is common to the premises and which does not appear in the conclusion, is called the *middle term*.

Since x, y and z may assume any one of the four values, α, β, γ, ϵ, there will be sixty-four ways in each figure, called the *moods* of the syllogism, in which $xy \angle z$ may be expressed. True syllogistic variants are called *valid moods*. Those that are untrue are called *invalid moods*.

It will be convenient to deduce in the first place all of the valid moods of syllogistic form that exist and to establish later on the invalidity of those moods that remain. In what follows we shall suppose that x, y and z stand only for the unprimed letters. Thus: we shall refer to $x(a, b)y(b, c) \angle z(ca)$, $x'(a, b)y(b, c) \angle z'(ca)$,

$$x(a, b)y(b, c) \angle z'(ca), \text{ etc.,}$$

as specific types.

§ 16. The valid moods of the syllogism,

$$x(a,\ b)y(b,\ c) \ \angle\ z(ca),$$

twenty-nine in all, which are not set down among the assumptions of § 5 and § 8, may be derived at once by the following principles:

$$(xy \ \angle\ z)(w \ \angle\ x) \ \angle\ (wy \ \angle\ z),$$
$$(x \ \angle\ y)(y \ \angle\ z) \ \angle\ (x \ \angle\ z),$$
$$(xy \ \angle\ z) \ \angle\ (yx \ \angle\ z).$$

Thus, from postulate (x), by the second principle,
$\{\epsilon(ba)\gamma(cb) \ \angle\ \epsilon(ca)\}\{\epsilon(ca) \ \angle\ \epsilon(ac)\} \ \angle\ \{\epsilon(ba)\gamma(cb) \ \angle\ \epsilon(ac)\}$, and, since the term-order in the conclusion is now reversed, so that the major term has become the minor term and the minor term has become the major term, it will be necessary to employ the third principle to restore the normal order of the premises. Accordingly,

$$\{\epsilon(ba)\gamma(cb) \ \angle\ \epsilon(ac)\} \ \angle\ \{\gamma(cb)\epsilon(ba) \ \angle\ \epsilon(ac)\},$$

and it will be seen that the term order in this result is that of the fourth figure. The second principle (above) thus enables us to *convert simply* in the conclusion and the effect of simple conversion in the conclusion is to change the first figure to the fourth.

Similarly, since the third principle enables us to arrange the premises in either order, the first principle will allow us to convert simply in either premise, if that premise be not in the γ-form. Thus, from postulate (i) of § 5,

$$\{\alpha(ba)\alpha(bc) \ \angle\ \alpha(ca)\} \ \angle\ \{\alpha(bc)\alpha(ba) \ \angle\ \alpha(ca)\},$$

by the third principle (above);

$$\{\alpha(bc)\alpha(ba) \ \angle\ \alpha(ca)\}\{\alpha(cb) \ \angle\ \alpha(bc)\} \ \angle\ \{\alpha(cb)\alpha(ba) \ \angle\ \alpha(ca)\},$$

by the first principle (above);

$$\{\alpha(cb)\alpha(ba) \ \angle\ \alpha(ca)\} \ \angle\ \{\alpha(ba)\alpha(cb) \ \angle\ \alpha(ca)\},$$

by the third principle (above); and this result is a valid mood of the first figure. However, when it is desired to convert simply in the minor premise, it will be more convenient to employ at once the principle,

$$(xy \angle z)(w \angle y) \angle (xw \angle z),$$

and avoid two of the three steps, that would otherwise be necessary.

EXERCISE

From postulates (i–iv) of § 5 and postulate (x) of § 8 derive twenty-three valid moods of the syllogism by the aid of the principles,

$$(xy \angle z)(w \angle x) \angle (wy \angle z),$$
$$(xy \angle z)(w \angle y) \angle (xw \angle z),$$
$$(xy \angle z)(z \angle w) \angle (xy \angle w),$$
$$(xy \angle z) \angle (yx \angle z).$$

§ 17. The valid moods of the syllogism,

$$x(a, b)y(b, c) \angle z'(ca),$$

one hundred and forty-two in number, as well as those of the syllogisms, $x(a, b)y'(b, c) \angle z'(ca)$ and $x'(a, b)y(b, c) \angle z'(ca)$, which number thirty-one and twenty-seven respectively, may now be obtained from the results of § 16 and the forms of immediate inference contained in § 8, by the aid of the additional principles,

$$(xy \angle z') \angle (xz \angle y'),$$
$$(xy \angle z') \angle (zy \angle x').$$

The examples, which follow, will be enough to illustrate the method.

(1) $\{\gamma(ba)\gamma(cb) \angle \gamma(ca)\} \angle \{\gamma(ba)\gamma'(ca) \angle \gamma'(cb)\},$

by $(xy \angle z) \angle (xz' \angle y')$.

(2) $\{\gamma(ab)\gamma'(cb) \angle \gamma'(ca)\}\{\epsilon(cb) \angle \gamma'(cb)\}$
$\angle \{\gamma(ab)\epsilon(cb) \angle \gamma'(ca)\},$

by $(xy \angle z)(w \angle y) \angle (xw \angle z)$.

(3) $\{\gamma(ab)\epsilon(cb) \angle \gamma'(ca)\} \angle \{\gamma(ab)\gamma(ca) \angle \epsilon'(cb)\}$

by $(xy \angle z') \angle (xz \angle y')$.

No other valid moods of syllogistic form exist, except those that have now been enumerated, as will appear in the sequel, when all of the remaining variants shall have been declared untrue.

EXERCISES

1. From postulate (x) of § 8 above deduce six valid implications of the form, $x(a, b)y'(b, c) \angle z'(ca)$.

2. From postulate (x) of § 8 above deduce thirty-three valid implications of the form, $x(a, b)y(b, c) \angle z'(ca)$.

3. Assuming the special conditions mentioned at the end of § 8 to hold true, show that postulate (ix) of § 8 yields only thirteen valid implications of the form given in the last exercise.

§ 18. It will be convenient in establishing the invalid moods of the syllogism to begin with the form

$$x(a, b)y(b, c) \angle z'(ca).$$

Any invalid mood under this head, which contains an α-premise or an α-conclusion, may be shown to be invalid by identifying terms in the α-form. Thus:

1. Suppose $\alpha(ba)\gamma(cb) \angle \gamma'(ca)$ were valid, and identify terms in the major premise.

$$\{\alpha(aa)\gamma(ca) \angle \gamma'(ca)\}\{i \angle \alpha(aa)\} \angle \{\gamma(ca) \angle \gamma'(ca)\},$$

by $(xy \angle z)(w \angle x) \angle (wy \angle z)$.

$$\therefore \quad \alpha(ba)\gamma(cb) \angle \gamma'(ca) \text{ is invalid.}$$

2. Suppose $\gamma(ab)\gamma(cb) \angle \alpha'(ca)$ were valid and identify terms in the conclusion.

$$\{\gamma(ab)\gamma(ab) \angle \alpha'(aa)\}\{\alpha'(aa) \angle o\} \angle \{\gamma(ab)\gamma(ab) \angle o\},$$

by $(xy \angle z)(z \angle w) \angle (xy \angle w)$;

$$\{\gamma(ab)\gamma(ab) \angle o\} \angle \{\gamma(ab) \angle \gamma'(ab)\},$$

by $(xy \angle o) \angle (x \angle y')$.

$$\therefore \quad \gamma(ab)\gamma(cb) \angle \alpha'(ca) \text{ is invalid.}$$

EXERCISE

Establish the invalidity of the thirty-four moods of the syllogism, $x(a, b)y(b, c) \angle z'(ca)$, which are invalid and which contain an α-premise or an α-conclusion.

In order to deduce the invalid moods, which remain, eighty in all, it will be necessary to add eleven postulates to the ones already set down. These assumptions are:

(xvii)	$\{\beta(ba)\beta(cb) \angle \beta'(ca)\}'$,
(xviii)	$\{\beta(ba)\beta(cb) \angle \gamma'(ca)\}'$,
(xix)	$\{\beta(ba)\beta(cb) \angle \epsilon'(ca)\}'$,
(xx)	$\{\gamma(ab)\gamma(cb) \angle \beta'(ca)\}'$,
(xxi)	$\{\gamma(ba)\gamma(bc) \angle \beta'(ca)\}'$,
(xxii)	$\{\gamma(ba)\gamma(cb) \angle \gamma'(ca)\}'$,
(xxiii)	$\{\gamma(ab)\gamma(cb) \angle \epsilon'(ca)\}'$,
(xxiv)	$\{\epsilon(ba)\gamma(bc) \angle \beta'(ca)\}'$,
(xxv)	$\{\epsilon(ba)\gamma(bc) \angle \epsilon'(ca)\}'$,
(xxvi)	$\{\epsilon(ba)\epsilon(cb) \angle \beta'(ca)\}'$,
(xxvii)	$\{\epsilon(ba)\epsilon(cb) \angle \epsilon'(ca)\}'$.

EXERCISE

From postulates (xvii–xxvii) deduce sixty-nine other non-implications of the same form, by the aid of the additional principles,

$$(xy \angle z)'(w \angle z) \angle (xy \angle w)',$$
$$(xy \angle z)'(x \angle w) \angle (wy \angle z)',$$
$$(xy \angle z)'(y \angle w) \angle (xw \angle z)',$$
$$(xy \angle z')' \angle (xz \angle y')',$$
$$(xy \angle z')' \angle (zy \angle x')',$$
$$(xy \angle z)' \angle (yx \angle z)'.$$

§ 19. All of the invalid moods of the syllogistic form, $x(a, b)y(b, c) \angle z(ca)$, $x'(a, b)y(b, c) \angle z'(ca)$ and

$$x(a, b)y'(b, c) \angle z'(ca),$$

may be deduced at once from the results that have now been established. A few examples will be enough to illustrate the method.

$$\{\epsilon(ba)\gamma(bc) \angle \beta'(ca)\}' \angle \{\beta(ca)\epsilon(ba) \angle \gamma'(bc)\}',$$

by $(xy \angle z')' \angle (zx \angle y')'$;

$$\{\beta(ab)\epsilon(cb) \angle \gamma'(ca)\}'\{\beta(ab) \angle \gamma'(ab)\} \angle \{\gamma'(ab)\epsilon(cb) \angle \gamma'(ca)\}',$$

by $(xy \angle z)'(x \angle w) \angle (wy \angle z)'$;

$$\{\gamma'(ab)\epsilon(cb) \angle \gamma'(ca)'\} \angle \{\epsilon(cb)\gamma(ca) \angle \gamma(ab)\},'$$

by $(x'y \angle z')' \angle (yz \angle x)'$;

$$\{\gamma'(ab)\epsilon(cb) \angle \gamma'(ca)\}' \angle \{\gamma(ca)\gamma'(ab) \angle \epsilon'(cb)\}',$$

by $(xy \angle z')' \angle (zx \angle y')'$;

$$\{\epsilon(ba)\gamma(bc) \angle \beta'(ca)\}'\{\epsilon(ca) \angle \beta'(ca)\} \angle \{\epsilon(ba)\gamma(bc) \angle \epsilon(ca)\}',$$

by $(xy \angle z)'(w \angle z) \angle (xy \angle w)'$;

$$\{\epsilon(ba)\gamma(bc) \angle \epsilon(ca)\}'\{\epsilon(ac) \angle \epsilon(ca)\} \angle \{\gamma(bc)\epsilon(ba) \angle \epsilon(ac)\}',$$

by $(xy \angle z)'(w \angle z) \angle (yx \angle w)'$.

EXERCISES

1. Show that there exist no valid implications of the form, $x'(a, b)y(b, c) \angle z(ca)$ or $x(a, b)y'(b, c) \angle z(ca)$, and consequently none of the form, $x'(a, b)y'(b, c) \angle z(ca)$ or $x'(a, b)y'(b, c) \angle z'(ca)$.

2. Show that as a result of a complete induction of the moods in question. (*a*) a valid mood of the syllogism, whose premises and conclusion are all unprimed forms and one of whose premises is of the same form as the conclusion, will remain valid, when the other premise is put in the α-form; and (*b*) a valid mood of the syllogism, whose premises are unprimed forms and whose conclusion is a primed form and one of whose premises is of a different form from the conclusion, will remain valid, when the other premise is put in the α-form.

§ 20. It will be well at this point to indicate the equivalence of the logic, whose system has now been partially developed, with the classical science, perfected in the *Organon* of Aristotle.

The four categorical forms employed by the traditional logic and denoted by the letters, A, E, I, O, are:

$$A(ab) = \text{All } a \text{ is } b,$$
$$E(ab) = \text{No } a \text{ is } b,$$
$$I(ab) = \text{Some } a \text{ is } b,$$
$$O(ab) = \text{Some } a \text{ is not } b,$$

the word *some*, which is expressed before the subject of I and O and understood before the predicate of A and I, being interpreted to mean, *some at least, possibly all.*

This set of four forms satisfies certain conditions, which are characteristic of the system, *i.e.*,

1. Corresponding to each member of the set, there is another, which stands for its contradictory;

2. The relation of subalternation, $A(ab)$ *implies* $I(ab)$, holds true;

3. $A(ab)$ becomes true, when subject and predicate have been identified;

4. The subject and predicate of $E(ab)$ and $I(ab)$ alone are simply convertible,

We should have to have,

to satisfy (1), $A(ab)O(ab) \angle o$,
$E(ab)I(ab) \angle o$,
$A'(ab)O'(ab) \angle o$,
$E'(ab)I'(ab) \angle o$;

to satisfy (2), $A(ab)E(ab) \angle o$;

to satisfy (3), $A'(aa) \angle o$,

to satisfy (4), $I(ab) \angle I(ba)$.*

Today it is all but universally taken for granted that not all of these conditions hold true, if the terms are allowed to take on the meanings *nothing* and *universe*, and it is usual to retain (1), (3) and (4) and to assert that the relation of subalternation of the classical logic is false. This modern

* It would in the end economize assumptions to take $E(ab)A(cb) \angle E(ca)$ for granted instead of this last form of immediate inference, for

$E(ab)A(cb) \angle E(ca)$ yields $E(ab)A(bb) \angle E(ba)$, for $b = c$;
$\{E(ab)A(bb) \angle E(ba)\}\{i \angle A(bb)\} \angle \{E(ab) \angle E(ba)\}$, by (3);
$\{E(ab) \angle E(ba)\} \angle \{E'(ba) \angle E'(ab)\}$;
$\{I(ba) \angle E'(ba)\}\{E'(ba) \angle E'(ab)\} \angle \{I(ba) \angle E'(ab)\}$, by (1);
$\{I(ba) \angle E'(ab)\}\{E'(ab) \angle I(ab)\} \angle \{I(ba) \angle I(ab)\}$, by (1).

tradition is, however, based upon a misapprehension, upon the supposed necessity of retaining $A(ab) = E(ab')$. The implications set down in the following table will be seen to satisfy not only conditions (1–4) but also the definition of the null-class and the consequences, that follow from that definition. These implications are:

$A'(oo) \angle o,$	$A'(oi) \angle o,$	$A(io) \angle o,$	$A'(ii) \angle o,$
$E(oo) \angle o,$	$E(oi) \angle o,$	$E(io) \angle o,$	$E(ii) \angle o,$
$I'(oo) \angle o,$	$I'(oi) \angle o,$	$I'(io) \angle o,$	$I'(ii) \angle o,$
$O(oo) \angle o,$	$O(oi) \angle o,$	$O'(io) \angle o,$	$O(ii) \angle o.$

We may now state the connection between the traditional propositions, A, E, I, O, and the special categorical forms, which have been employed in the text. This connection is expressed by the following equalities:

$$\begin{aligned} A(ab) &= \alpha(ab) + \gamma(ab), \\ E(ab) &= \epsilon(ab), \\ I(ab) &= \alpha(ab) + \beta(ab) + \gamma(ab) + \gamma(ba), \\ O(ab) &= \epsilon(ab) + \beta(ab) + \gamma(ba). \end{aligned}$$

and it will be readily seen that these values of A, E, I and O satisfy conditions (1–4).

If, now, we express α, β, γ and ϵ in the members of the set, A, E, I, O, we should have

$$\begin{aligned} \alpha(ab) &= A(ab)A(ba), \\ \beta(ab) &= I(ab)O(ab)O(ba), \\ \gamma(ab) &= A(ab)O(ba), \\ \epsilon(ab) &= E(ab), \end{aligned}$$

which may be verified by actually multiplying out these products as in ordinary algebra and allowing the product $k(ab)w(ab)$ to drop out whenever it occurs (see ex. 4 at the end of § 14).

It only remains to be pointed out that these equalities and those, which precede them, are satisfied by the implications in the table given above and by those contained in the similar table in § 13.

CHAPTER III

§ 21. In the remarks at the end of § 8 and elsewhere we have referred to a system of inference, in which not all of the implications of the common logic hold true. It is proposed now, as a further illustration of method, to construct in some detail another system, some of whose characteristic postulates stand in contradiction to those of § 5 and § 8. Without doing violence to the fundamental conditions described in § 12 and § 13, we may assume:

$$\alpha'(aa) \angle \alpha(aa), \qquad \text{(i)}$$
$$\beta'(aa) \angle \beta(aa), \qquad \text{(ii)}$$
$$\gamma'(aa) \angle \gamma(aa), \qquad \text{(iii)}$$
$$\epsilon\,(aa) \angle \epsilon'(aa), \qquad \text{(iv)}$$

which yield at once the equivalent set:

$$i \angle \alpha(aa),$$
$$i \angle \beta(aa),$$
$$i \angle \gamma(aa),$$
$$\epsilon\,(aa) \angle o.$$

§ 22. In order to frame an image of the possibility of $\alpha(aa)$, $\beta(aa)$ and $\gamma(aa)$ standing for true propositions, imagine the subject-class and the predicate-class, a and b, to approach connotative identity. It will then be understood, how it might become a question, as to whether $\alpha(ab)$, $\beta(ab)$ and $\gamma(ab)$ are to be regarded as true or false in the limiting case. This image would, of course, not serve to guide us, unless the assumptions we have made had an analytic justification. It is, too, in a sense misleading, for we shall have to conceive of $\beta(ab)$ and $\gamma(ab)$ becoming empirically untrue for special concrete meanings of a and b that render $\alpha(ab)$ empirically true, without making it impossible to regard $\beta(aa)$ and $\gamma(aa)$ as true for all meanings of a. In interpreting $\beta(aa)$ and $\gamma(aa)$, the part of a, which is not a, is taken to be the null-part.

§ 23. The other postulates, by the aid of which we shall effect our solution, are:

(v)	$\beta(ab) \angle \beta(ba)$,	(xi)	$\{\alpha(ab) \angle \beta(ab)\}'$,
(vi)	$\beta(ab) \angle \epsilon'(ab)$,	(xii)	$\{\alpha(ab) \angle \gamma(ab)\}'$,
(vii)	$\alpha(ba)\alpha(bc) \angle \alpha(ca)$,	(xiii)	$\{\gamma(ab) \angle \beta(ab)\}'$,
(viii)	$\alpha(ba)\,\epsilon(bc) \angle \epsilon(ca)$,	(xiv)	$\{\beta(ba)\beta(cb) \angle \epsilon'(ca)\}'$,
(ix)	$\gamma(ba)\gamma(cb) \angle \gamma(ca)$,	(xv)	$\{\gamma(ab)\gamma(cb) \angle \epsilon'(ca)\}'$,
(x)	$\gamma(ab)\,\epsilon(bc) \angle \epsilon(ca)$,	(xvi)	$\{\beta(ba)\gamma(cb) \angle \epsilon'(ca)\}'$,

(xvii) $\{\epsilon(ba)\epsilon(cb) \angle \epsilon'(ca)\}'$.

EXERCISES

1. Employing the method of § 5, deduce six valid moods of the form $x(a, b) \angle y(a, b)$.

2. Deduce twenty-two invalid moods of the form given in the last example.

Thus, suppose $\gamma(ab) \angle \gamma(ba)$ were valid,

$$\{\gamma(ab) \angle \gamma(ba)\} \angle \{\gamma(ab)\gamma(cb) \angle \gamma(ba)\gamma(cb)\},$$

by $(x \angle y) \angle (wx \angle wy)$;

$$\{\gamma(ab)\gamma(cb) \angle \gamma(ba)\gamma(cb)\}\{\gamma(ba)\gamma(cb) \angle \epsilon'(ca)\} \angle \{\gamma(ab)\gamma(cb) \angle \epsilon'(ca)\},$$

by $(x \angle y)(y \angle z) \angle (x \angle z)$ and a valid syllogism to be obtained later; but this result is invalid, by (xv);

$$\therefore \quad \gamma(ab) \angle \gamma(ba) \text{ is invalid.}$$

Again, suppose $\epsilon(ab) \angle \beta(ab)$ were valid.

$$\{\epsilon(ba) \angle \beta(ba)\} \angle \{\epsilon(ba)\alpha(cb) \angle \beta(ba)\alpha(cb)\},$$

by $(x \angle y) \angle (wx \angle wy)$;

$$\{\epsilon(ba)\alpha(cb) \angle \beta(ba)\alpha(cb)\}\{\beta(ba)\alpha(cb) \angle \epsilon'(ca)\} \angle \{\epsilon(ba)\alpha(cb) \angle \epsilon'(ca)\},$$

by the last result, a valid syllogism and

$$(x \angle y)(y \angle z) \angle (x \angle z).$$

Now, $\epsilon(ba)\alpha(cb) \angle \epsilon(ca)$ is valid and

$$\therefore \quad \epsilon(ba)\alpha(cb) \angle o,$$

by $(x \angle y)(x \angle y') \angle (x \angle o)$; and

$$\epsilon(ba)\alpha(cb)\beta(cd) \angle o,$$

by $(x \angle o) \angle (xy \angle o)$;

$$\therefore \epsilon(ba)\alpha(cb)\beta(cd) \angle \beta'(da),$$

by $(x \angle o) \angle (x \angle y)$;

Identifying c and b and suppressing the α-premise we should have

$$\epsilon(ba)\beta(bd) \angle \beta'(da),$$

which yields

$$\beta(da)\beta(bd) \angle \epsilon'(ba);$$

but this result contradicts (xiv);

$$\therefore \epsilon(ab) \angle \beta(ab) \text{ is invalid.}$$

3. Employing the method of § 8 deduce eleven valid moods in addition to (vi) of the form, $x(a, b) \angle y'(a, b)$.

4. Deduce twenty invalid moods of the form given in the last example.

The only invalid mood of this type, which offers any difficulty, is $\epsilon(ab) \angle \epsilon'(ab)$. Suppose, then, that this mood were valid.

$$\{\epsilon(ba) \angle \epsilon'(ba)\} \angle \{\epsilon(ba)\alpha(cb) \angle \epsilon'(ba)\alpha(cb)\},$$

by $(x \angle y) \angle (wx \angle wy)$;

$$\{\epsilon(ba)\alpha(cb) \angle \epsilon'(ba)\alpha(cb)\}\{\epsilon'(ba)\alpha(cb) \angle \epsilon'(ca)\} \angle \{\epsilon(ba)\alpha(cb) \angle \epsilon'(ca)\},$$

by $(x \angle y)(y \angle z) \angle (x \angle z)$ and a valid syllogism to be established later;

$$\{\epsilon(ba)\alpha(cb) \angle \epsilon'(ca)\}\{\epsilon(ba)\alpha(cb) \angle \epsilon(ca)\} \angle \{\epsilon(ba)\alpha(cb) \angle o\},$$

by $(x \angle y)(x \angle y') \angle (x \angle o)$ and a valid syllogism to be established later;

$$\{\epsilon(ba)\alpha(cb) \angle o\} \angle \{\epsilon(ba)\alpha(cb)\beta(cd) \angle o\},$$

by $(x \angle o) \angle (xy \angle o)$.

If, now, we identify b and c and suppress the α-premise, we should have $\epsilon(ba)\beta(bd) \angle o$;

$$\{\epsilon(ba)\beta(bd) \angle o\} \angle \{\epsilon(ba)\beta(bd) \angle \beta'(da)\},$$

by $(x \angle o) \angle (x \angle y)$;

$$\{\epsilon(ba)\beta(bd) \angle \beta'(da)\} \angle \{\beta(da)\beta(bd) \angle \epsilon'(ba)\}$$

by $(xy \angle z') \angle (zy \angle x')$; but this result contradicts (xiv) above, and

$$\therefore \quad \epsilon(ab) \angle \epsilon'(ab) \text{ is invalid.}$$

5. Establish the invalidity of all the thirty-two moods of the form, $x'(a, b) \angle y(a, b)$.

If we assume $\beta(oi) \angle o$ to be true, the results of the following table will then be forced upon us by what has gone before. That is,

$\alpha'(oo) \angle o$,	$\alpha(oi) \angle o$,	$\alpha(io) \angle o$,	$\alpha'(ii) \angle o$,
$\beta'(oo) \angle o$,	$\beta(oi) \angle o$,	$\beta(io) \angle o$,	$\beta'(ii) \angle o$,
$\gamma'(oo) \angle o$,	$\gamma'(oi) \angle o$,	$\gamma(io) \angle o$,	$\gamma'(ii) \angle o$,
$\epsilon(oo) \angle o$,	$\epsilon(oi) \angle o$,	$\epsilon(io) \angle o$,	$\epsilon(ii) \angle o$,

and it will be seen that postulate (xiii) above may now be saved.

6. Derive thirteen valid moods of the syllogism,

$$x(a, b)y(b, c) \angle z(ca),$$

as in § 16, from postulates (viii–x) above.

7. Prove that two hundred and eleven of the invalid moods of the type given in the last exercise may be shown to be invalid by the aid of the characteristic postulates (i–iv) above.

8. Show that the twenty-eight invalid moods not accounted for in the last exercise may be made to depend on one or the other of postulates (xiv–xvii) above.

9. Derive the eighty-one valid moods of the syllogism, $x(a, b)$ $y(b, c) \angle z'(ca)$.

10. Prove that one hundred and forty-four of the invalid moods of the type given in the last exercise may be reduced to invalid forms of immediate inference already established, and so shown to be invalid, by the aid of the characteristic postulates (i–iv) above.

11. Show that twenty-seven invalid moods not accounted for in the last exercise may be made to depend on postulates (xiv–xvii) above.

12. As in exercise (1), § 19, show that no other valid syllogistic variations exist, except those contained in the syllogisms, $x'(a, b)y(b, c) \angle z'(ca)$ and $x(a, b)y'(b, c) \angle z'(ca)$.

CHAPTER IV

§ 24. A sorites is a form of implication of the general type,*

$$x(1, 2)y(2, 3) \cdots z(n-1, n) \angle w(n1).$$

Certain valid moods of the sorites can be constructed from chains of valid syllogisms. For example the chain,

$$\gamma(21)\gamma(32) \angle \gamma(31),$$
$$\gamma(31)\gamma(43) \angle \gamma(41),$$
$$\gamma(41)\gamma(54) \angle \gamma(51),$$

will yield a valid mood, viz.,

$$\gamma(21)\gamma(32)\gamma(43)\gamma(54) \angle \gamma(51),$$

for

$$\{\gamma(41)\gamma(54) \angle \gamma(51)\}\{\gamma(31)\gamma(43) \angle \gamma(41)\} \angle \{\gamma(31)\gamma(43)\gamma(54) \angle \gamma(51)\},$$

and

$$\{\gamma(31)\gamma(43)\gamma(54) \angle \gamma(51)\}\{\gamma(21)\gamma(32) \angle \gamma(31)\} \angle \{\gamma(21)\gamma(32)\gamma(43)\gamma(54) \angle \gamma(51)\}.$$

Again,

$$\{\alpha(21)\alpha(32) \angle \alpha(31)\} \angle \{\alpha(21)\alpha(32)\alpha(43) \angle \alpha(31)\alpha(43)\}.$$
$$\therefore \quad \alpha(21)\alpha(32)\alpha(43) \angle \alpha(41).$$

The valid mood of the sorites is, accordingly, built up out of the chain,

$$\alpha(21)\alpha(32) \angle \alpha(31),$$
$$\alpha(31)\alpha(43) \angle \alpha(41).$$

It remains to be proven that the only valid moods of the sorites that exist can be built up from chains of valid syllogisms.

It will be convenient to take the conclusion successively in each one of the eight possible forms.

* In what follows it will be convenient to employ the ordinal numbers for class-terms instead of the initial letters of the alphabet. The solution given here belongs to the logic of the last chapter.

Conclusion α' and All Premises Unprimed

Suppose in the first instance that no ϵ-premise is present. Then no α-premise is present, for suppose $x(s, s + 1)$ to be an α-premise. Identifying terms in the α-, β- and γ-premises except x, the mood of the sorites will reduce to an invalid mood of immediate inference, viz.,

$$\alpha(s, s + 1) \angle \alpha'(s, s + 1).$$

Similarly no β-premise can occur, by

$$\beta(s, s + 1) \angle \alpha'(s, s + 1),$$

and no γ-premise can occur, by

$$\gamma(s, s + 1) \angle \alpha'(s, s + 1).$$

Consequently at least one ϵ-premise is present if the mood of the sorites is valid.

Not more than a single ϵ-premise can be present, for if two or more ϵ-premises were present, the mood of the sorites could, by identifying terms and suppressing the α-, β- and γ-premises, be reduced to the form,

$$\epsilon(1, 2)\epsilon(2, 3) \cdots \epsilon(n - 1, n) \angle \alpha'(n1),$$

and the validity of this mood can be made to depend on that of a mood in which all but two of the premises are absent,* viz.,

$$\epsilon(s, s + 1)\epsilon(s + 1, s + 2) \angle \alpha'(s + 2\, s),$$

which is invalid.

Contradict the ϵ-premise and the conclusion and interchange and the sorites reduces to the case of an ϵ'-conclusion, which will be considered later on.

* Thus, $\epsilon(12)\epsilon(23)\epsilon(34)\epsilon(45) \angle \alpha'(51)$ yields $\epsilon(14)\epsilon(43)\epsilon(34)\epsilon(45) \angle \alpha'(51)$ for $2 = 4$, and

$$\{\epsilon(14)\epsilon(43)\epsilon(34)\epsilon(45) \angle \alpha'(51)\}\{\epsilon(34) \angle \epsilon(43)\epsilon(34)\} \angle \{\epsilon(14)\epsilon(34)\epsilon(45) \angle \alpha'(51)\}.$$

Thus we obtain in succession,

$$\epsilon(14)\epsilon(34)\epsilon(45) \angle \alpha'(51), \text{ for } 2 = 4,$$
$$\epsilon(34)\epsilon(45) \angle \alpha'(53), \text{ for } 1 = 3.$$

Conclusion β' or γ' and All Premises Unprimed

Exactly as in the last case it can be shown that at least one ϵ-premise must be present and that not more than one ϵ-premise can occur. Contradicting and interchanging as before the ϵ-premise and the conclusion, the sorites reduces to the case of an ϵ'-conclusion.

Conclusion ϵ' and All Premises Unprimed

No ϵ-premise can occur, for, if one or more ϵ-premises were present, the mood of the sorites would reduce to the form,

$$\epsilon(1, 2)\epsilon(2, 3) \cdots \epsilon(n-1, n) \angle \epsilon'(n1),$$

by identifying terms in all of the α-, β- and γ-premises, and this form is reducible to the invalid syllogism,

$$\epsilon(s, s+1)\epsilon(s+1, s+2) \angle \epsilon'(s+2\ s),$$

or an invalid mood of immediate inference,

$$\epsilon(s, s+1) \angle \epsilon'(s\ s+1).$$

Not more than a single β-premise can occur, for, if two or more β-premises were present, the mood of the sorites would, by identifying terms in all of the premises but two of the β-premises, be reducible to an invalid syllogism of the form,

$$\beta(s, s+1)\beta(s+1, s+2) \angle \epsilon'(s+2\ s).$$

Suppose that no β- or γ-premise is present. Then all of the premises are in the α-form and the sorites becomes,

$$\alpha(1, 2)\alpha(2, 3) \cdots \alpha(n-1, n) \angle \epsilon'(n1),$$

which can be constructed from the chain of valid syllogisms,

$$\begin{aligned}
&\alpha(1, 2)\alpha(2, 3) \angle \alpha(31),\\
&\alpha(31)\alpha(3, 4) \angle \alpha(41),\\
&\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\\
&\alpha(n-2\ 1)\alpha(n-2, n-1) \angle \alpha(n-1\ 1),\\
&\alpha(n-1\ 1)\alpha(n-1, n) \angle \epsilon'(n\ 1).
\end{aligned}$$

If no γ-premise is present and all but one β-premise is in the α-form, the sorites then becomes,

$$\alpha(1, 2)\alpha(2, 3) \cdots \alpha(s-1, s)\beta(s, s+1)\alpha(s+1, s+2) \cdots \alpha(n-1, n) \angle \epsilon'(n1),$$

which can be built up from the chain of valid syllogisms,

$$\begin{aligned}
&\alpha(1, 2)\alpha(2, 3) \angle \alpha(31),\\
&\alpha(31)\alpha(3, 4) \angle \alpha(41),\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\alpha(s-1\ 1)\alpha(s-1, s) \angle \alpha(s1),\\
&\alpha(s1)\beta(s, s+1) \angle \epsilon'(s+1\ 1),\\
&\epsilon'(s+1\ 1)\alpha(s+1, s+2) \angle \epsilon'(s+2\ 1),\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\epsilon'(n-1\ 1)\alpha(n-1, n) \angle \epsilon'(n1).
\end{aligned}$$

Suppose, again, that all of the premises are in the γ-form, *i.e.*, that the sorites is

$$\gamma(1, 2)\gamma(2, 3) \cdots \gamma(n-1, n) \angle \epsilon'(n1).$$

The first premise, which presents the term-order $(s-1\ s)$, *i.e.*, with the smaller ordinal number appearing as subject, establishes that order in each one of the premises which follow. For, suppose that the term order $(s-1\ s)$, having once occurred, should appear reversed later on. The sorites would, by identifying terms, be reducible to an invalid syllogism, viz.,

$$\gamma(s-1\ s)\gamma(s+1\ s) \angle \epsilon'(s+1\ s-1).$$

The sorites becomes, consequently,

$$\gamma(21)\gamma(32) \cdots \gamma(s\ s-1)\gamma(s\ s+1) \cdots \gamma(n-1\ n) \angle \epsilon'(n1),$$

which can be derived from the chain,

$$\begin{aligned}
&\gamma(21)\gamma(32) \angle \gamma(31),\\
&\gamma(31)\gamma(43) \angle \gamma(41),\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\gamma(s1)\gamma(s\ s+1) \angle \epsilon'(s+1\ 1),\\
&\epsilon'(s+1\ 1)\gamma(s+1\ s+2) \angle \epsilon'(s+2\ 1),\\
&\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot\\
&\epsilon'(n-1\ 1)\gamma(n-1\ n) \angle \epsilon'(n1).
\end{aligned}$$

If the mood of the sorites contains only α- and γ-premises, the term order in each γ-premise is established as above, the first γ-premise, which presents the term order $(s - 1\ s)$' establishing that order in each γ-premise, which follows. The generating chain of syllogisms will be the same as the last, except that each minor premise in the chain, which corresponds to an α-premise of the sorites, will appear in the α-form.

If all of the premises, except a single β-premise, be in the γ-form, the sorites becomes,

$$\gamma(1, 2)\gamma(2, 3) \cdots \gamma(s - 1, s)\beta(s, s + 1)\gamma(s + 1, s + 2) \cdots \gamma(n - 1, n) \angle \epsilon'(n1).$$

The term-order in each premise, which precedes the β-premise, is established as $(s\ s - 1)$. For, if the term-order in a γ-premise coming before the β-premise should appear as $(s - 1\ s)$, then, by identifying terms, the mood of the sorites would be reducible to an invalid syllogism of the form

$$\gamma(s - 1\ s)\beta(s, s + 1) \angle \epsilon'(s + 1\ s - 1).$$

Each premise, which follows the β-premise, must present the term-order $(s - 1\ s)$, for otherwise the mood of the sorites would be reducible to an invalid syllogism, vz.,

$$\beta(s - 2, s - 1)\gamma(s\ s - 1) \angle \epsilon'(s\ s - 2).$$

The term-order being now unambiguously established, the sorites becomes

$$\gamma(21)\gamma(32) \ldots \gamma(s\ s - 1)\beta(s, s + 1)\gamma(s + 1\ s + 2) \cdots \gamma(n - 1\ n) \angle \epsilon'(n1),$$

which may be generated from the chain,

$$\gamma(21)\gamma(32) \angle \gamma(31),$$
$$\gamma(31)\gamma(43) \angle \gamma(41),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\gamma(s - 1\ 1)\gamma(s\ s - 1) \angle \gamma(s1),$$
$$\gamma(s1)\beta(s, s + 1) \angle \epsilon'(s + 1\ 1),$$
$$\epsilon'(s + 1\ 1)\gamma(s + 1\ s + 2) \angle \epsilon'(s + 2\ 1),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\epsilon'(n - 1\ 1)\gamma(n - 1\ n) \angle \epsilon'(n1),$$

or, if the initial premise be in the β-form, from

$$\beta(1, 2)\gamma(23) \angle \epsilon'(31),$$
$$\epsilon'(31)\gamma(34) \angle \epsilon'(41),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\epsilon'(n - 1\ 1)\gamma(n - 1\ n) \angle \epsilon'(n1).$$

The remaining moods of valid sorites which contain α-, β- and γ-premises and an ϵ'-conclusion are obtained from the last type by replacing one or more γ-premises by α-premises in every possible way. Each type so obtained can be constructed from one of these last chains of valid syllogisms, except that now the minor premise of each member of the chain, that corresponds to an α-premise of the sorites, will appear in the α-form.

There exist, consequently, no valid moods of the sorites, in which the conclusion is in the ϵ'-form and in which none of the premises is a primed form, which cannot be constructed from chains of valid syllogisms. All the other moods, in which the premises are unprimed and the conclusion is a primed form, are gotten from the valid moods already established by the aid of the principle,

$$(xy \angle z) \angle (xz' \angle y').$$

§ 25. *Conclusion α and All Premises Unprimed*

It will be easy to show that no β-, γ- or ϵ-premise can occur, if the mood of the sorites is valid, and that, consequently, the general form of implication will be

$$\alpha(1, 2)\alpha(2, 3) \ldots \alpha(n - 1, n) \angle \alpha(n1),$$

whose chain of generating syllogisms is

$$\alpha(1, 2)\alpha(2, 3) \angle \alpha(31),$$
$$\alpha(31)\alpha(3, 4) \angle \alpha(41),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\alpha(n - 1\ 1)\alpha(n - 1, n) \angle \alpha(n1).$$

Conclusion β and All Premises Unprimed

Under this head it will be found that no α-, γ- or ϵ-premise can occur and that consequently all of the premises are in the β-form. But such a sorites may be reduced to an invalid syllogism, viz.,

$$\beta(s - 1, s)\beta(s, s + 1) \angle \beta(s + 1\ s - 1).$$

There exist, consequently, no valid moods of this type.

Conclusion γ and All Premises Unprimed

Here it can be shown at once that no α-, β- or ϵ-premise can occur and that, consequently, all of the premises are in the γ-form. Moreover, the term-order in each γ-premise is established as $(s\ s - 1)$, *i.e.*, with the larger ordinal number coming first; for, suppose one of the premises should appear as $\gamma(s - 1\ s)$. The mood of the sorites would then be reducible to an invalid syllogism of one of the forms,

$$\gamma(s - 1\ s)\gamma(s, s + 1) \angle \gamma(s + 1\ s - 1),$$
$$\gamma(s - 2, s - 1)\gamma(s - 1\ s) \angle \gamma(s\ s - 2).$$

The sorites becomes, then,

$$\gamma(21)\gamma(32) \cdots \gamma(n\ n - 1) \angle \gamma(n\ 1),$$

and its chain of generating syllogisms is

$$\gamma(21)\gamma(32) \angle \gamma(31),$$
$$\gamma(31)\gamma(43) \angle \gamma(41),$$
$$\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot$$
$$\gamma(n - 1\ 1)\gamma(n\ n - 1) \angle \gamma(n\ 1).$$

Conclusion ϵ and All Premises Unprimed

Just as in the cases already considered, it will be easy to show that one and only one ϵ-premise must be present and that no β-premise can occur. One form of this sorites is, accordingly,

$$\alpha(1, 2)\alpha(2, 3) \cdots \alpha(s - 1, s)\epsilon(s, s + 1)\alpha(s + 1, s + 2) \cdots \alpha(n - 1, n) \angle \epsilon(n1),$$

which can be constructed from the chain of valid syllogisms,

$$\alpha(1, 2)\alpha(2, 3) \angle \alpha(31),$$
$$\alpha(31)\alpha(3, 4) \angle \alpha(41),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\alpha(s - 1\ 1)\alpha(s - 1, s) \angle \alpha(s1),$$
$$\alpha(s1)\epsilon(s, s + 1) \angle \epsilon(s + 1\ 1),$$
$$\epsilon(s + 1\ 1)\alpha(s + 1, s + 2) \angle \epsilon(s + 2\ 1),$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot$$
$$\epsilon(n - 1\ 1)\alpha(n - 1, n) \angle \epsilon(n1).$$

The other valid sorites of this type are gotten by replacing one or more α-premises, coming before the ϵ-premise, by a $\gamma(s - 1\ s)$, one or more α-premises, coming after the ϵ-premise, by a $\gamma(s\ s - 1)$, and it will be easy in each case to construct the generating chain of syllogisms. There exist, consequently, no valid moods of the sorites, whose premises and conclusion are all unprimed forms, which cannot be built up from chains of valid syllogisms. It only remains to be shown that the valid types already established are the only valid types that exist, except those immediately derived from these by the principle, $(xy \angle z) \angle (xz' \angle y')$.

All valid moods of the sorites, in which the conclusion is a primed form and a single one of the premises is a primed form, follow at once from the valid moods already established by the aid of the principle $(xy \angle z) \angle (xz' \angle y')$. For, suppose a valid mood of this type, but not so derived, should exist. Then a mood of the sorites already established as invalid would appear as valid upon application of the same principle, *i.e.*, $(xz' \angle y') \angle (xy'' \angle z'')$, or $(xy' \angle z') \angle (xz \angle y)$.

No valid implications exist, in which a single premise is a primed form and the other premises and conclusion are unprimed forms. Suppose in the first instance that the conclusion is α or γ. Then if the primed premise is ϵ it may be strengthened* to an unprimed β-premise and, if the primed premise be α, β or γ, it may be strengthened

* If $x \angle y$, then x is said to be a *strengthened* form of y and y is said to be a *weakened* form of x.

to ϵ. In either case the resulting mood is one already shown to be invalid. If the conclusion is β and the primed premise be strengthened to any unprimed premise, the resulting mood is invalid. If, finally, the conclusion is ϵ, any primed ϵ-premise may be strengthened to β, any primed α-, β-, or γ-premise to ϵ, if another ϵ-pemise is present, and the resulting mood is again invalid. If the conclusion is ϵ and no ϵ-premise is present the mood will reduce to $x(s-1, s)\, y'(s, s+1) \angle \epsilon(s+1\, s-1)$. Continuing this same line of reasoning, it will be seen that no valid moods of the sorites exist, in which the conclusion is unprimed and *two or more* premises are primed.

Finally no valid moods of the sorites exist, in which the conclusion is a primed form and in which two or more of the premises are primed forms. For suppose such a mood to exist. Then, by contradicting and interchanging one of the primed premises and the conclusion, the validity of a mood already found to be invalid would follow.

All of the valid implications of the general form,

$$x(1, 2)y(2, 3) \cdots z(n-1, n) \angle w(n1),$$

$x \cdots z$, w, standing for either primed or for unprimed letters, have, accordingly, been established, without introducing any assumptions except those essential, to the solution of the forms of immediate inference and of the syllogism. This rather general type of inference may be expressed conveniently in the form of the product of n premises containing a cycle of n terms and implying zero, thus:

$$x(1, 2)y(2, 3) \cdots z(n, 1) \angle 0.$$

That the solution of this last type is exactly equivalent to the solution just given follows from the principles,

$$(x \angle y) \angle (xy' \angle 0),$$
$$(xy' \angle 0) \angle (x \angle y).$$

EXERCISES

1. Construct a valid mood of the sorites from the chain of syllogisms,

$$\alpha(21)\gamma(32) \angle \gamma(31),$$
$$\gamma(31)\alpha(43) \angle \gamma(41),$$
$$\gamma(41)\gamma(54) \angle \gamma(51),$$

which are valid in the common logic (§§ 1–19).

2. If $\epsilon\gamma\epsilon$ (first and second figure) and $\gamma\epsilon\epsilon$ (second and fourth figure) be regarded as invalid moods of the syllogism (see the concluding remarks of § 8) establish the invalidity of the sorites,

$$\gamma(12)\gamma(23) \cdots \gamma(s-1\ s)\epsilon(s,\ s+1)\gamma(s+2\ s+1) \cdots \gamma(n\ n-1) \angle \epsilon(n1),$$

by the aid of the following

Principle.—A valid mood of the sorites, whose premises and conclusion are all unprimed forms and which has one premise of the same form as the conclusion, will remain valid, when as many other β- and γ-premises as we desire, are put in the α-form.

3. Prove, that there exists no valid mood of the sorites, in which none of the unprimed premises has the same form as the unprimed conclusion, by the aid of the following

Principle.—A valid mood of the sorites, whose premises and conclusion are all unprimed forms and none of whose premises has the same form as the conclusion, will remain valid, when as many β- and γ-premises as we desire are put in the α-form.

4. Employing the principle of exercise 2 reduce the sorites $\alpha(21)\gamma(32)\alpha(43)\gamma(54) \angle \gamma(51)$, which is valid in the common logic (§§ 1–19), successively to each one of the three valid syllogisms of exercise 1.

5. Employing the same principle, establish the invalidity of the sorites,

$$\gamma(21)\gamma(32)\gamma(34)\gamma(54) \angle \gamma(51).$$

6. From what chain of valid syllogisms (§§ 15–19) can the sorites,

$$\gamma(12)\gamma(23)\epsilon(3,\ 4)\gamma(54)\gamma(65) \angle \epsilon(61),$$

be built up?

7. By the aid of the principle of exercise 2, solve the sorites of the common logic for the case, in which all of the premises and the conclusion are unprimed forms.

8. Complete the solution of the sorites begun in exercise 7, taking for granted the following principles:

(*a*) A valid mood of the sorites, whose premises are all unprimed forms, whose conclusion is a primed form and all of whose premises and conclusion are of the same form, will remain valid, when as many premises, as we desire, but one, are put in the α-form,

(*b*) A valid mood of the sorites, whose premises are all unprimed forms, whose conclusion is a primed form and one of whose premises is a form different from the conclusion, will remain valid, when as many other β- or γ-premises, as we desire, are put in the α-form.

CHAPTER V

§ 26. In § 12 are laid down certain conditions, which must be taken account of in setting down the foundations of any system of inference. The conditions are

$$\alpha(io) + \gamma(io) \angle o, \qquad \alpha'(oi)\gamma'(oi) \angle o, \qquad \text{I}$$

which contains as a consequence $\alpha'(aa)\gamma'(aa) \angle o$, or in particular,

$$\alpha'(oo)\gamma'(oo) \angle o, \qquad \alpha'(ii)\gamma'(ii) \angle o. \qquad \text{II}$$

Thus, we should have to have

(*a*) $\gamma(oi)$ is a true proposition,
(*b*) $\alpha(oi)$, $\alpha(io)$ and $\gamma(io)$ are false propositions,
(*c*) either $\alpha(oo)$ or $\gamma(oo)$ is a true proposition,
(*d*) either $\alpha(ii)$ or $\gamma(ii)$ is a true proposition.

These results, which are forced upon us as a matter of definition, leave us a number of choices as to the truth or falsity of β and ϵ, where subject and predicate are allowed to take on the meanings *nothing* and *universe* in every possible way.

In order to determine another of these systems, we might, by introducing a series of postulates, remove one possibility after another, until no choice among alternatives remains. As one further illustration of method, we shall determine the system, which appears the most paradoxical to ordinary intuition, the one, namely, which asserts the untruth, for all meanings of a, of the proposition *all a is all a*.

We shall assume in the first place that $\alpha(ab)$, $\gamma(ab)$ and the product, $\beta'(ab)\epsilon'(ab)$, are convertible by *contraposition*, *i.e.*, denoting *non-a* by a',

(1) $$\beta'(ab)\epsilon'(ab) \angle \beta'(b'a')\epsilon'(b'a'),$$
$$\alpha(ab) \angle \alpha(b'a'),$$
$$\gamma(ab) \angle \gamma(b'a').$$

Our other postulates will be:

(2) $$\alpha(ab) \angle \alpha'(ab')\gamma'(ab'),$$

which yields $\alpha(oo) \angle o$, for $a = b = o$, by I. Consequently, $\gamma'(oo) \angle o$, by II; $\alpha(ii) \angle o$, by (1); $\gamma'(ii) \angle o$, by II or (1);

(3) $$\beta(ab) \angle \alpha'(ab')\gamma'(ab'),$$

which yields $\beta(oo) \angle o$, for $a = b = o$, by I; and $\beta(oi) \angle o$, $\beta(io) \angle o$, for $a = o$, $b = i$, by II;

(4) $$\epsilon'(ab) \angle \alpha'(ab')\gamma'(ab'),$$

which yields $\epsilon'(oo) \angle o$, for $a = b = o$, by I; and $\epsilon'(oi) \angle o$, $\epsilon'(io) \angle o$, for $a = b = i$, by I.

(5) $$\alpha'(ab')\gamma'(ab') \angle \epsilon'(ab),$$

which yields $\epsilon(ii) \angle o$, for $a = b = i$, by I.

The only case, which remains unsettled, is that of $\beta(ii)$ and it may now be seen, from the first member of (1) that $\beta'(ii) \angle o$. For convenience of reference we may now summarize our results:

$\alpha\,(oo) \angle o$	$\alpha\,(oi) \angle o$	$\alpha(io) \angle o$	$\alpha\,(ii) \angle o$
$\beta\,(oo) \angle o$	$\beta\,(oi) \angle o$	$\beta(io) \angle o$	$\beta'(ii) \angle o$
$\gamma'(oo) \angle o$	$\gamma'(oi) \angle o$	$\gamma(io) \angle o$	$\gamma'(ii) \angle o$
$\epsilon'\,(oo) \angle o$	$\epsilon'\,(oi) \angle o$	$\epsilon'(io) \angle o$	$\epsilon\,(ii) \angle o$

It only remains to add to what has gone before, viz., $\alpha(oo) \angle o$, $\alpha(ii) \angle o$, the more general postulate,

$$\alpha(aa) \angle o.$$

Without this postulate it still remains unsettled, whether we intend to deny, merely, the truth of *all a is all a*, or to assert its untruth *for all meanings of a*.

In the exercises below, it will be taken for granted that the forms of immediate inference, which are untrue in the common logic, are invalid in the system, whose foundations are set down here.

EXERCISES

1. Deduce the valid-moods of the syllogism,

$$x(a,\ b)y(b,\ c)\ \angle\ z(ca),$$

which are twenty-one in number, from the following postulates:

(i) $\alpha(ba)\beta(cb)\ \angle\ \beta(ca)$, (ii) $\alpha(ba)\epsilon(cb)\ \angle\ \epsilon(ca)$,
(iii) $\gamma(ba)\gamma(cb)\ \angle\ \gamma(ca)$, (iv) $\gamma(ab)\epsilon(bc)\ \angle\ \epsilon(ca)$,
(v) $\beta(ab)\ \angle\ \beta(ba)$, (vi) $\alpha(ab)\ \angle\ \alpha(ba)$.

2. Deduce the valid moods of the syllogisms, $x'(a,\ b)y(b,\ c)\ \angle\ z'(ca)$ and $x(a,\ b)y'(b,\ c)\ \angle\ z'(ca)$, of which there are nineteen and twenty-three respectively, from the results of exercise 1.

3. Deduce the valid moods of the syllogism, $x(a,\ b)y(b,\ c)\ \angle\ z'(ca)$, there being one hundred and fourteen of this type, from the postulates and results of exercise 1, by the aid of the additional postulates:

(vii) $\alpha(ba)\beta(cb)\ \angle\ \gamma'(ca)$,
(viii) $\beta(ba)\alpha(cb)\ \angle\ \gamma'(ca)$,
(ix) $\alpha(ba)\,\epsilon(cb)\ \angle\ \gamma'(ca)$,
(x) $\alpha(ab)\ \angle\ \gamma'(ab)$,
(xi) $\beta(ab)\ \angle\ \epsilon'(ab)$.

4. Show that the members of the following set may be made to depend upon the implications that have already been obtained:

$\alpha(aa)\ \angle\ \alpha'(aa)$, $\{\alpha'(aa)\ \angle\ \alpha\,(aa)\}'$,
$\{\beta(aa)\ \angle\ \beta'(aa)\}'$, $\{\beta'(aa)\ \angle\ \beta\,(aa)\}'$,
$\{\gamma(aa)\ \angle\ \gamma'(aa)\}'$, $\gamma'(aa)\ \angle\ \gamma\,(aa)$,
$\{\epsilon(aa)\ \angle\ \epsilon'(aa)\}'$, $\{\epsilon'(aa)\ \angle\ \epsilon\,(aa)\}'$,
$\{\beta(ab)\ \angle\ \gamma'(ab)\}'$, $\{\gamma(ab)\ \angle\ \beta'(ab)\}'$,
$\{\gamma(ab)\ \angle\ \epsilon'(ab)\}'$, $\{\epsilon(ab)\ \angle\ \gamma'(ab)\}'$.
$\{\gamma(ab)\ \angle\ \gamma'(ba)\}'$,

5. Prove that ninety-six of the invalid moods of the syllogism, $x(a,\ b)y(b,\ c)\ \angle\ z'(ca)$, may be reduced to simpler invalid forms of inference already established and so shown to be invalid, (a) either by identifying terms in a γ-premise or a γ-conclusion and suppressing the part $\gamma(aa)$, or, (b) by replacing the subject and predicate of a β-premise or a β-conclusion by unity and suppressing the part $\beta(ii)$.

6. Show that the remaining forty-six invalid moods not ac-

counted for in exercise 5 may be obtained by the aid of the additional postulates:

(xii) $\{\alpha(ba)\alpha(cb) \angle \alpha'(ca)\}'$, (xiii) $\{\alpha(ba)\epsilon(cb) \angle \epsilon'(ca)\}'$,
(xiv) $\{\alpha(ba)\beta(cb) \angle \beta'(ca)\}'$, (xv) $\{\beta(ba)\beta(cb) \angle \epsilon'(ca)\}'$,
(xvi) $\{\alpha(ba)\gamma(cb) \angle \gamma'(ca)\}'$, (xvii) $\{\gamma(ba)\alpha(cb) \angle \gamma'(ca)\}'$.

7. Derive the invalidity of all but eight of the two hundred and thirty-five invalid moods of the syllogism $x(a, b)y(b, c) \angle z(ca)$ from the results established in exercises 5 and 6.

8. Employ the results of exercise 7 in order to show that there exist no valid syllogistic variations of the form $x'(a, b)y(b, c) \angle z(ca)$, $x(a, b)y'(b, c) \angle z(ca)$, $x'(a, b)y'(b, c) \angle z(ca)$, or $x'(a, b)y'(b, c) \angle z'(ca)$.