ANALYTICAL
CONICS

SOMMERVILLE



BELL

BELL'S MATHEMATICAL SERIES

ADVANCED SECTION

*General Editor*: WILLIAM P. MILNE, M.A., D.Sc.

FORMERLY PROFESSOR OF MATHEMATICS, UNIVERSITY OF LEEDS

# ANALYTICAL CONICS

## BELL'S MATHEMATICAL SERIES

ADVANCED SECTION

General Editor: WILLIAM P. MILNE, M.A., D.Sc.
Emeritus Professor of Mathematics,
University of Leeds

---

AN ELEMENTARY TREATISE ON DIFFERENTIAL EQUATIONS AND THEIR APPLICATIONS. By H. T. H. Piaggio, M.A., D.Sc.

ELEMENTARY VECTOR ANALYSIS. By C. E. Weatherburn, M.A., D.Sc.

ADVANCED VECTOR ANALYSIS. By C. E. Weatherburn, M.A., D.Sc.

ANALYTICAL CONICS. By D. M. Y. Sommerville, M.A., D.Sc.

A FIRST COURSE IN STATISTICS. By D. Caradog Jones, M.A., F.S.S.

A FIRST COURSE IN NOMOGRAPHY. By S. Brodetsky, M.A., Ph.D.

---

LONDON: G. BELL AND SONS, LTD
York House, Portugal Street, W.C.

# ANALYTICAL CONICS

BY

D. M. Y. SOMMERVILLE

M.A., D.Sc., F.N.Z.Inst.

LATE PROFESSOR OF PURE AND APPLIED MATHEMATICS
VICTORIA UNIVERSITY COLLEGE, WELLINGTON, N.Z.

LONDON
G. BELL AND SONS, LTD.
1961

First published, 1924.
Second Edition 1929.
Third Edition, revised, 1933.
Reprinted 1937, 1941, 1944, 1945, 1946, 1949, 1951, 1956, 1961.



PRINTED IN GREAT BRITAIN BY ROBERT MACLEHOSE AND CO. LTD
THE UNIVERSITY PRESS, GLASGOW

# PREFACE

In this text-book on Analytical Conics I have studied to present the principles of analytical geometry as applied to the conic sections, keeping always in view the fact that the subject matter is geometry and not merely algebra with a geometrical interpretation. The book starts with the elements and presupposes no previous knowledge of analytical geometry or conics, but it is advisable that the beginner should have already some knowledge of graphs.

Some explanation of the order of treatment may appear necessary. It has become almost conventional in English text-books to define a conic by the focus-directrix property, and to treat the three varieties in the order: parabola, ellipse, hyperbola. I have chosen another definition, which is an immediate extension of the familiar rectangle theorem for a circle; next to defining the curves as sections of a cone—and I had reluctantly to abandon this method of approach as leading too far into solid geometry—this definition seems to me to be the simplest, as it was one of the earliest properties to be discovered, and it leads most directly to the equations in the standard forms. The focus-directrix property, while no doubt simple superficially, is by no means an obvious one, as appears from the fact that it was not discovered until 600 years after Apollonius wrote his celebrated treatise, and even the existence of a focus in the case of the parabola was not suspected by Apollonius. For these reasons, therefore, I have postponed the treatment of the focal properties and presented their investigation in the form of a little piece of research. As regards the order, I have taken the ellipse first, because, for a great many properties, it may be taken as the typical conic; the hyperbola then supplements the ellipse and provides the theory of the asymptotes. The parabola, though it is in many respects the simplest of the conics, fails just for this reason to be a typical conic, and some of the common terms and definitions, if applied first to the parabola, are devoid of proper meaning. Thus the term " diameter " has only a forced meaning when applied to the parabola, but its application is readily understood when it has first been studied in relation to the ellipse and the hyperbola. Again, the term " eccentricity " means primarily the relative distance of the focus from the centre; for the parabola the term has only its secondary meaning, viz. the focus-directrix ratio.

I have ventured to introduce the idea of homogeneous coordinates very early (Chap. II.). This greatly facilitates the treatment of parallel lines and points at infinity. In fact it is scarcely possible to treat points at infinity without homogeneous coordinates. The dual aspect of geometry, whereby the straight line and the point can serve equally as primitive elements, should probably be emphasized from the beginning, but as most writers and teachers seem to agree in the conclusion that there is an inherent difficulty in the process of thinking in terms of lines, I have followed the usual practice in postponing the introduction of line-coordinates until the student has had considerable experience with ordinary cartesian point-coordinates. The early introduction of the circular points at infinity seemed to be advisable, as it provides a clear geometrical interpretation of results which would otherwise have only an algebraic meaning.

At Professor Milne's suggestion I have included a chapter on "Systems of points on a conic." The subject matter of this chapter is the algebra of forms or quantics, which was largely created by Sylvester and Cayley, but it is so clothed in geometrical language that it may be regarded as part of geometry, and is so treated by Clebsch in his *Vorlesungen*; it thus forms a useful introduction to the theory of invariants which is treated in the concluding chapter.

In the preparation of the work I have benefited more than I can say by the constant advice and criticism of Professor Milne. I am indebted also to Mr. J. Milne, M.A., Master of Method (Mathematics and Science), Provincial Training Centre, Aberdeen, and Mr. Peter Fraser, M.A., B.Sc., University of Bristol, for very useful criticisms. My thanks are due to the Syndics of the Cambridge University Press for permission to include many examples from the Cambridge Entrance Scholarship and Tripos papers, due acknowledgment of which is given in each case. The remainder of the examples, which have been carefully selected and arranged, have been drawn from various sources for which acknowledgment is impossible, and many have been specially devised. Some of the earlier chapters have been provided with two sets of examples, A and B, the second set being of a somewhat harder character and suitable for the more advanced student when revising. It is hoped that not many errors will be found among the answers.

In conclusion, I desire to thank the publishers for their unfailing courtesy, and Messrs. MacLehose for the excellence of the printing.

D. M. Y. SOMMERVILLE.

Victoria Univ. Coll., Wellington, N.Z.,
*July*, 1923.

# PREFACE TO THE THIRD EDITION

In the second edition (1929) most of the typographical and other slight errors of the first edition were corrected. The opportunity has now been taken in the third edition to subject the book to a thorough revision, without, however, disturbing the stereotype plates more than was absolutely necessary. With the exception of Chapter XII, therefore, very little change has been made in the page-numbering, and the numbering of the examples has not been tampered with except occasionally to replace an unsuitable example or add a few new ones at the ends of collections. The inconvenience attached to the use of different editions of the same text-book has thus been minimised.

Chapter XII, on homogeneous coordinates, has been almost entirely rewritten, bringing this matter more into line with the modern treatment which is found especially in the Italian text-books. It is recognised that the general system of homogeneous coordinates is applicable primarily to projective geometry. Projective coordinates are therefore explained from the outset, and it is shown how the various metrical systems, areals, trilinears, homogeneous cartesians, are derived as particular cases. The transition from projective to metrical coordinates has, however, been left in a somewhat abrupt state, with the assumption of the metrical definition of cross-ratio; a full explanation of the projective measure of distance would have required too much additional matter. In Chapter IV there has been inserted a discussion of the sections of a cone, the omission of which from the original edition was a matter of regret to the author. The only other substantial change is an expansion of the paragraph in Chapter II relating to points at infinity and homogeneous cartesian coordinates.

D. M. Y. S.

Victoria Univ. Coll.,
Wellington, N.Z.

# CONTENTS


| CHAPTER | | PAGE |
|---|---|---|
| I. | COORDINATES | 1 |
| II. | THE STRAIGHT LINE | 9 |
| III. | THE CIRCLE | 25 |
| IV. | THE ELLIPSE | 36 |
| V. | THE HYPERBOLA | 60 |
| VI. | THE PARABOLA | 70 |
| VII. | SYSTEMS OF CIRCLES AND INVERSION | 84 |
| VIII. | OBLIQUE AXES AND TRANSFORMATION OF COORDINATES | 101 |
| IX. | TRACING OF CONICS | 112 |
| X. | THE GENERAL CONIC | 122 |
| XI. | LINE-COORDINATES AND ENVELOPES | 130 |
| XII. | PROJECTIVE GEOMETRY AND HOMOGENEOUS COORDINATES | 145 |
| XIII. | THE CONIC IN HOMOGENEOUS COORDINATES | 168 |
| XIV. | THE LINE AT INFINITY AND THE CIRCULAR POINTS | 179 |
| XV. | CONFOCAL CONICS AND SIMILAR CONICS | 195 |
| XVI. | PENCILS AND RANGES OF CONICS | 209 |
| XVII. | PARAMETRIC REPRESENTATION | 216 |
| XVIII. | CORRESPONDENCE, HOMOGRAPHY, AND INVOLUTION | 227 |
| XIX. | SYSTEMS OF POINTS ON A CONIC | 247 |
| XX. | INVARIANTS | 265 |
| | ANSWERS | 295 |
| | INDEX | 305 |

# CHAPTER I.

## COORDINATES.

**1. Coordinate Network.** Picture a sheet of paper ruled in squares, a sheet of "squared paper." Taking two of the edges, the position of any point of the network can be fixed by counting the number of divisions which separate the point from the two edges. Thus corresponding to every point there is a pair of numbers, and conversely if the two numbers are noted the point can be again found. Two such numbers, which fix the position of a point, are called its *coordinates*.

The position is not determined *absolutely*, for it is obvious that if a sheet of squared paper of unlimited size is given, and two numbers are assigned, no particular point on the sheet is specified. Two lines on the sheet must first be assigned—to take the place of the two edges—and then the two numbers fix the position of a point *relative* to these two lines or *axes*.

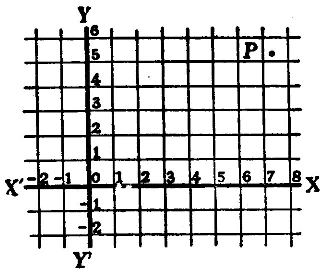


FIG. 1.

Denote the two chosen axes of reference by $X'OX$ and $Y'OY$. Their point of intersection $O$ is called the *origin*. We may then attach numbers to the two sets of lines of the network. Attach the cipher 0 to the axis $Y'OY$, and the numbers 1, 2, 3, ... to the successive parallels going towards the right. The parallels to the left of $Y'OY$ must also have numbers attached to them, and it is convenient to denote them by the negative numbers $-1$, $-2$, $-3$, ... . With this arrangement we have a regular sequence of numbers, from negative through zero to positive, such that, without exception, each is obtained from the preceding by adding on unity.

Similarly the other set of lines parallel to $X'OX$ will have the positive numbers 1, 2, 3, ... attached to them when they lie above $X'OX$, and the negative numbers $-1$, $-2$, $-3$, ... when they lie below.

It is a convention that positive numbers refer to the right and above. It would do equally well to attach positive numbers to the left, negative numbers being then attached to the right.

We must next imagine the network made indefinitely fine by the introduction of intermediate rulings, so that through every point there is a line of each system carrying a definite number.

Two numbers, positive, negative or zero, being assigned, a point is now uniquely determined as the point of intersection of the two lines which carry these numbers. The two axes of reference divide the whole plane into four quadrants. In the first, $XOY$, the signs of the coordinates, naming always $x$ first, are $++$. In the second, $YOX'$, they are $-+$; in the third, $X'OY'$, $--$; in the fourth, $Y'OX$, $+-$.

**2. Coordinates.** We have now established a system of coordinates based upon a rectangular network of lines. We may consider the two coordinates, geometrically, as being the distances of the point from each axis. If $NP$ is $\perp OY$ and $MP \perp OX$, the coordinates are

$$x = NP = OM, \quad y = MP = ON;$$

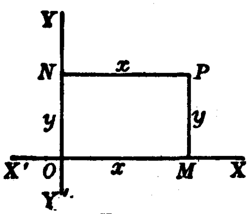



FIG. 2.

and these are called respectively the *abscissa* and the *ordinate* of $P$.

The first systematic application of numbers to geometry was made by Descartes (1637). His system of coordinates was just that described, and is consequently called a *Cartesian System*, or, to distinguish it from a system of *oblique coordinates* in which the two axes of reference are not at right angles, it is called a system of *rectangular cartesian coordinates*.

The same principle is of wide application, and can be used to establish a much more general coordinate-system. All that is necessary is to have two distinct systems of lines, straight or curved, having the property that every line of the one system cuts every line of the other system in just one point, and no two lines of the same system intersect. One line of each system is chosen and has 0 attached to it. The other lines are arranged in order, and numbered ... $-3$, $-2$, $-1$, $+1$, $+2$, $+3$, ... and intermediately 1·1, 1·2, etc. Every point is then the intersection of two lines, one of each system, and is completely determined by two numbers, $x$, $y$. There may be isolated exceptions to the rule that no two lines of the same system intersect. Thus we may take as the two systems of lines: (1) a system of concentric circles, (2) a system of straight lines through their common centre. The numbers may be assigned so that the coordinates of a point are: (1) the radius of the circle $= r$, (2) the angle $\theta$ which the radius makes with a fixed radius. This is called a system of *Polar Coordinates*. The common centre is called the *pole*, the distance from the pole is called the *radius vector*, and the angle $\theta$ is called the *vectorial angle*. The zero axis from which the angle $\theta$ is measured is called the *initial line* or *initial vector*.

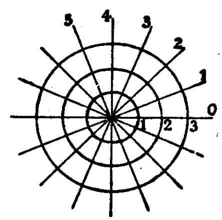



FIG. 3.

Another example is found in the meridians and parallels on the surface of the earth. The coordinates of a point on the earth are then the latitude and longitude, the axes of reference being the equator and a fixed meridian, say the meridian through Greenwich. Other examples will occur later.

In the cartesian system the coordinate lines are all straight, and it is said to be a type of *rectilinear coordinates*, as distinguished from a system like polar coordinates, in which some of the coordinate lines are curved, and which is therefore a type of *curvilinear coordinates*.

**3. Relation between cartesian and polar coordinates.** From the figure we have, arrows denoting positive directions,

$$x = OM = r\cos\theta,$$

$$y = MP = r\sin\theta,$$

which express $x$, $y$ in terms of $r$, $\theta$.

Conversely, if $x$, $y$ are given, we have

$$r^2 = x^2 + y^2,$$

$$\tan\theta = \frac{y}{x}.$$

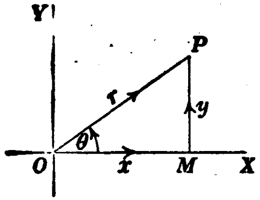


FIG. 4.

These equations do not determine $r$, $\theta$ uniquely, for $r = \pm\sqrt{x^2+y^2}$, and $\theta$ has an indefinite number of values. If we fix

$$r = +\sqrt{x^2+y^2},$$

then the two equations $\cos\theta = x/r$, $\sin\theta = y/r$

determine $\theta = \alpha + 2n\pi$,

where $\alpha$ is any one value of the angle. The principal value is taken to be that value which lies between $-\pi$ and $+\pi$.

Thus if $x = 3$, $y = -4$, then $r = \pm 5$. Taking $r = +5$, we have $\cos\theta = \frac{3}{5}$, $\sin\theta = -\frac{4}{5}$, and the principal value of $\theta$ is $-\tan^{-1}\frac{4}{3} = -53°\ 08'$ approximately. Taking $r = -5$, we have $\cos\theta = -\frac{3}{5}$, $\sin\theta = \frac{4}{5}$, and the principal value of $\theta$ is $\pi - \tan^{-1}\frac{4}{3} = 126°\ 52'$ approximately. Hence the polar coordinates of the point $(3, -4)$ may be $(5, -53°\ 08')$ or $(-5, 126°\ 52')$.

We may note that $(r, \theta)$ and $(-r, \pi+\theta)$ represent the same point.

**4. Distance between two points.** Let

$$P \equiv (x_1, y_1) \quad \text{and} \quad Q \equiv (x_2, y_2)$$

be two points. Draw the ordinates $PL$, $QM$, and draw $PK \parallel OX$ meeting $QM$ in $K$. Then

$$PK = LM = x_2 - x_1,$$

$$KQ = MQ - MK = y_2 - y_1,$$

and therefore

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2.$$

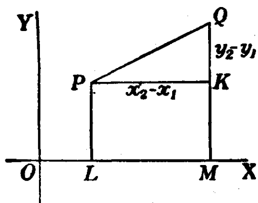


FIG. 5.

**5. Distance between two points in polar coordinates.** Let

$$P \equiv (r_1, \theta_1) \quad \text{and} \quad Q \equiv (r_2, \theta_2),$$

so that

$$OP = r_1, \quad OQ = r_2, \quad \angle AOP = \theta_1, \quad \angle AOQ = \theta_2,$$

and therefore $\angle POQ = \theta_2 - \theta_1$.

Then $PQ^2 = r_1{}^2 + r_2{}^2 - 2r_1r_2\cos(\theta_2 - \theta_1)$.

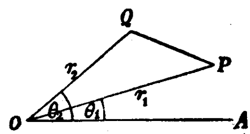


FIG. 6.

**Examples.**

1. Find rectangular coordinates for the following points whose polar coordinates are given: (i) $(4, \frac{1}{3}\pi)$; (ii) $(-2, -\frac{1}{4}\pi)$; (iii) $(-4, 4\frac{5}{7}$ radians).

2. Find polar coordinates for the points (i) (1, 1); (ii) (−5, 12); (iii) $(-2, -\sqrt{3})$; (iv) (0, 0).

3. Calculate the lengths of the sides of the triangle whose vertices are (i) (8, 9), (−4, 4), (4, −2); (ii) (5, 20°), (12, 80°), (−8, 40°); and write down the value of the perimeter.

4. Find the nature of each of the following triangles, whether isosceles or equilateral: (i) (2, 3), (−1, −2), (−2, 2); (ii) (2, 2), (−2, −2), $(-2\sqrt{3}, 2\sqrt{3})$; (iii) (1, 1), (5, 3), $(3+\sqrt{3}, 2-2\sqrt{3})$; (iv) (2, 4), (7, 9), (9, 2); (v) (−4, −2), $(2, 6\sqrt{3}-2)$, (8, −2), (vi) (5, 3), (−4, 2), (1, −2).

5. Prove that each of the following sets of points forms

(*a*) a rhombus: (i) (2, 5), (6, 2), (2, −1), (−2, 2);
(ii) (2, −1), (3, 4), (−2, 3), (−3, −2);
(iii) (1, −1), (−4, 4), (3, 6), (−6, −3).

(*b*) a square: (i) (1, 2), (−3, 1), (−2, −3), (2, −2);
(ii) (0, 2), (3, 8), (9, 5), (6, −1);
(iii) (3, 2), (−2, −1), (1, −6), (6, −3).

**6. Area of a triangle.** Let one of the vertices be at the origin. Let the polar coordinates of the other two vertices $P$ and $Q$ be $(r_1, \theta_1)$ and $(r_2, \theta_2)$. Then

$$\begin{aligned}\Delta OPQ &= \tfrac{1}{2}r_1r_2\sin(\theta_2-\theta_1)\\ &= \tfrac{1}{2}r_1r_2(\sin\theta_2\cos\theta_1-\cos\theta_2\sin\theta_1)\\ &= \tfrac{1}{2}(x_1y_2-x_2y_1).\end{aligned}$$

**7. Sign of an area.** If the points $P$, $Q$ are interchanged the area is $\frac{1}{2}(x_2y_1-x_1y_2)$, that is, it is the same with its sign changed. The area of a triangle must therefore be regarded as capable of having a sign, and we have the result: *if the order of the vertices is reversed the sign of the area is changed.* When $\theta_2-\theta_1$ is a positive angle less than 180° the area is positive, and the vertices $OPQ$ have the same cyclic order as the positive direction of measurement of the angle $\theta$, viz. counter-clockwise.

The positive direction can also be defined in this way: if an observer be supposed to walk round the perimeter in the positive sense he will always have the area on his left.

**8. Area of any triangle.** This expression for the area of the triangle $OPQ$ can be applied to find the area of any closed polygon. Consider a triangle $ABC$. Taking into account the signs of the areas, we have

$$\Delta ABC = \Delta OAB + \Delta OBC + \Delta OCA.$$

Hence we have the formula

$$\begin{aligned}\Delta ABC &= \tfrac{1}{2}\{(x_1y_2-x_2y_1)+(x_2y_3-x_3y_2)+(x_3y_1-x_1y_3)\}\\ &= \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}.\end{aligned}$$

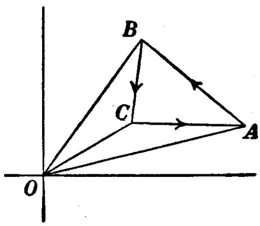



FIG. 7.

**9. Collinear points.** The expression for the area of a triangle can be applied to determine whether three points are collinear. If the points $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, $C \equiv (x_3, y_3)$ are collinear, the area of the triangle $ABC$ is zero, and conversely.

Hence the condition for collinearity is

$$\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix} \equiv (x_1y_2 - x_2y_1) + (x_2y_3 - x_3y_2) + (x_3y_1 - x_1y_3) = 0.$$

**Examples.**

1. Calculate the areas of the triangles: (i) (2, 4), (7, 9), (9, 2); (ii) ( − 2, 3), ( − 7, 5), (3, − 5); (iii) $OPQ$, where $P \equiv (5, 30°)$, $Q \equiv (7, 60°)$.

2. Calculate the areas of the polygons: (i) (2, 2), ( − 1, − 6), ( − 3, 4), ( − 7, − 1); (ii) (4, 4), ( − 6, − 1), (5, − 2), ( − 3, 5), ( − 1, − 5); (*a*) the vertices being taken in the order that will produce a simple, *i.e.* non-intersecting polygon, (*b*) the vertices being taken strictly in the order given. In case (*b*) name the actual parts of the area which have been calculated, each with its proper sign and the number of times that it is counted.

3. Show by areas that the following sets of points are collinear: (i) (2, 2), (4, − 4), (3, − 1); (ii) (2, 9), ( − 3, 12), (7, 6), ( − 8, 15); (iii) ( − 5, 5), ( − 1, 3), (5, 0), (7, − 1).

**10. Joachimsthal's section-formulae.** *To find the coordinates of the point which divides the join of two given points in a given ratio.* Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be the two given points, and $R \equiv (x, y)$ a point dividing the segment $PQ$ so that

$$PR : RQ = l : m.$$

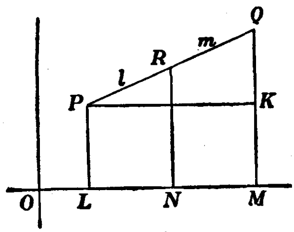



FIG. 8.

Draw the ordinates $PL$, $QM$, $RN$. Then

$$OL = x_1, \quad OM = x_2, \quad ON = x;$$
$$LP = y_1, \quad MQ = y_2, \quad NR = y;$$

and $LN : NM = PR : RQ = l : m.$

Therefore $$\frac{x - x_1}{x_2 - x} = \frac{l}{m}.$$

Hence $$x = \frac{lx_2 + mx_1}{l + m}.$$

Similarly $$y = \frac{ly_2 + my_1}{l + m}.$$

These are called Joachimsthal's formulae.

**11. Position-ratio of a point.** The ratio $PR : RQ$ or $l : m$ is called the *position-ratio* of the point $R$, referred to the base-points $P$, $Q$. When $R$ lies in the segment $PQ$ the ratio is a positive number. If $R$ lies outside the segment $PQ$ the ratio is negative, one of the segments $PR$, $RQ$ being positive and the other negative. As $R$ approaches $P$, $k \to 0$, and as $R$ approaches $Q$, $k \to \infty$. Since

$$k = PR/RQ = (PQ + QR)/RQ = PQ/RQ - 1,$$

we see that $k \to -1$ as $RQ \to \infty$, *i.e.* as $R$ tends to infinity in either direction along the line.

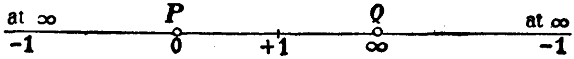



FIG. 9.

We can now mark the position-ratio of some of the most important points on the line (Fig. 9).

Although there is no point whose position-ratio is exactly equal to $-1$ it is convenient to postulate a "*point at infinity*" which corresponds to this value of the position-ratio. This so-called point is to be regarded as being at both ends of the line at once.

**12. Mean points and centres of gravity.** When the ratio $l : m = 1$ the point $R$ is the middle point of $PQ$, and its coordinates are $\frac{1}{2}(x_1 + x_2)$, $\frac{1}{2}(y_1 + y_2)$, or the arithmetic means of the coordinates of $P$ and $Q$. $R$ may be called the *mean point* of the two points $P$, $Q$. If particles of equal weight were placed at $P$ and $Q$, $R$ would be their centre of gravity. If particles of weights $m_1$ and $m_2$ are placed at $P$ and $Q$, the centre of gravity divides $PQ$ in the ratio $m_2 : m_1$, and its coordinates are

$$\frac{m_1 x_1 + m_2 x_2}{m_1 + m_2}, \quad \frac{m_1 y_1 + m_2 y_2}{m_1 + m_2}.$$

*To find the mean point of three points.*

Let $A \equiv (x_1, y_1)$, $B \equiv (x_2, y_2)$, $C \equiv (x_3, y_3)$ be three points, forming a triangle. Let $L$, $M$, $N$ be the mid-points of the sides $BC$, $CA$, $AB$. The coordinates of $L$ are $\frac{1}{2}(x_2 + x_3)$, $\frac{1}{2}(y_2 + y_3)$. Find $G$ dividing the median $AL$ in the ratio $2 : 1$. The coordinates of $G$ are

$$\frac{x_1 + (x_2 + x_3)}{1 + 2}, \text{ etc.,}$$

*i.e.* $\qquad \frac{1}{3}(x_1 + x_2 + x_3), \quad \frac{1}{3}(y_1 + y_2 + y_3).$

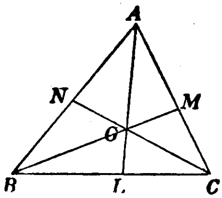



FIG. 10.

From the symmetry of this result it follows that we get the same point dividing $BM$ and $CN$ in the ratio $2 : 1$. This proves that the three medians of the triangle divide each other at the same point in the ratio $2 : 1$.

The point $G$, whose coordinates are the arithmetic means of the coordinates of the three points $A$, $B$, $C$, is called the *mean point* of these three points, or the *centroid* of the triangle $ABC$.

Similarly, if we have any number of points, the point whose coordinates are the arithmetic means of their coordinates is the mean point of the system of points.

If equal particles are placed at the vertices of a triangle, the mean point of the vertices is the centre of gravity of the triangle.*

By placing particles of weights $m_1$, $m_2$, $m_3$ at the vertices we get the centre of gravity of the system

$$\frac{m_1x_1+m_2x_2+m_3x_3}{m_1+m_2+m_3}, \quad \frac{m_1y_1+m_2y_2+m_3y_3}{m_1+m_2+m_3},$$

which is called the mean point of the three points for multiples $m_1$, $m_2$, $m_3$. This may also be extended for any number of points.

**Examples.**

1. Write down the coordinates of the points dividing the join of $(-8, 3)$ and $(4, 9)$ in the ratio (i) $2:1$, (ii) $1:5$, (iii) $4:1$ externally, (iv) $-2:1$, (v) $1:-3$.

2. Prove that the point $(6, 2)$ is collinear with $(-2, -2)$ and $(12, 5)$; and find the ratio in which it divides the segment joining them.

3. Find the ratios in which the diagonals of the following quadrilaterals divide one another: (i) $(4, 9)$, $(1, 5)$, $(4, 1)$, $(7, 5)$; (ii) $(10, 10)$, $(14, 2)$, $(7, -2)$, $(2, 2)$; (iii) $(2, -3)$, $(12, -6)$, $(7, -13)$, $(-4, -8)$.

4. Find in what ratios the join of $(-2, 2)$ and $(4, 5)$ is cut by the axes.

## EXAMPLES I.

1. Calculate the sides and the perimeter of the triangle $(20, 50)$, $(-20, -46)$, $(48, 5)$. Calculate also the area, and verify the formula

$$\Delta=\sqrt{\{s(s-a)(s-b)(s-c)\}}.$$

2. Given the quadrilateral $A(4, 7)$, $B(-5, 3)$, $C(2, -3)$, $D(1, 2)$, calculate the areas of the quadrilaterals $ABCD$, $BCAD$, $CABD$, and verify that their sum is equal to twice the area of the triangle $ABC$.

3. Prove that the four points $(-3, 11)$, $(5, 9)$, $(8, 0)$, $(6, 8)$ lie on a circle with centre $(-1, 2)$.

4. Find in what ratios the join of $(-3, 2)$ and $(4, 6)$ is cut by the axes.

5. What are the coordinates of $B$ if $P(3, 5)$ divides the join of $A(-1, 3)$ and $B$ in the ratio $2:3$?

6. Prove that the point whose coordinates are

$$x=x_1+t(x_2-x_1), \quad y=y_1+t(y_2-y_1)$$

divides the join of $(x_1, y_1)$ and $(x_2, y_2)$ in the ratio $t/(1-t)$.

7. Find in what ratios the diagonals of the following quadrilaterals divide one another: (i) $(5, 5)$, $(-6, 5)$, $(-7, -1)$, $(9, -4)$; (ii) $(8, 1)$, $(-6, 3)$, $(-7, -4)$, $(6, -3)$.

8. Show that $(2, -1)$ is the centre of the circle which passes through the points $(-3, -1)$, $(-1, 3)$, $(6, 2)$; and find its radius.

9. Given the triangle $A(1, 2)$, $B(8, 4)$, $C(4, 10)$, find the coordinates of a point $P$ such that the triangles $PCB$, $PCA$ and $PAB$ have the same area in magnitude and sign. Interpret the result geometrically.

* This is proved in any text-book on Statics. The centre of gravity of a heavy lamina, like a sheet of tin, in the case where it has more than three corners, does not in general coincide with that of a system of equal particles placed at the corners.

10. Prove that the mean point of the vertices of a quadrilateral coincides with the mid-point of the line joining the mid-points of the diagonals.

11. Prove that the medians of a quadrilateral (the lines joining the mid points of opposite sides) bisect one another.

12. Apply Ptolemy's Theorem ($AB \,.\, CD + BC \,.\, AD = AC \,.\, BD$) to prove that the points (1, −2), (−2, −1), (4, 7), (6, 3) are concyclic.

13. Apply Ptolemy's Theorem in the same way to the points (−1, −5), (1, −1), (2, 1), (3, 3). Plot the points, and show by applying the test $AB + BC = AC$ that the four points are collinear.

14. Show that the point (7, 5) has the same position-ratio with respect to the two pairs of points (1, 2), (5, 4) and (−5, −1), (3, 3).

15. Find the coordinates of a point which has the same position-ratio with respect to the two pairs of collinear points (−4, 11), (16, −4) and (−16, 20), (24, −10).

16. Show that there are two points whose position-ratios with respect to the two pairs of points (−9, −2), (−3, 1) and (−1, 2), (5, 5) are equal, but of opposite sign.

17. Given two pairs of points $A$, $B$ and $C$, $D$ in a line, show that there are two points (real, coincident, or imaginary) whose position-ratios with respect to $A$, $B$ and $C$, $D$ have a given ratio. Explain the apparent abnormality when this ratio = 1.

# CHAPTER II.

## THE STRAIGHT LINE.

**1. Equation of a locus.** If the coordinates $(x, y)$ of a point $P$ always satisfy a fixed equation, the point $P$ is restricted to lie on a certain locus or curve. The coordinates of every point of the curve satisfy the equation, and every point whose coordinates satisfy the equation lies on the curve. To every equation there corresponds a locus (the locus of the equation), which may be drawn by plotting the points whose coordinates satisfy the equation; and to every locus which is defined by a definite geometrical relation there corresponds an equation (the equation of the locus), which is the algebraic expression of the geometrical relation.

**2.** *To find the equation of the straight line passing through the point $(a, b)$ and inclined at the angle $\psi$ to the axis of $x$.*

Let $\quad C \equiv (a, b), \quad P \equiv (x, y),$

and draw $CK \parallel Ox$. Then

$$CK = x - a, \quad KP = y - b,$$

and $\angle KCP =$ the fixed angle $\psi$. (This is the geometrical property of the straight line.)

Hence $\quad y - b = (x - a) \tan \psi.$

*Def.* The tangent of the angle which a straight line makes with the positive direction of the $x$-axis is called the *gradient* of the line.

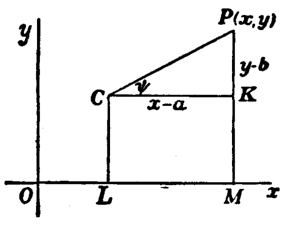



FIG. 11.

Denoting the gradient of the line by $\mu$, the equation takes the form

$$y - b = \mu(x - a).$$

**3.** Conversely, *an equation of the first degree in $x$, $y$ always represents a straight line.* The general equation of the first degree in $x$, $y$ is

$$lx + my + n = 0.$$

Let $C \equiv (a, b)$ be any point on the locus, so that

$$la + mb + n = 0.$$

Subtracting these, we have

$$l(x - a) + m(y - b) = 0.$$

Hence, if $P \equiv (x, y)$ is any point on the locus, and $\psi$ is the angle which $CP$ makes with the axis of $x$, we have

$$\tan\psi = \frac{KP}{CK} = \frac{y-b}{x-a} = -\frac{l}{m} = \text{constant}.$$

The locus of $P$ is therefore a straight line of gradient $-l/m$, passing tnrough $(a, b)$.

**4. Special forms of the equation of a straight line.**

(1) *Gradient forms.* Let the straight line make an angle $\psi$ with the axis of $x$. Then the gradient is

$$\mu = \tan\psi.$$

The line will be fixed if we know the point in which it cuts the $y$-axis. Let $OB = b$, and let $P \equiv (x, y)$ be any point on the line. Draw the ordinate $MP$, and draw $BK \parallel Ox$ cutting $MP$ in $K$. Then $BK = x$, $KP = y - b$, and $KP = BK \tan\psi$. Therefore

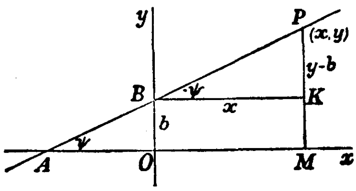


FIG. 12.

$$y = x\tan\psi + b = \mu x + b. \qquad \text{.........................(1)}$$

The line will also be fixed if we know its gradient and the coordinates of any point on the line, $C \equiv (x_1, y_1)$. Then we have

$$y - y_1 = (x - x_1)\tan\psi = \mu(x - x_1). \qquad \text{...................(2)}$$

Let $CP = r$, then we have

$$\frac{x - x_1}{\cos\psi} = \frac{y - y_1}{\sin\psi} = r, \qquad \text{...............................(3)}$$

or

$$\left.\begin{aligned} x &= x_1 + r\cos\psi, \\ y &= y_1 + r\sin\psi. \end{aligned}\right\} \qquad \text{...............................(4)}$$

The first part of (3) is the *constraint equation* corresponding to the *freedom-equations* (4). The constraint equation is the equation in $x$, $y$ which restricts the point to the locus. The freedom equations are equations in which the coordinates $x$, $y$ are expressed in terms of a single variable or *parameter* (in this case $r$) and express the freedom of the point to move as the parameter varies; they are also called *parametric equations.* When $r$ is given a succession of values we get successive points on the line.

(2) *Intercept form.* The line will be fixed if we know the points $A$, $B$ in which it cuts the axes. The distances $OA = a$, $OB = b$ are called the intercepts upon the axes. Let $P \equiv (x, y)$ be any point on the line. Join $OP$. Then

$$\triangle OAP + \triangle OPB = \triangle OAB;$$

therefore

$$ay + bx = ab$$

or

$$\frac{x}{a} + \frac{y}{b} = 1. \qquad \text{................(5)}$$

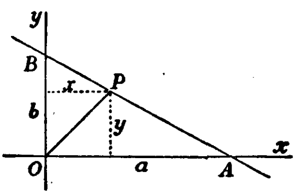


FIG. 13.

(3) *Line through two points.* Let $P_1 \equiv (x_1, y_1)$, $P_2 \equiv (x_2, y_2)$, $P \equiv (x, y)$. Draw $P_1HK \parallel Ox$. Then, since the triangles $P_1HP_2$ and $P_1KP$ are similar, we have

$$\frac{P_1K}{P_1H} = \frac{KP}{HP_2};$$

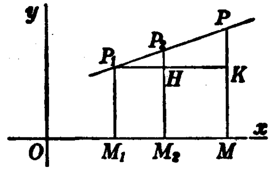


FIG. 14.

therefore

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1}. \quad \text{.............(6)}$$

Putting each of these ratios equal to $t$ and solving for $x$ and $y$, we get the freedom-equations, with parameter $t$,

$$\left.\begin{aligned} x &= x_1 + t(x_2 - x_1), \\ y &= y_1 + t(y_2 - y_1). \end{aligned}\right\} \quad \text{.............(7)}$$

Otherwise: let the equation of the straight line be $lx + my + n = 0$. Then, since the line passes through each of the points $(x_1, y_1)$ and $(x_2, y_2)$, we have

$$lx_1 + my_1 + n = 0, \quad lx_2 + my_2 + n = 0.$$

Eliminating $l, m, n$ between these three equations, we obtain the equation of the line in determinant form,

$$\begin{vmatrix} x & y & 1 \\ x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \end{vmatrix} = 0. \quad \text{.............(8)}$$

(4) *Polar equation.* Draw the perpendicular from the origin upon the straight line. Let $ON = p$ and $\angle xON = \alpha$. The signs of $p$ and $\alpha$ are fixed according to the usual convention for polar coordinates. Let $P$ be any point on the line with rectangular coordinates $(x, y)$ and polar coordinates $(r, \theta)$. Then $ON = OP \cos(\theta - \alpha)$, *i.e.*

$$r \cos(\theta - \alpha) = p. \quad \text{.............(9)}$$

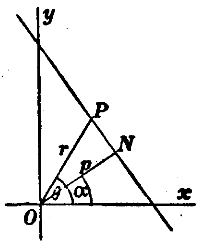


FIG. 15.

(5) *Normal or canonical form.* Expressing the equation (9) in rectangular coordinates, we have

$$x \cos \alpha + y \sin \alpha = p \quad \text{.............(10)}$$

**Examples.**

1. Find the gradient and the intercepts upon the axes of the following lines, and reduce each to the normal form: (i) $3x - 4y + 12 = 0$, (ii) $12x + 5y = 39$, (iii) $15x - 8y + 34 = 0$, (iv) $11x + 60y = 61$, (v) $x - y = 8$.

2. Find the equation of the locus of a point which moves under the following conditions: (i) equidistant from the points (2, 3) and (−4, 1); (ii) making with the points (5, 1) and (−1, 2) a triangle of area 6; (iii) difference of squares of distances from (−1, 3) and (2, 4) equal to 8.

3. Find freedom-equations for a line (i) through (2, 3) with gradient 2; (ii) through (1, 4) with gradient $-\frac{1}{3}$; (iii) through (1, 3) and (4, 2).

4. Find the constraint equations from the following freedom-equations:

$$\text{(i)}\ \left.\begin{aligned} x &= 2 + 3t, \\ y &= -1 + 4t; \end{aligned}\right\} \quad \text{(ii)}\ \left.\begin{aligned} x &= \tfrac{1}{2} + \tfrac{3}{4}t, \\ y &= -3 + t; \end{aligned}\right\} \quad \text{(iii)}\ x = \frac{7 + t}{2 - t},\ y = \frac{2 + 5t}{2 - t}.$$

5. Find freedom-equations (with integral coefficients, if possible) for the straight lines: (i) $x + 3y = 7$; (ii) $3x - 4y = 13$; (iii) $7x - 3y = 8$; (iv) $2x + 6y = 5$.

5. We shall now apply these forms to the investigation of some metrical relations between straight lines.

**Angle between two straight lines.** If the lines make angles $\psi_1, \psi_2$ with the axis of $x$, then the angle between the lines is

$$\varphi = \psi_1 - \psi_2.$$

Let the gradients of the lines be $\mu_1$ and $\mu_2$, so that $\tan \psi_1 = \mu_1$ and $\tan \psi_2 = \mu_2$, then

$$\tan \varphi = \frac{\tan \psi_1 - \tan \psi_2}{1 + \tan \psi_1 \tan \psi_2} = \frac{\mu_1 - \mu_2}{1 + \mu_1 \mu_2}.$$

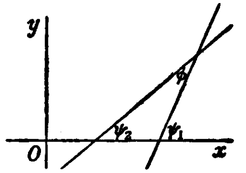


FIG. 16.

If special values of $\mu_1$ and $\mu_2$ are put in this formula it may give a positive or a negative result, and the sign will be changed if $\mu_1$ and $\mu_2$ are interchanged. The positive sign gives the tangent of the acute angle, the negative sign that of the obtuse angle.

If the equations of the lines are

$$l_1 x + m_1 y + n_1 = 0$$

and

$$l_2 x + m_2 y + n_2 = 0,$$

the gradients are $-l_1/m_1$ and $-l_2/m_2$, so that

$$\tan \varphi = \frac{l_1 m_2 - l_2 m_1}{l_1 l_2 + m_1 m_2}.$$

**Condition that two lines may be parallel.** The lines will be parallel if they are equally inclined to the axis of $x$, *i.e.* if

$$\mu_1 = \mu_2.$$

In the case of the general equation, the lines will be parallel when

$$l_1 m_2 = l_2 m_1 \quad \text{or} \quad \frac{l_1}{m_1} = \frac{l_2}{m_2},$$

*i.e.* if the coefficients of $x$ and $y$ are proportional. The angle $\varphi$ is then zero.

As a special case two lines will be parallel when their equations in $x$ and $y$ differ only by a constant term. If $k$ has any value whatever the line

$$lx + my = k$$

is parallel to the line

$$lx + my + n = 0.$$

**Condition that two lines may be at right angles.** The condition that $\varphi = \frac{\pi}{2}$ is that $\psi_1 = \frac{\pi}{2} + \psi_2$; therefore

$$\mu_1 = \tan \psi_1 = -\cot \psi_2 = -\frac{1}{\mu_2},$$

*i.e.*

$$\mu_1 \mu_2 + 1 = 0.$$

In the case of the general equation the condition is

$$l_1 l_2 + m_1 m_2 = 0.$$

In particular the lines

$$lx + my + n = 0$$

and

$$mx - ly = k$$

are perpendicular for all values of $k$.

**6. Distance of a point from a line.** Let the equation of the line in the normal form be

$$x \cos \alpha + y \sin \alpha = p,$$

and let $P \equiv (x', y')$ be the given point. Draw the ordinate $PM$. Then, projecting $OM$, $MP$ upon $ON'$ (see Fig. 17), we have, if $P$ is on the opposite side of the line from $O$,

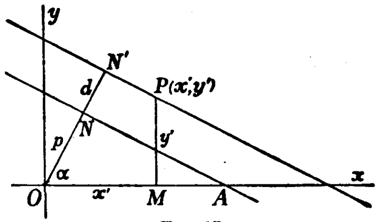


FIG. 17.

$$x' \cos \alpha + y' \sin \alpha = ON' = p + d,$$

or $= p - d$, if $P$ and $O$ are on the same side of the line. Hence

$$\pm d = x' \cos \alpha + y' \sin \alpha - p.$$

If the equation of the line is given in the form

$$lx + my + n = 0,$$

we have, by equating the ratios of the coefficients

$$\frac{\cos \alpha}{l} = \frac{\sin \alpha}{m} = \frac{-p}{n} = \frac{1}{\sqrt{l^2 + m^2}}.$$

Therefore $$d = \pm(lx' + my' + n)/\sqrt{l^2 + m^2}.$$

7. Let the expression $lx + my + n$ be denoted by $u$, so that the equation of the line can be denoted by $u = 0$; also let $lx_1 + my_1 + n \equiv u_1$. It can be shown that the expression $u$ changes sign as the point $(x, y)$ crosses the line $u = 0$.

Let $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ be two points whose join cuts the line $u = 0$ at $R \equiv (x, y)$. Let $PR : RQ = k$. Then

$$x = \frac{kx_2 + x_1}{k + 1}, \quad y = \frac{ky_2 + y_1}{k + 1},$$

and since these values satisfy the equation of the line, we have

$$l(kx_2 + x_1) + m(ky_2 + y_1) + n(k + 1) = 0,$$

whence $$k = -\frac{lx_1 + my_1 + n}{lx_2 + my_2 + n} = -\frac{u_1}{u_2}.$$

If $u_1$ and $u_2$ have the same sign, $k$ is negative, and $R$ lies outside the segment $PQ$, *i.e.* $P$ and $Q$ lie on the same side of the line.

If $u_1$ and $u_2$ are of opposite signs, $k$ is positive, $R$ lies between $P$ and $Q$, and $P$ and $Q$ lie on opposite sides of the line.

The line $u = 0$ therefore divides the plane into two regions, such that for one region $u > 0$ and for the other region $u < 0$. In order to determine the different regions it is generally most convenient to determine the sign of $u$ for the origin.

Two lines $u$ and $v$ divide the plane into four regions, corresponding to the four combinations of sign of $u$ and $v$.

Three lines $u$, $v$, $w$ divide the plane into seven regions. One is the interior of the triangle formed by the three lines, and corresponds to a certain set of signs, *e.g.* $+ + +$, of $u$, $v$, $w$. The other six regions correspond to all the other combinations of sign except $- - -$.

**Examples.**

1. Find the angle between the following pairs of lines: (i) $y=4x-2$, $5y=3x+1$; (ii) $y=2x+3$, $y=3x+2$; (iii) $3x+4y=7$, $2x-3y=8$.

2. Find the distance of $3x+4y=5$ from (i) (2, 6); (ii) (4, 2); (iii) (−1, 12).

3. Find the distance of (2, 3) from

(i) $4x+3y=7$; (ii) $5x+12y=20$; (iii) $15x-8y+11=0$.

4. Show that (2, −1) and (1, 1) are on opposite sides of $3x+4y=6$.

5. Show that (1, 1), (2, 3), (0, 7) and (−2, 4) are in the four regions of the plane made by $3x+2y=6$ and $2x-y+2=0$.

6. Find the equation of the straight line (i) through (4, 5), parallel to $y=5x+6$; (ii) perpendicular to $y=3x-2$ and making an intercept 4 on the axis of $y$; (iii) parallel to $2x+3y+11=0$, and such that the algebraic sum of the intercepts on the axes is 15; (iv) perpendicular to $4x+7y+9=0$, and such that the triangle formed with the axes has area $3\frac{1}{2}$; (v) parallel to $x/3+y/4=1$, and such that the perpendicular from the origin is 8.

**8. Intersection of two lines.** A point of intersection of two loci is a point which lies upon both loci, and whose coordinates therefore satisfy the equations of both loci. The coordinates of the points of intersection are therefore found by solving the two equations simultaneously.

In the case of two straight lines

$$l_1x+m_1y+n_1=0, \quad \text{.....(1)}$$
$$l_2x+m_2y+n_2=0, \quad \text{.....(2)}$$

the coordinates of the point of intersection are given by

$$\frac{x}{m_1n_2-m_2n_1}=\frac{y}{n_1l_2-n_2l_1}=\frac{1}{l_1m_2-l_2m_1}. \quad \text{.....(3)}$$

This gives a unique solution and a definite point of intersection, provided $l_1m_2 \neq l_2m_1$.

**9. Parallel lines. Points at infinity.** Suppose $l_1m_2=l_2m_1$, and let $l_1/l_2=m_1/m_2=k$. Then equations (1) and (2) can be written

$$l_1x+m_1y+\ \ n_1=0,$$
$$l_1x+m_1y+kn_2=0.$$

Regarded as algebraic equations, these are inconsistent unless $kn_2=n_1$, in which case the two lines would coincide. Hence, if the two lines do not coincide they have no point of intersection. The condition $l_1m_2=l_2m_1$ is in fact the condition that the lines should be parallel.

Considering, however, the general solution (3), it appears that in this case the values of $x$ and $y$ for the point of intersection become infinite, and it is convenient to say that parallel lines determine a "point at infinity."

With the one exception of parallel lines, two straight lines in a plane have always a point in common. Two parallel lines have no point in common, but they have a common *direction*. Two intersecting lines have no direction in common, but they have a common point. Hence two lines in a plane have always in common either a point or a direction. The use of the phrase "point at infinity," which is to be regarded as synonymous with "direction," enables us to make the general statement that two coplanar lines have always a "point" in common, which may be a "point at infinity."

The point at infinity or direction of a line is determined by its gradient, which is the value of the ratio $-l/m$. When the line is parallel to the axis of $y$, $m=0$ and the gradient becomes infinite, but the direction is still properly determined by the two numbers $l$, $0$; similarly when the line is parallel to the axis of $x$ its direction is determined by the two numbers $0$, $m$. There is thus an advantage in using the two numbers $-l, m$ to represent the direction instead of the single number $\mu = -l/m$.

A similar device applied to the coordinates of a point has the remarkable result of enabling us actually to represent a point at infinity by definite coordinates just as if it were an ordinary point. This we shall now explain.

For the cartesian coordinates $x$, $y$ of a point let us write the ratios $x=X/Z$, $y=Y/Z$. The point is then determined by the three numbers $X$, $Y$, $Z$ which are called its *homogeneous cartesian coordinates*. If $\rho$ is any factor, not zero, the homogeneous coordinates $(\rho X, \rho Y, \rho Z)$ always represent the same point as $(X, Y, Z)$. In order that the cartesian coordinates $x$, $y$ should be definite and finite we have to postulate that $Z \neq 0$. With this condition any set of values of $X$, $Y$, $Z$ determines a unique point, viz. the point whose cartesian coordinates are $x=X/Z$, $y=Y/Z$.

The equation of a straight line $lx+my+n=0$ becomes $lX+mY+nZ=0$ which is homogeneous in $X$, $Y$, $Z$. The point of intersection of two lines $l_1X+m_1Y+n_1Z=0$ and $l_2X+m_2Y+n_2Z=0$ is given by

$$(X, Y, Z)=(m_1n_2-m_2n_1,\ n_1l_2-n_2l_1,\ l_1m_2-l_2m_1)$$

(or any multiple of these). When the two lines are parallel, $l_1m_2-l_2m_1=0$, *i.e.* $Z=0$. If therefore we now remove the restriction that $Z\neq 0$ we can represent points at infinity as definitely as ordinary points; they correspond to sets of values of $X$, $Y$, $Z$ in which $Z=0$.

The common "point" of the two parallel lines $lx+my+n=0$ and $lx+my+kn=0$ is represented by the homogeneous coordinates

$$\begin{aligned}(X, Y, Z)&=\{mn(k-1),\ nl(1-k),\ 0\}\\ &=(-m,\ l,\ 0).\end{aligned}$$

We therefore consider $(-m, l, 0)$ as the homogeneous coordinates of the point at infinity on the line $lx+my+n=0$, or on any parallel line.

When $l$ and $m$ are not both zero the general homogeneous equation $lX+mY+nZ=0$ represents a definite straight line. It cuts the axes $Y=0$ and $X=0$ respectively in the points $(-n, 0, l)$ and $(0, -n, m)$. When $l=0$ the line is parallel to the axis of $x$ and the "point" $(-n, 0, 0)$ is a point at infinity. When $l$ and $m$ both vanish the "points" in which it meets the axes are both points at infinity. A line, however, cannot be parallel to two intersecting lines, but, just as the term "point" has been extended to include point at infinity, so now we extend the term "line" to include the "locus" of the equation $Z=0$, and we call this the "line at infinity." The *line at infinity* is the locus of all points at infinity in the plane. The point at infinity on any line $lX+mY+nZ=0$ can now be regarded as its point of intersection with the line at infinity, for putting $Z=0$ we obtain $lX+mY=0$ which is satisfied by $X:Y=-m:l$, *i.e.* the homogeneous coordinates of the point of intersection are $(-m, l, 0)$.

**10. Equation of a straight line through the intersection of two given lines.** Let the equations of the given lines be

$$l_1x+m_1y+n_1=0, \quad \text{.....................................(1)}$$

$$l_2x+m_2y+n_2=0. \quad \text{.....................................(2)}$$

We could, as in § 8, find the coordinates of the point of intersection of these lines, and then write down the equation of a line passing through this point. But the result can be much more expeditiously reached as follows.

The equation $\quad (l_1x+m_1y+n_1)-k(l_2x+m_2y+n_2)=0$ .......................(3)

is an equation of the first degree, and therefore represents a straight line. Also it is satisfied by the coordinates of the point whose coordinates satisfy both (1) and (2). Hence it is the equation of a line through the intersection of the given lines. For shortness we may write the equations (1) $u=0$, (2) $v=0$, and (3) $u-kv=0$.

By giving the proper value to $k$ the equation can be made to represent any line through the intersection of the given lines. Regarding $k$ as a variable parameter the equation (3) represents a pencil of lines through the intersection of (1) and (2).

**11. Condition for concurrency.** The three lines

$$l_1x+m_1y+n_1=0, \quad \text{.....................................(1)}$$

$$l_2x+m_2y+n_2=0, \quad \text{.....................................(2)}$$

$$l_3x+m_3y+n_3=0 \quad \text{.....................................(3)}$$

will be concurrent if the coordinates of the point of intersection of the first two satisfy the third. Substituting for $x$ and $y$ the values

$$\frac{m_1n_2-m_2n_1}{l_1m_2-l_2m_1}, \quad \frac{n_1l_2-n_2l_1}{l_1m_2-l_2m_1},$$

we have, provided $l_1m_2-l_2m_1 \neq 0$,

$$l_3(m_1n_2-m_2n_1)+m_3(n_1l_2-n_2l_1)+n_3(l_1m_2-l_2m_1)=0, \quad \text{............(4)}$$

which may be written also in the form of a determinant

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

This is therefore a *necessary* condition that the three lines should be concurrent.

Conversely, it may be shown that, allowing for points at infinity, the condition (4) is also the *sufficient* condition that the three lines should be concurrent.

Denoting the lines (1), (2), (3) for shortness by $u=0$, $v=0$, $w=0$, *the three lines will be concurrent, if we can find multiples* $\lambda$, $\mu$, $\nu$, *such that*

$$\lambda u+\mu v+\nu w$$

*vanishes identically.*

For $\lambda u+\mu v=0$ represents a line through the intersection of $u$ and $v$, and if $w$ is concurrent with $u$ and $v$, we can choose the values of $\lambda$ and $\mu$, so that this line may coincide with $w$. Then $\lambda u+\mu v$ will be some multiple of $w$, say $-\nu w$. Hence

$$\lambda u+\mu v+\nu w=0$$

identically.

This is often useful as a test in simple cases when the multiples $\lambda$, $\mu$, $\nu$ can be found by inspection.

**Ex.** Prove that the altitudes of the triangle whose vertices are $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ are concurrent.

The equations of the altitudes are

$$(x-x_1)(x_2-x_3)+(y-y_1)(y_2-y_3)=0,$$
$$(x-x_2)(x_3-x_1)+(y-y_2)(y_3-y_1)=0,$$
$$(x-x_3)(x_1-x_2)+(y-y_3)(y_1-y_2)=0.$$

Adding the three expressions on the left-hand side the sum vanishes identically.

**12. Bisectors of the angles between two straight lines.** Let

$$x\cos\alpha+y\sin\alpha=p,$$
$$x\cos\beta+y\sin\beta=q$$

be the equations of two lines $SA$, $SB$, in the normal form, and let $P\equiv(x, y)$ be any point lying on the bisector of either the angle $ASB$ or the angle $BSA'$. Then, if $PM\perp SA$ and $PN\perp SB$, $PM=PN$. But, having regard only to the magnitudes of the perpendiculars,

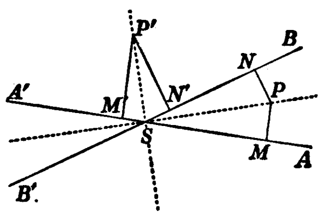



FIG. 18.

$$PM=x\cos\alpha+y\sin\alpha-p,$$
$$PN=x\cos\beta+y\sin\beta-q.$$

Equating these, we get

$$x(\cos\alpha-\cos\beta)+y(\sin\alpha-\sin\beta)=p-q,$$

which is the equation of a straight line and represents either the bisector of the angle $ASB$ or that of the angle $BSA'$. Having regard now to the signs of the perpendiculars, we see that while $PM$ and $P'M'$ have the same sign, $PN$ and $P'N'$ are of opposite sign, hence for the one bisector we have $PM=PN$ and for the other $PM=-PN$. The equation of the other bisector is then

$$x(\cos\alpha+\cos\beta)+y(\sin\alpha+\sin\beta)=p+q.$$

The two bisectors are at right angles, since

$$\angle PSB + BSP' = \tfrac{1}{2}(ASB + BSA') = \frac{\pi}{2};$$

or, applying the analytical test to the equations, we have

$$(\cos^2\alpha - \cos^2\beta) + (\sin^2\alpha - \sin^2\beta) = 0.$$

In any numerical case the two bisectors can be easily distinguished by considering their gradients, since these are of opposite sign.

If the equations of the two lines are given in the general form

$$l_1x + m_1y + n_1 = 0,$$
$$l_2x + m_2y + n_2 = 0,$$

we must first reduce them to the normal form by dividing by $\sqrt{l_1^2 + m_1^2}$ and $\sqrt{l_2^2 + m_2^2}$. Then the equations of the two bisectors are

$$\frac{l_1x + m_1y + n_1}{\sqrt{l_1^2 + m_1^2}} = \pm\frac{l_2x + m_2y + n_2}{\sqrt{l_2^2 + m_2^2}}.$$

**Examples.**

1. Find the coordinates of the point of intersection of each line with the next: (i) $y=3x+4$; (ii) $2x-3y=5$; (iii) $3x+y+2=0$; (iv) $x=2+3t$, $y=1-t$; (v) $x=4+4t$, $y=1-2t$; (vi) $x=-3-t$, $y=2+3t$; (vii) $y=3x+4$.

2. Find the equation of the line (i) joining the origin to the point of intersection of $3x-2y+4=0$ and $5x+3y-3=0$, (ii) joining (2, 3) to the point of intersection of $x-2y+3=0$ and $3x-4y=5$, (iii) parallel, (iv) perpendicular to $x+2y=4$ and through the intersection of $3x+4y=8$ and $2x-5y+3=0$.

3. Show that the following pairs of equations represent the same pencil of lines: (i) $2x+3y-8+\lambda(4x-7y+10)=0$ and $3x+4y-11+\mu(2x-5y+8)=0$, (ii) $2x+4y-12+\lambda(x-3y+9)=0$ and $3x-2y+6+\mu(2x+3y-9)=0$.

4. Write down the equations of the bisectors of the angles between the lines: (i) $x+2y+3=0$ and $2x-y-5=0$, (ii) $4x+3y+10=0$ and $12x-5y+2=0$, (iii) $3x+2y+2=0$ and $18x-y-1=0$.

5. Prove that the two pairs of lines $x+y+2=0$, $x-7y=2$ and $6x+8y+13=0$, $2y+1=0$ have the same angle-bisectors.

**13. Quadratic equation representing a pair of straight lines.** It is often convenient to represent by a single equation a locus consisting of a pair of straight lines, such as the bisectors of the angles between two given lines.

Consider any two straight lines $lx+my+n=0$ and $l'x+m'y+n'=0$. The equation

$$(lx+my+n)(l'x+m'y+n')=0$$

is satisfied by the coordinates of any point on either of the given lines, and by no other values. It therefore represents the two straight lines together as a single locus of the second degree. Thus, to form the equation which represents two straight lines whose equations are separately given, take all the terms to the left-hand side of the equations, multiply together the two expressions on the left and equate to zero.

Conversely, if a quadratic expression in $x$, $y$ can be broken up into factors of the first degree, the equation obtained by equating the expression to zero represents two straight lines. Thus the equation

$$x^2 - xy - 2y^2 + 2x + 5y = 3$$

can be written $(x+y-1)(x-2y+3)=0$, and represents the two lines $x+y=1$ and $x-2y+3=0$. On the other hand, the equation $x^2+y^2-1=0$ cannot be reduced, and does not represent two straight lines; it represents the locus of a point at unit distance from the origin, *i.e.* a circle.

**14. Condition that the general equation of the second degree should represent two straight lines.** The most general equation of the second degree in $x$, $y$ is

$$ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

Suppose $a$ is not zero. Multiply each term by $a$, and complete the square with all the terms which involve $x$, and we get

$$(ax+hy+g)^2 - \{(h^2-ab)y^2 + 2(gh-af)y + (g^2-ac)\} = 0.$$

In order that the second part may be a perfect square in $y$, we must have

$$(gh-af)^2 = (h^2-ab)(g^2-ac).$$

Expanding this, and cancelling $a$, which is not zero, we have

$$abc + 2fgh - af^2 - bg^2 - ch^2 = 0. \quad \ldots\ldots\ldots\ldots(\Delta)$$

Conversely, if this is true, and $a \neq 0$, the left-hand side can be written as the difference of two squares, and therefore factorizes.

If $a=0$, and $b \neq 0$, we may complete the square in $y$. Then proceeding as before, we get the condition $2fgh - bg^2 - ch^2 = 0$, which is the same condition as before with $a=0$.

If $a=0$ and $b=0$, but $h \neq 0$, multiplying by $h$ the factors must be of the form

$$2(hx+f)(hy+g);$$

hence we must have $2fg = ch$, which is again the same condition with $a=0$ and $b=0$.

If $a=0$, $b=0$, and $h=0$, the expression is not of the second degree.

The expression on the left of the condition, which is denoted by $\Delta$ is called the *discriminant* of the equation. It may be written in the form of a symmetrical determinant

$$\Delta \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

**15. The homogeneous equation of the second degree.** The general homogeneous equation of the second degree is

$$ax^2 + 2hxy + by^2 = 0.$$

The left-hand side breaks up into the two factors

$$y + \frac{h \pm \sqrt{h^2 - ab}}{b}x,$$

and these, equated to zero, represent *two straight lines through the origin.*

If $h^2 - ab$ is negative, we say that the equation represents two imaginary straight lines, and if $h^2 - ab = 0$, the two lines are coincident. Imaginary straight lines which arise from equations with real coefficients always occur in pairs, and are called *conjugate imaginary lines*. Similarly we may have conjugate imaginary points as, for example, when we try to form the points of intersection of a circle with a line at a greater distance from its centre than the length of the radius.

**Examples.**

1. Examine whether each of the following equations represents two straight lines:

(i) $xy=0$, (ii) $xy=2y$, (iii) $xy+3x^2=0$,
(iv) $9x^2-25y^2=0$, (v) $x^2+4y^2=0$, (vi) $10xy+8x-15y-12=0$,
(vii) $2x^2-xy+5x-2y+2=0$, (viii) $6x^2-17xy-3y^2+22x+10y=8$,
(ix) $10x^2-23xy-5y^2=29x-32y-21$, (x) $6x^2-15y^2-xy+16x+24y=0$,
(xi) $4x^2+9=12x$, (xii) $3x^2+5xy-2y^2-8x+5y-3=0$,
(xiii) $x^2-2xy \operatorname{cosec}\theta+y^2=0$, (xiv) $x^2-2xy\cot\theta-y^2=0$,
(xv) $4x^2+9y^2+12x-42y+58=0$, (xvi) $2x^2+26xy+17y^2-22x-8y-7=0$,
(xvii) $4x^2+12xy+9y^2-20x-30y+25=0$,
(xviii) $10x^2-4xy-5y^2+12x-6y+3=0$,
(xix) $2x^2-2xy-y^2-2x+4y-1=0$, (xx) $x^2+y^2-xy-x-y+1=0$.

2. Find the values of $\lambda$ in order that the following equations may represent pairs of straight lines:

(i) $x^2+xy+3x+\lambda y=0$, (ii) $2x^2+9xy+4y^2=\lambda x+2y$,
(iii) $\lambda xy+5x+3y+2=0$, (iv) $4x^2-9y^2-2(8+\lambda)x-18y=29+2\lambda$.

**16. Angle between the two straight lines represented by the homogeneous equation.** Let

$$by^2+2hxy+ax^2 \equiv b(y-\mu_1 x)(y-\mu_2 x);$$

then, equating coefficients, we have

$$\mu_1+\mu_2=-2h/b, \quad \mu_1\mu_2=a/b.$$

Let $\varphi$ be the angle between the two lines; then

$$\tan\varphi=\frac{\mu_1-\mu_2}{1+\mu_1\mu_2}.$$

But $$(\mu_1-\mu_2)^2=(\mu_1+\mu_2)^2-4\mu_1\mu_2=4(h^2-ab)/b^2.$$

Hence $$\tan\varphi=\frac{2\sqrt{h^2-ab}}{a+b}.$$

*Cor.* The two lines are at right angles if $a+b=0$; hence the equation

$$ax^2+2hxy-ay^2=0$$

always represents two straight lines at right angles.

**17.** *If the general equation* $ax^2+2hxy+by^2+2gx+2fy+c=0$ *represents two straight lines they are parallel respectively to the two straight lines represented by the homogeneous equation* $ax^2+2hxy+by^2=0$. Let the factors of the left-hand side be $$b(y-\mu_1 x-b_1)(y-\mu_2 x-b_2).$$

Multiplying out and comparing with the given equation, we have

$$\mu_1\mu_2 = a/b, \quad \mu_1 + \mu_2 = -2h/b.$$

Hence $ax^2 + 2hxy + by^2 \equiv b(y - \mu_1 x)(y - \mu_2 x)$, and the lines are parallel.

*Cor.* 1. The angle $\varphi$ between the two straight lines represented by the general equation is given by

$$\tan\varphi = \frac{2\sqrt{h^2 - ab}}{a+b}.$$

*Cor.* 2. The two lines will be at right angles if $a + b = 0$.

*Cor.* 3. The two lines will be parallel if $h^2 - ab = 0$.

**18. Lines joining the origin to the points of intersection of two loci.** As an example consider the loci

$$x^2 + y^2 - 2x + 4y - 4 = 0$$

and $$x + 2y + 5 = 0,$$

*i.e.* a circle and a straight line.

Make the equations homogeneous by introducing powers of $z$; thus

$$x^2 + y^2 - 2(x - 2y)z - 4z^2 = 0,$$
$$x + 2y + 5z = 0.$$

Then eliminate $z$, and we get the equation

$$31x^2 - 16xy - 31y^2 = 0.$$

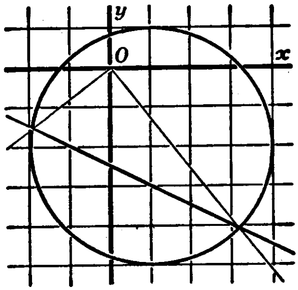



FIG. 19.

This equation is homogeneous in $x$, $y$, and is satisfied by the coordinates of any point which satisfy the two given equations simultaneously. Hence it represents the two straight lines joining the origin to the points of intersection of the two loci. We see also that, since the sum of the coefficients of $x^2$ and $y^2$ is zero, these two lines are at right angles.

**Examples.**

1. Find the equation of the lines joining the origin to the points of intersection of the line $x + y = 1$ with the curve $4x^2 + 4y^2 + 4x - 2y - 5 = 0$, and show that they are at right angles.

2. Write down the equation of the line joining the origin to the point of intersection of the curves $y^2 = a^3x$, $x^2 = b^3y$.

3. Show that the lines joining the origin to the points of intersection of the line $2x - 3y + 4 = 0$ with the curve $x^2 + 4xy + 2y^2 + 12x + 4y = 0$ are at right angles.

4. Find the angle between the lines joining the origin to the intersections of the line $x - 3y + 2 = 0$ with the curve $y^2 - 17xy + 16y - 12 = 0$.

5. Find the equation of the lines joining the origin to the points of intersection of the line $l(x-2) + m(y-3) = 0$ with the locus $x^2 - y^2 + 2xy - 10x + 2y + 7 = 0$, and interpret the result.

6. Find the angle between the following pairs of lines:

(i) $3x^2 + 7xy + 2y^2 = 0$, (ii) $11x^2 + 16xy - y^2 = 0$, (iii) $3x^2 + 2xy - y^2 = 0$,

(iv) $x^2 + 4xy + y^2 - 6x - 3 = 0$, (v) $6x^2 + xy - y^2 - 21x - 8y + 9 = 0$.

(vi) $x^2 - 5xy + 4y^2 + 3x - 4 = 0$.

7. Find the coordinates of the point of intersection of the lines (iv), (v), (vi) of Ex. 6.

**19. Harmonic ranges and pencils.**

*Def.* 1. If the points $P$, $Q$ divide the segment $XY$ internally and externally in equal ratios, $(XY, PQ)$ is said to be a *harmonic range*, and $P$, $Q$ are *harmonic conjugates* with regard to $X$, $Y$; the two pairs of points are also said to be *apolar*.

It may be proved that $X$, $Y$ are then harmonic conjugates with regard to $P$, $Q$.

*Def.* 2. If $(XY, PQ)$ is a harmonic range, and $O$ is any point not on the line $XY$, the pencil $O(XY, PQ)$, formed by joining $O$ to the four points, is said to be a *harmonic pencil*, and the rays $OP$, $OQ$ are *harmonic conjugates* with regard to $OX$, $OY$; the two pairs of lines are also said to be *apolar*.

It is proved in text-books on pure geometry that if any straight line cuts the four rays of a harmonic pencil in $X'Y'$, $P'Q'$, the range $(X'Y', P'Q')$ is harmonic.

A particular case of a harmonic range, which is extremely useful, is formed by taking the ratio equal to unity. Then *one of the points $P$ is the mid-point of the segment $XY$, and the other point $Q$ is the point at infinity on the line $XY$.*

**20. Condition that the two pairs of lines through the origin, $y=\lambda x$, $y=\mu x$ and $y=\lambda' x$, $y=\mu' x$ should be apolar.** Draw a straight line cutting the lines in $L$, $M$, $L'$, $M'$. If this line is parallel to the first line, $L$ is a point at infinity, and therefore $M$ must be the mid-point of $L'M'$. Let the equation of the straight line be $y=\lambda x+c$. Then the abscissae of the points $L'$, $M$, $M'$ are
$$c/(\lambda'-\lambda), \quad c/(\mu-\lambda), \quad c/(\mu'-\lambda).$$
Hence the condition that the two pairs of lines be apolar, or that $M$ should be the mid-point of $L'M'$, is
$$\frac{2}{\mu-\lambda}=\frac{1}{\lambda'-\lambda}+\frac{1}{\mu'-\lambda}.$$
This is more usually written in the form
$$\frac{\lambda-\lambda'}{\lambda-\mu'}\bigg/\frac{\mu-\lambda'}{\mu-\mu'}=-1.$$

*Def.* The expression on the left-hand side (which has a definite value for any four lines through the origin) is called the *cross-ratio* of the four lines taken in the given order, and is written $(\lambda\mu, \lambda'\mu')$.

**21. Condition that the two pairs of lines**
$$ax^2+2hxy+by^2=0,$$
$$a'x^2+2h'xy+b'y^2=0,$$
**should be apolar.** Let the two pairs of lines be
$$\left.\begin{matrix} y=\lambda x, \\ y=\mu x, \end{matrix}\right\} \quad \left.\begin{matrix} y=\lambda' x, \\ y=\mu' x; \end{matrix}\right\}$$

then the condition in § 20 can be written in the form

$$2(\lambda\mu+\lambda'\mu')=(\lambda+\mu)(\lambda'+\mu').$$

Now $$\lambda+\mu=-2h/b, \qquad \lambda\mu=a/b,$$
$$\lambda'+\mu'=-2h'/b', \quad \lambda'\mu'=a'/b'.$$

Therefore the condition reduces to

$$2\left(\frac{a}{b}+\frac{a'}{b'}\right)=\frac{4hh'}{bb'},$$

*i.e.* $$ab'+a'b-2hh'=0.$$

**22. Equation of the bisectors of the angles between the two straight lines** $$\mathbf{ax^2+2hxy+by^2=0.}$$

It is easily seen that the two bisectors are harmonic conjugates with regard to the two straight lines, and they are at right angles. The equation of any pair of perpendicular straight lines through the origin is

$$x^2+2\lambda xy-y^2=0.$$

Since these are also to be harmonic conjugates with regard to the given pair, we have $$-a+b-2\lambda h=0;$$

therefore $$\lambda=-\tfrac{1}{2}(a-b)/h.$$

Hence the equation of the bisectors is

$$hx^2-(a-b)xy-hy^2=0.$$

**Examples.**

1. Obtain the equation of the bisectors of the angles between the lines

   (i) $3x^2+4xy+2y^2=0$, (ii) $2x^2-3xy+y^2=0$,

   (iii) $2(x-1)^2-6(x-1)(y+2)+(y+2)^2=0$,

   (iv) $x^2-5xy+3y^2+11x-21y+27=0$.

2. Find the equations of the lines which form with the three lines $y=0$, $y=2x$, $y=-x$, in some order, a harmonic pencil.

3. Find the coordinates of the points which form with the three points $(1, 0)$, $(4, 0)$, $(6, 0)$ a harmonic range.

4. Find the harmonic conjugate of the line $y=\mu x$ with regard to the pair $ax^2+2hxy+by^2=0$.

## EXAMPLES II. A.

1. Given the triangle $A(10, 4)$, $B(-4, 9)$, $C(-2, -1)$, find (i) the equation of the median through $A$, (ii) the equation of the altitude through $B$, (iii) the length of this altitude.

2. Prove that $(-4, -1)$ is the centre of one of the escribed circles of the triangle $3x-4y=17$, $y=4$, $12x+5y=12$.

3. The vertices of a triangle are $(2, 1)$, $(5, 2)$, $(3, 4)$. Find the coordinates of the centroid $G$, the orthocentre $O$, and the circumcentre $S$, and show that $G$ divides $OS$ in the ratio $2:1$.

4. Find the equations of the interior bisectors of the angles of the triangle $11x+2y=13$, $22x-19y=3$, $x-2y=119$, and verify that they are concurrent.

5. Mark the regions of the plane according to the signs of the three expressions $1-y$, $y+1+x$, $y+1-x$, and show that the signs cannot be all negative.

6. Show that the following pairs represent the same pencil of lines, and find for each the relation between $\lambda$ and $\mu$ in order that they may represent the same individual line:

$$\text{(i)}\ x+2y-3+\lambda(3x-y+4)=0 \quad \text{and} \quad 8x-5y+15-\mu(5x-4y+11)=0,$$
$$\text{(ii)}\ 3x-y-7+\lambda(x+y-1)=0 \quad \text{and} \quad 2x+3y-1+\mu(5x-6y-16)=0.$$

7. Two equal circles of radius 1 have their centres at the points (0, 2) and (1, 0); find the equations of their parallel common tangents.

8. A line moves so that the sum of the reciprocals of its intercepts on the axes is constant; show that it passes through a fixed point.

9. A line moves so that the ratio of the perpendiculars upon it from two fixed points is constant; show that it passes through a fixed point.

10. Show, without reducing to the constraint form, that the following pairs of freedom-equations represent the same straight line:

$$\text{(i)}\ \left.\begin{aligned} x&=2+3t,\\ y&=-1+4t,\end{aligned}\right\}\ \text{and}\ \left.\begin{aligned} x&=-4+6u,\\ y&=-9+8u;\end{aligned}\right\} \quad \text{(ii)}\ \left.\begin{aligned} x&=(3+t)/(2-t),\\ y&=(1-3t)/(2-t),\end{aligned}\right\}\ \text{and}\ \left.\begin{aligned} x&=1+u,\\ y&=1-u.\end{aligned}\right\}$$

11. Find for each of the pairs of freedom-equations in Ex. 10 the relation between the parameters in order that the equations may represent the same point.

12. Find the coordinates of the in- and ex-centres of the triangle $11x+2y=0$, $22x-19y=0$, $x-2y=120$.

13. Find the coordinates of the in- and ex-centres of the triangle (50, 20), (−13, 20), (2, −16).

14. $A$, $A'$ are two points on the $x$-axis, and $B$, $B'$ two points on the $y$-axis. $AB'$, $A'B$ meet in $P$, and $AB$, $A'B'$ in $Q$. Prove that $OP$, $OQ$ are harmonic conjugates with regard to the axes, and are equally inclined to each of the axes.

15. Find three values of $k$ for which $k(x^2+y^2-25)+(3x+4y)(x-2y-5)=0$ represents two straight lines. How are these pairs of lines related to the circle $x^2+y^2=25$?

16. Express by a single equation the two lines through (10, 2) making an angle 45° with the line $2x+3y=4$.

17. Express by a single equation the two lines through (2, 3) which make with the axes a triangle in the first quadrant of area $12\frac{1}{2}$.

18. Prove that the equation of the two lines through the origin perpendicular to the two lines $ax^2+2hxy+by^2=0$ is $bx^2-2hxy+ay^2=0$.

19. Show that the lines joining the origin to the intersections of

$$3x^2+5xy-3y^2+2x+3y=0 \quad \text{and} \quad 3x-2y=1$$

are at right angles.

20. Write down the equation of the lines drawn from the origin to the intersections of the curve $y=x^2$ with the straight line $lx+my+n=0$, and show that if these two lines are at right angles the line $lx+my+n=0$ must pass through a certain fixed point.

21. Show that all chords of the curve $3x^2-y^2-2x+4y=0$ which subtend a right angle at the origin pass through a fixed point.

22. Show that it is impossible for both pairs of points of a harmonic range to be conjugate imaginaries.

23. Given the following pair of opposite vertices of a square, find the other pair: (i) (1, 3), (7, 1); (ii) (5, 3), (−3, 1); (iii) (3, 1), (−1, 7); (iv) $(a, b)$, $(-a, -b)$; (v) $(x_1, y_1)$, $(x_2, y_2)$.

## EXAMPLES II. B.

1. The reciprocals of the intercepts which a line makes on the axes are connected by an equation of the first degree; show that the line passes through a fixed point. Examine the case when the intercepts have a constant ratio.

2. A line moves so that the algebraic sum of its distances from a number of given points is zero; show that it passes always through the mean point of the given points.

3. Prove that the perpendiculars from the points (−8, 10), (1, 2), (1, 11) on the lines $y=3x-5$, $2y=x$, $x+y=15$ respectively are concurrent, and show that the reciprocal relation also holds for these two triangles.

4. Prove that if the perpendiculars from the vertices $A$, $B$, $C$ on the sides $B'C'$, $C'A'$, $A'B'$ of another triangle are concurrent, the perpendiculars from $A'$, $B'$, $C'$ on the sides $BC$, $CA$, $AB$ of the first triangle are also concurrent.

5. Two straight rulers with inches marked on them are laid across one another at a given angle, so that the zero points do not coincide. Show that perpendiculars drawn to the rulers at points having the same marks intersect on a line parallel to the bisector of the angle between the rulers. (Selwyn, 1914.)

6. Prove that the vertices of the quadrilateral whose sides are given by the equations $l_1x+m_1y+n_1=0$, etc., are concyclic if

$$(l_1m_2-l_2m_1)(l_3l_4+m_3m_4)+(l_3m_4-l_4m_3)(l_1l_2+m_1m_2)=0,$$

and explain why this condition does not involve $n_1$, $n_2$, $n_3$.

7. Find the gradient of a line which, along with the lines $x+y=1$, $x-2y+3=0$, $2x-3y+1=0$ forms a cyclic quadrangle; and explain the three solutions which can be obtained.

8. Two straight lines making a fixed angle $\alpha$ intercept segments on the coordinate axes which are each equal to the fixed segment $k$. Find the locus of their point of intersection. (Selwyn, 1907.)

9. Show that the equation of any line-pair whose angle-bisectors are $ax^2+2hxy-ay^2=0$ is $(A-h)x^2+2axy+(A+h)y^2=0$.

10. Show that the four lines joining the origin to the points of intersection of the two curves represented by $u_2+u_1+u_0=0$ and $v_2+v_1+v_0=0$, where each letter denotes a homogeneous expression in $x$, $y$ whose degree is indicated by the suffix, are represented by the equation

$$(u_2v_0-v_2u_0)^2=(u_1v_0-v_1u_0)(u_2v_1-v_2u_1).$$

Show for the two curves

$$ax^2+2hxy+by^2+2gx+2fy+c=0 \text{ and } a'x^2+2h'xy+b'y^2+2gx+2fy+c=0$$

that these four lines reduce to two.

11. Show that the three lines joining the origin to the points of intersection, other than the origin, of the two curves represented by $u_2+u_1=0$ and $v_2+v_1=0$ are represented by the equation $u_2v_1=v_2u_1$.

12. $A$, $A'$ and $B$, $B'$ are points on the axes of $x$ and $y$ respectively, each at a distance $a$ from the origin. Find the locus of a point $P$ such that the pencil $P(AA', BB')$ is harmonic.

13. A variable circle cuts $OY$ in fixed points $B$, $B'$ and $OX$ in points $A$, $A'$. Show that the equation of the locus of the point of intersection of $AB$ and $A'B'$ or of $AB'$ and $A'B$ is $x^2=(y-b)(y-b')$ where $OB=b$ and $OB'=b'$.

14. Show that $x^2-y^2-2xy\tan\theta+2ay\sec\theta-a^2=0$ represents, for all values of $\theta$, two straight lines which pass through two fixed points, and find, as $\theta$ varies, the locus of their point of intersection.

15. $A$, $A'$ are two fixed points on $OX$, and $B$, $B'$ are two fixed points on $OY$. Show that the equation of the locus of a point $P$ such that $P(AA', BB')$ is harmonic is

$$bb'x^2+aa'y^2+\tfrac{1}{2}(a+a')(b+b')xy-bb'(a+a')x-aa'(b+b')y+aa'bb'=0.$$

# CHAPTER III.

## THE CIRCLE.

**1. To find the equation of a circle of radius r with its centre at the origin.** Let $P \equiv (x, y)$ be any point on the circle. Then, since $\sqrt{x^2+y^2}$ is the distance of $P$ from the centre, and this distance is equal to $r$, we have $\sqrt{x^2+y^2}=r$. Hence the equation of the circle is

$$x^2+y^2=r^2.$$

**2. The tangent at a given point.** A tangent to a curve is the limiting case of a secant when two of the points of intersection come to coincide. This suggests a general method of finding the tangent at a given point $P$. Take a point $Q$ on the curve, not far removed from $P$; find the equation of the secant $PQ$, and obtain the limiting form of this equation as $Q$ approaches $P$.

To find the equation of the tangent at $P \equiv (x_1, y_1)$ to the circle

$$x^2+y^2=r^2.$$

The equation of any line through $P$ is

$$y-y_1=\mu(x-x_1).$$

If this line cuts the circle again in $Q \equiv (x_2, y_2)$, the gradient

$$\mu=\frac{y_2-y_1}{x_2-x_1}.$$

We have to find the limiting value of this ratio as $Q$ moves along the circle into coincidence with $P$.

Since $P$ and $Q$ both lie on the circle,

$$x_1^2+y_1^2=r^2=x_2^2+y_2^2,$$

therefore

$$y_2^2-y_1^2=x_1^2-x_2^2,$$

and hence

$$\frac{y_2-y_1}{x_2-x_1}=-\frac{x_1+x_2}{y_1+y_2}.$$

The limiting value of the gradient, as $x_2 \to x_1$ and $y_2 \to y_1$, is therefore

$$-\frac{2x_1}{2y_1}=-\frac{x_1}{y_1}.$$

The equation of the tangent is therefore

$$y-y_1=-\frac{x_1}{y_1}(x-x_1),$$

which becomes, on multiplying up,

$$xx_1 + yy_1 = x_1^2 + y_1^2,$$

or finally $$xx_1 + yy_1 = r^2.$$

The student familiar with the differential calculus will recognise that the gradient of the tangent, viz. $\text{Lim}\,\dfrac{y_2 - y_1}{x_2 - x_1} = \text{Lim}\,\dfrac{\delta y_1}{\delta x_1} = \left(\dfrac{dy}{dx}\right)_1$, *i.e.* the value of the differential coefficient when $x = x_1$ and $y = y_1$.

**3. The normal.** The normal to a curve at any point $P$ on the curve is the line through $P$ perpendicular to the tangent at $P$. Its equation is therefore easily found from the equation of the tangent.

To find the equation of the normal at $(x_1, y_1)$ to the circle

$$x^2 + y^2 = r^2,$$

write down first the equation of the tangent

$$xx_1 + yy_1 = r^2.$$

Then the equation of a line through $(x_1\ y_1)$ perpendicular to this line is

$$\frac{x - x_1}{x_1} = \frac{y - y_1}{y_1},$$

*i.e.* $$y_1 x - x_1 y = 0.$$

Hence, *all the normals pass through the centre.*

**Examples.**

Find the equations of the tangents and normals to the following circles: (i) $x^2 + y^2 = 25$ at (3, 4), (ii) $x^2 + y^2 = 13$ where $x = -3$, (iii) $x^2 + y^2 = a^2$ at the point where the radius makes an angle $\alpha$ with the axis of $x$.

**4. Pole and polar.** The equation

$$xx_1 + yy_1 = r^2 \quad \text{.................................(1)}$$

represents a definite straight line whether $P \equiv (x_1, y_1)$ lies on the circle or not. What is its relation to the point when $P$ does not lie on the circle?

When $P$ lies on the circle there is just one tangent to the circle which passes through this point, and its equation is given by (1). When $P$ is outside the circle, on the other hand, two tangents pass through it. Let $U \equiv (x', y')$ be the point of contact of one of the tangents. The equation of this tangent is $$xx' + yy' = r^2,$$

but this line passes through $(x_1, y_1)$; therefore

$$x_1 x' + y_1 y' = r^2.$$

This relation also expresses the condition that the point $U$ should lie on the line

$$xx_1 + yy_1 = r^2.$$

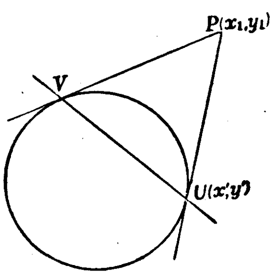


FIG. 20.

Hence this line passes through the point of contact of any tangent from $P$ to the circle. Since two tangents can be drawn from $P$ to the circle it follows that this line is the join of the two points of contact, or the *chord of contact*.

If $P$ lies within the circle so that the tangents from $P$ are imaginary, the line $UV$ is still real, though the points $U$, $V$ are now conjugate imaginary points. Hence:

*To every point* $(x_1, y_1)$ *there is related, with respect to a given circle* $x^2+y^2=r^2$, *a definite straight line* $xx_1+yy_1=r^2$. This line is called the *polar* of the point with respect to the circle, and the point is called the *pole* of the line.

**5. Harmonic property of pole and polar.** Consider any straight line through $P$, cutting the circle in $U$ and $V$. Let $Q\equiv(x, y)$ be any point on this line, and let one of the points, say $U$, divide $PQ$ in the ratio $k:1$. Then the coordinates of $U$ are

$$\frac{kx+x_1}{k+1}, \quad \frac{ky+y_1}{k+1}.$$

Substituting in the equation of the circle, we get

$$(kx+x_1)^2+(ky+y_1)^2=r^2(k+1)^2,$$

*i.e.* $$k^2(x^2+y^2-r^2)+2k(xx_1+yy_1-r^2)+(x_1^2+y_1^2-r^2)=0.$$

The roots of this quadratic in $k$, which we shall call *Joachimsthal's equation*, correspond to the two points $U$, $V$.

Now, if $Q$ lies on the polar of $P$, then $xx_1+yy_1-r^2=0$, and the roots are equal but of opposite sign, *i.e.* $U$, $V$ divide $PQ$ internally and externally in the same ratio. Hence $(PQ, UV)$ is a harmonic range. Conversely, *the locus of harmonic conjugates of $P$ with respect to the circle is a straight line, the polar of $P$.*

6. The polar of any point $P\equiv(x_1, y_1)$ with respect to a circle with centre $O$ is perpendicular to $OP$. For, taking $O$ as origin, the gradient of $OP$ is $y_1 : x_1$, and the gradient of the polar is $-x_1 : y_1$.

Further, let $OP$ cut the circle in $A$, $A'$. Then, since $(AA', NP)$ is a harmonic range, and $O$ is the mid-point of $AA'$,

$$ON \,.\, OP=OA^2=r^2.$$

These two results fix the position of the polar of a given point.

Q. What is the polar of the centre?

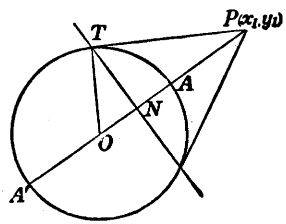


FIG. 21.

**7. To find the pole of the line $lx+my+n=0$ with respect to the circle $x^2+y^2=r^2$.** If the coordinates of the pole are $(x_1, y_1)$, the equation of the polar is

$$xx_1+yy_1=r^2.$$

Comparing this with the equation of the given straight line, we have

$$\frac{x_1}{l}=\frac{y_1}{m}=\frac{r^2}{-n}.$$

Hence

$$x_1 = -\frac{l}{n} r^2, \quad y_1 = -\frac{m}{n} r^2.$$

The homogeneous coordinates of this point may be written $(l : m : -n/r^2)$.

Q. What is the pole of a diameter?

**8. Reciprocal property of pole and polar. Conjugate points and lines.** If $Q \equiv (x_2, y_2)$ lies on the polar of $P \equiv (x_1, y_1)$, we have

$$x_1 x_2 + y_1 y_2 = r^2.$$

This is a symmetrical relation in $(x_1, y_1)$ and $(x_2, y_2)$, and hence *if Q lies on the polar of P, P lies on the polar of Q.* The points $P$, $Q$ are called *conjugate points* with respect to the circle.

It follows also that if the pole of a line $p$ lies on the line $q$, the pole of $q$ lies on $p$. The two lines $p$, $q$ are called *conjugate lines* with respect to the circle. If $l_1 x + m_1 y + n_1 = 0$ and $l_2 x + m_2 y + n_2 = 0$ are conjugate lines with respect to the circle, then, substituting the coordinates of the pole of the first line in the equation of the second, we have

$$l_1 l_2 + m_1 m_2 = n_1 n_2 / r^2.$$

If $P$ and $Q$ are conjugate points the polar of $P$ passes through $Q$; but the polar of $P$ is the locus of harmonic conjugates of $P$ with regard to the circle; hence, by the reciprocal property of a harmonic range, the two points, real or imaginary, in which $PQ$ cuts the circle are harmonic conjugates with regard to $P$ and $Q$. This is expressed by saying that *the join of two conjugate points is cut harmonically by the circle.*

Let us find what is the corresponding property of conjugate lines. Let $p$ be a given line and $P$ its pole; then any line through $P$ is conjugate to $p$. Let $q$ be a line through $P$, and $Q$ its pole. Let $p$, $q$ intersect in $O$. Then, since $O$ lies both on $p$ and on $q$, its polar passes through both $P$ and $Q$, and is therefore the line $PQ$. But if $O$ is outside the circle, the polar of $O$ is the chord of contact of tangents $OS$, $OT$ from $O$. Since $P$, $Q$ are conjugate points $(PQ, ST)$ is harmonic; therefore the pencil $O(PQ, ST)$ is harmonic, *i.e.* *$p$, $q$ are harmonic conjugates with respect to the tangents drawn from their point of intersection to the circle.*

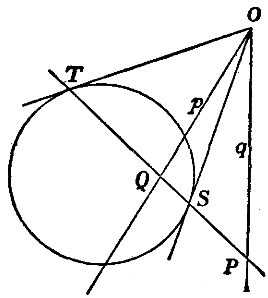


FIG. 22.

**Examples.**

1. Write down the equations of the polars of $(1, 3)$, $(2, 1)$, $(3, -1)$ with respect to the circle $x^2 + y^2 = 4$, and show that they are concurrent.

2. Find the coordinates of the pole of the line $3x - y = 1$ with regard to the circle $x^2 + y^2 = 8$.

3. $S$ is a fixed point and $p$ a variable line through $S$ cutting a circle in $U$, $V$. The tangents at $U$, $V$ intersect in $P$. Show that the locus of $P$ is a straight line, the polar of $S$.

**9. Conjugate triangles.** Let $PQR$ be any triangle, and let $P'$ be the pole of $QR$, $Q'$ the pole of $RP$, and $R'$ the pole of $PQ$, with respect to a given circle; then the polars of $P'$ and $Q'$ both pass through $R$; therefore the polar of $R$ is $P'Q'$. Similarly $P$ is the pole of $Q'R'$, and $Q$ is the pole of $R'P'$. The two triangles $PQR$ and $P'Q'R'$ are therefore related so that each vertex of the one triangle is the pole of a side of the other triangle. Such pairs of triangles are said to be *conjugate* with regard to the circle.

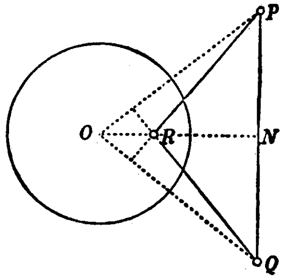


FIG. 23.

If $QR$ is the polar of $P$ and $RP$ the polar of $Q$, then $PQ$ is the polar of $R$, and the triangle $PQR$ is said to be *self-conjugate* with regard to the circle.

If $O$ is the centre of the circle, $QR \perp OP$, $RP \perp OQ$, $PQ \perp OR$, and $O$ is the orthocentre of the triangle $PQR$.

**Examples.**

1. Prove that a self-conjugate triangle has one vertex within the circle and the other two outside.

2. Show that there is just one circle with respect to which a given triangle is self-conjugate. (This unique circle is called the *polar circle* of the given triangle.)

**10.** The circle $x^2+y^2=r^2$ divides the plane into two regions. If $P \equiv (x_1, y_1)$ lies inside the circle, its distance from the centre is less than the radius, *i.e.* $x_1^2+y_1^2<r^2$, whence $x_1^2+y_1^2-r^2<0$; if $P$ lies outside the circle $x_1^2+y_1^2-r^2>0$. The two regions, the interior and the exterior of the circle, are characterized by the sign of $x^2+y^2-r^2$. In passing from one region to the other this expression vanishes in crossing the circle, and changes sign. (Cf. Chap. II. § 7.)

**11. Geometrical meaning of the expression $x^2+y^2-r^2$.** Suppose $P \equiv (x_1, y_1)$ is outside the circle. Draw the tangent $PT$. Then

$$PT^2 = PC^2 - CT^2 = x_1^2+y_1^2-r^2.$$

Hence $x_1^2+y_1^2-r^2$, if it is positive, is equal to the square of the length of the tangent from $P$ to the circle.

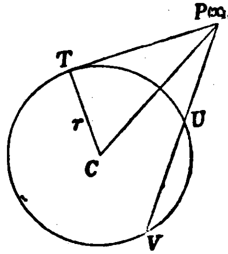


FIG. 24.

Draw any line through $P$ cutting the circle in $U$, $V$. The freedom-equations of this line are

$$x = x_1 + \rho \cos \psi,$$
$$y = y_1 + \rho \sin \psi,$$

where $\tan \psi$ is the gradient and $\rho$ is the distance of $(x, y)$ from $P$. Substituting these values for $x$, $y$ in the equation of the circle, we have

$$(x_1+\rho \cos \psi)^2 + (y_1 + \rho \sin \psi)^2 = r^2,$$

*i.e.*
$$\rho^2 + 2\rho(x_1 \cos \psi + y_1 \sin \psi) + (x_1^2+y_1^2-r^2) = 0.$$

Let the roots of this equation be $\rho_1$, $\rho_2$; then $\rho_1\rho_2 = x_1^2 + y_1^2 - r^2$. But $\rho_1 = PU$ and $\rho_2 = PV$; hence

$$PU \cdot PV = x_1^2 + y_1^2 - r^2.$$

This product, which depends only upon the position of the point $P$, is called the *power* of $P$ with respect to the circle. It is positive and equal to $PT^2$ when $P$ is outside the circle, negative when $P$ is inside the circle, and zero when $P$ lies on the circle.

**12. Equation of the tangents to a given circle from a given point.** Let the equation of the circle be $x^2+y^2=r^2$, and the coordinates of the point $P \equiv (x', y')$. Draw any line through $P$ cutting the circle in $U$, $V$, and let $Q \equiv (x, y)$ be any point on this line. Then, if $k$ is the position-ratio of one of the points $U$, $V$ with respect to $P$, $Q$, Joachimsthal's equation gives

$$k^2(x^2+y^2-r^2)+2k(xx'+yy'-r^2)+(x'^2+y'^2-r^2)=0.$$

If $PQ$ is a tangent, $U$ and $V$ coincide, and this equation has equal roots; hence

$$(xx'+yy'-r^2)^2=(x^2+y^2-r^2)(x'^2+y'^2-r^2).$$

This is a relation connecting the coordinates $(x, y)$ of any point on a tangent through $(x', y')$, and therefore expresses by one equation the two tangents from $(x', y')$.

**13. Condition that a straight line should touch a given circle.** The points of intersection of the straight line $lx+my+n=0$, with the circle $x^2+y^2=r^2$, are found by solving these two equations simultaneously. Eliminating $x$ we have, provided $l \neq 0$,

$$l^2(r^2-y^2)=(my+n)^2,$$

*i.e.*

$$(l^2+m^2)y^2+2mny+(n^2-l^2r^2)=0.$$

This quadratic equation gives two values for $y$, corresponding to the two points of intersection. The line will be a tangent if the equation has equal roots, *i.e.* if

$$m^2n^2=(l^2+m^2)(n^2-l^2r^2),$$

*i.e.*

$$0=l^2n^2-l^2m^2r^2-l^4r^2,$$

or, since $l \neq 0$,

$$(l^2+m^2)r^2=n^2.$$

This equation is called the *tangential equation* of the circle.

**Examples.**

1. Find the length of the tangents drawn from the point (3, 4) to the circle $x^2+y^2=16$.

2. Find the equations of the tangents to the circle $x^2+y^2=3$ inclined to the $x$-axis at (i) 60°, (ii) 45°.

3. Find the points of intersection of the circle $x^2+y^2=25$ and the straight line $x-7y+25=0$.

4. Show that the equation $(x-7y+25)^2=25(x^2+y^2-25)$ represents the tangents at the points of intersection of the circle $x^2+y^2=25$ with the line $x-7y+25=0$.

**14. To find the equation of the circle whose centre is $(\alpha, \beta)$ and radius r.** If $(x, y)$ is any point on the circle, the distance of $(x, y)$ from $(\alpha, \beta)$ is $r$. Hence

$$(x-\alpha)^2+(y-\beta)^2=r^2.$$

Written out in full, the general equation of the circle is

$$x^2+y^2-2\alpha x-2\beta y+\alpha^2+\beta^2-r^2=0,$$

an equation of the second degree. Hence the circle is a curve of the second degree; it is cut by every straight line in two points, real and distinct, coincident, or imaginary.

This equation is not the most general equation of the second degree, but involves certain conditions. We may make it a little more general by multiplying all through by any constant. Then we see that the following are *necessary* conditions that it should represent a circle: (1) there is no term in $xy$, (2) the coefficients of $x^2$ and $y^2$ are equal.

We shall next show that these conditions are *sufficient*, *i.e.* if these conditions are satisfied the equation of the second degree will represent a circle. The general equation of the second degree is

$$ax^2+by^2+2hxy+2gx+2fy+c=0.$$

With the two conditions (1) $h=0$, (2) $a=b$, the equation reduces to

$$ax^2+ay^2+2gx+2fy+c=0,$$

which may be written, if $a\neq 0$,

$$\left(x+\frac{g}{a}\right)^2+\left(y+\frac{f}{a}\right)^2=\frac{g^2+f^2-ac}{a^2}.$$

The left-hand side represents the square of the distance from $(x, y)$ to $(-g/a, -f/a)$, and therefore the equation represents a circle with centre $(-g/a, -f/a)$ and radius $\sqrt{g^2+f^2-ac}/a$.

If the coefficients $a, g, f, c$ are all real numbers, the centre is always a real point, but the radius is real only if $g^2+f^2>ac$.

If $g^2+f^2<ac$ the radius is purely imaginary, and no real values of $x$ and $y$ can satisfy the equation, for $(x+g/a)^2+(y+f/a)^2+(ac-g^2-f^2)/a^2$ is always positive. The locus consists therefore entirely of imaginary points. We shall call this a *virtual* circle.

In the intermediate case where $g^2+f^2=ac$, the radius is zero. The equation reduces to

$$(x+g/a)^2+(y+f/a)^2=0.$$

The only real values of $x$ and $y$ which satisfy this equation are $x=-g/a$, $y=-f/a$, and these are the coordinates of the centre. In this case the centre lies on the curve, and is the only real point on the locus. Every point, real or imaginary, which lies on the locus is at a zero distance from the centre, and the curve is called a *point-circle*.

Q. What becomes of the circle when $a=0$ ?

**Examples.**

1. Find the centre and radius of the following circles:

(i) $x^2+y^2-6x+2y-39=0$, (ii) $x^2+y^2-x-y=0$,
(iii) $4x^2+4y^2-4x-5y+1=0$, (iv) $3x^2=y(7-3y)$,
(v) $x^2+y^2=2ax$, (vi) $7x^2+3y^2-4y=(1-2x)^2$.

2. Write down the equations of the following circles: (i) centre $(-3, \frac{3}{2})$, radius $\frac{7}{2}$; (ii) centre $(-\frac{1}{2}, 1)$, radius $\frac{3}{2}$; (iii) through $(1, 0), (2, 3)$ and centre on $x+y=1$; (iv) through the origin, and centre at $(1, 2)$; (v) through $(0, 0)$, $(a, 0)$, $(0, b)$.

**15. Angle of intersection of two circles.** If two circles cut at a point $A$, the angle of intersection at that point is the angle between the tangents to the two circles at that point, and this is also equal to the angle between the radii. The circles cut at a second point $B$, and from elementary geometry the angle of intersection at $B$ is equal to the angle of intersection at $A$. We may therefore speak simply of the angle of intersection of the two circles. If the circles touch, the angle of intersection is zero. If the angle of intersection is a right angle, the two circles are said to cut orthogonally.

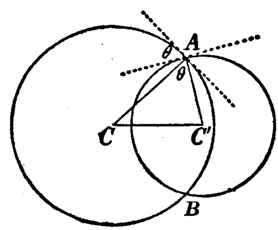


FIG. 25.

**16. Condition that two circles should cut orthogonally.** If the angle $CAC'$ is right, $CC'^2=CA^2+C'A^2=r^2+r'^2$. But $CC'$ is the distance between the points $(\alpha, \beta)$, $(\alpha', \beta')$; hence

$$(\alpha-\alpha')^2+(\beta-\beta')^2=(\alpha^2+\beta^2-c)+(\alpha'^2+\beta'^2-c'),$$

which reduces to $$2\alpha\alpha'+2\beta\beta'=c+c'.$$

**Examples.**

1. Prove that the equation of the polar of $(x_1, y_1)$ with respect to the circle $x^2+y^2+2gx+2fy+c=0$ is $xx_1+yy_1+g(x+x_1)+f(y+y_1)+c=0$; and, in the case in which $(x_1, y_1)$ lies on the circle, that this is the equation of the tangent.

2. Find the equations of the tangents and normals to the following circles: (i) $x^2+y^2-2x+4y=20$ at $(-2, 2)$, (ii) $2x^2+2y^2-4x+9y=0$ at the origin.

3. Show that the circles $x^2+y^2=2$ and $(x-3)^2+(y-3)^2=32$ touch at the point $(-1, -1)$, and find the equation of the tangent at the point of contact.

4. When the equation of a circle is given in the *canonical form*

$$S \equiv x^2+y^2+2gx+2fy+c=0,$$

prove that $S$ is equal to the power of the point $(x, y)$.

5. Find the power of the points $(1, 2)$ and $(3, -1)$ with respect to each of the circles (i) $x^2+y^2-6x-6y=0$, (ii) $x^2+y^2+2x+4y=10$, and state how the points are situated with regard to the circles.

6. Find how the points $(-5, 2)$, $(1, 7)$, $(7, 5)$, $(3, -2)$ lie with respect to the two circles $S \equiv x^2+y^2+8x-10y-8=0$, $S' \equiv x^2+y^2-8x-14y+35=0$; and deduce that the circles partially overlap.

7. Find the angle of intersection of the circles: (i) $x^2+y^2-4x+6y=12$ and $x^2+y^2+2x-2y=23$, (ii) $x^2+y^2=4$ and $x^2+y^2=2x+2y$.

8. Find the equations of the tangents from the origin to the circles
(i) $x^2+y^2-6x-10y+25=0$, (ii) $x^2+y^2+2x-14y+18=0$.

9. Find the equations of the circles touching the three lines
(i) $x=0$, $y=0$, $x=a$; (ii) $x=2$, $y=5$, $3x-4y=10$.

## EXAMPLES III. A.

1. Find the equations of the two circles touching the axis of $x$ at the origin and also touching the line $3y=4x+24$.

2. Prove that the following straight lines and circles touch, and find the coordinates of the point of contact:

(i) $3y+2x=0$ and $x^2+y^2+3y+2x=0$,
(ii) $3x-4y=10$ and $x^2+y^2+2x-6y=15$,
(iii) $5x-12y=45$ and $x^2+y^2+16x-14y=56$.

3. Show that the circles $x^2+y^2-4x-2y-4=0$ and $x^2+y^2-12x-8y+48=0$ touch at the point $(4\frac{2}{5}, 2\frac{4}{5})$.

4. At the points of intersection of the straight line $2x-y=3$ with the circle $x^2+y^2=4$ tangents are drawn cutting in $P$. Find the coordinates of $P$.

5. Show that the circles $x^2+y^2-4x+6y+8=0$ and $x^2+y^2-10x-6y+14=0$ touch at $(3, -1)$.

6. Show that the point $(1, 3)$ has the same polar with respect to the two circles $x^2+y^2-10x+10y+10=0$ and $x^2+y^2-8x+6y+10=0$, and deduce the existence of another point which has the same polar with respect to the two circles.

7. Show that the equation $(y-x+3)^2+2(x-2)(y+2)=0$ represents a circle Show that $x=2$ and $y+2=0$ are tangents, and find what is represented by the equation $x-y=3$.

8. A circle of radius 3 touches the circle $x^2+y^2-2x-2y-2=0$ and also the $x$-axis. Find its equation, and show that there are four such circles.

9. Find the equations of the two circles which touch both of the coordinate axes and pass through the point $(6, 3)$.

10. Prove that the equation of the circle whose diameter is the line joining the points $(a, b)$, $(a', b')$ is $(x-a)(x-a')+(y-b)(y-b')=0$.

11. In the last example, if $(a, b)$ is a fixed point, find the locus of $(a', b')$ if the circle always passes through the origin. Explain geometrically.

12. Two circles touch the axis of $y$ and intersect in the points $(1, 0)$, $(2, -1)$. Find their radii, and show that they will both touch the line $y+2=0$.

(Math. Tripos I., 1912.)

13. Find the locus of the point which moves so that the length of the tangent from it to the circle $x^2+y^2+2x=0$ is three times the length of the tangent to the circle $x^2+y^2=4$.

14. Find the equations of the tangents to the circle $x^2+y^2=25$ which pass through $(-1, 7)$, and show that they are at right angles.

15. Show that the tangents from the origin to the circle

$$x^2+y^2-14x+2y+25=0$$

are at right angles.

16. Prove that the locus of a point which moves so that the sum of the squares of its distances from a number of fixed points is constant is a circle whose centre is the centroid of the fixed points.

17. A point moves so that the sum of the squares of its distances from the sides of a square is constant; prove that its locus is a circle.

18. Find the locus of a point such that its polars with respect to two given circles make a given angle with one another.

19. Show that there are two points on the axis of $x$ from which the tangents to the circle $x^2+y^2-10x-8y+31=0$ are at right angles. Find their coordinates and the equations of the tangents.

## EXAMPLES III. B.

1. Find the condition that the line $x\cos\alpha+y\sin\alpha=p$ should touch the circle $x^2+y^2=2ax$. Hence prove that the locus of the foot of the perpendicular from the origin upon the tangent to the circle has for its equation in polar coordinates $r=a(1+\cos\theta)$.

2. The line $x\cos\alpha+y\sin\alpha=p$ cuts the circle $x^2+y^2=a^2$ in the points $F$, $G$. Prove that the equation of the circle described on $FG$ as diameter is

$$x^2+y^2-a^2=2p(x\cos\alpha+y\sin\alpha-p).$$

3. A circle is described on a chord of a given circle as diameter so as to cut another given circle orthogonally. Prove that the locus of the centre of the variable circle is a circle. (Queens', 1911.)

4. If a circle cuts the two circles

$$S\equiv(x-\alpha)^2+(y-\beta)^2-r^2=0,\quad S'\equiv(x-\alpha')^2+(y-\beta')^2-r'^2=0$$

at angles $\theta$, $\theta'$, prove that it will cut the circle $Sr'\cos\theta'-S'r\cos\theta=0$ orthogonally. (Queens', 1910.)

5. Write down the general equation of a circle cutting $x^2+y^2=c^2$ orthogonally, and show that if it passes through the point $(a, b)$ it will also pass through the point $c^2a/(a^2+b^2)$, $c^2b/(a^2+b^2)$.

6. Show that the ratio of the distances of any two points from the polar of the other with respect to a given circle is equal to the ratio of their distances from the centre of the circle.

7. If $S\equiv(x-\alpha)^2+(y-\beta)^2-r^2=0$ and $S'\equiv(x-\alpha')^2+(y-\beta')^2-r'^2=0$ are any two circles, prove that the two circles $S/r\pm S'/r'=0$, cut orthogonally.

8. Find the locus of points from which it is possible to draw two tangents, one to each of two fixed circles, which will be at right angles, and prove that the bisectors of the angles between the tangents always touch one or other of two fixed circles.

9. If $P$ and $Q$ are a pair of conjugate points with respect to a circle, prove that the sum of their powers with respect to the circle $=PQ^2$.

10. Three circles cut each other orthogonally. Prove that the centres of any two are conjugate with regard to the third circle.

11. A variable chord $PQ$ of a given circle subtends a right angle at a given point $A$. Find the locus of the pole of $PQ$ with regard to the circle. Interpret the case when $A$ lies on the given circle.

12. Find the locus of mid-points of chords of a given circle which pass through a fixed point.

13. Show that the equation of the chord of the circle $x^2+y^2-2\alpha x-2\beta y+c=0$ which is bisected at $(x', y')$ is

$$xx'+yy'-\alpha(x+x')-\beta(y+y')+c=x'^2+y'^2-2\alpha x'-2\beta y'+c.$$

14. Show that a point-circle cuts itself orthogonally; and conversely, if a circle is orthogonal to itself it must be a point-circle.

15. Show that two virtual circles cannot satisfy the condition for orthogonality.

16. If four circles cut in pairs orthogonally, prove that one and only one of them is virtual.

# CHAPTER IV.

## THE ELLIPSE.

**1.** The conics or conic sections were so named from the way in which they were first studied, as sections of a cone.

Consider a right circular cone with vertical angle $2\alpha$. It is generated by a line $OP$ which rotates about a fixed axis $OC$, the angle $POC$ being always equal to $\alpha$. The point $O$ is called the *vertex*. As the generating line $OP$ is unlimited in length, the conical surface extends to infinity on both sides of the vertex and consists of two *sheets* joined at the vertex.

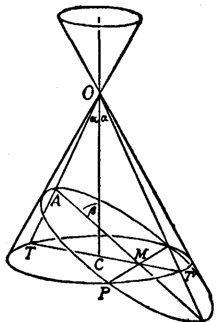


FIG. 25a.

Every section perpendicular to the axis (transverse section) is a circle, and every section through the vertex consists of two straight lines, real, coincident, or imaginary.

Consider an oblique plane making with the axis an angle $\beta$. Let the plane through the axis perpendicular to the plane of the section cut this plane in $AA'$. Take any point $P$ on the curve of section and draw $PM \perp AA'$. $PM$ is then perpendicular to the plane $OAA'$, and the transverse plane through $P$ contains $PM$ and cuts the plane $OAA'$ in $TT'$, which is a diameter of the transverse section. Now, the transverse section being a circle,

$$PM^2 = TM \,.\, MT'.$$

Also
$$\frac{TM}{AM} = \frac{\sin(\beta+\alpha)}{\cos\alpha}, \quad \frac{MT'}{MA'} = \frac{\sin(\beta-\alpha)}{\cos\alpha}.$$

Hence
$$PM^2 = k \,.\, AM \,.\, MA',$$

where $k$ is a constant $= \sin(\beta+\alpha)\sin(\beta-\alpha)/\cos^2\alpha$.

When $\beta > \alpha$, the plane cuts only one sheet of the cone (as in Fig. 25*a*) the section is a closed curve called the *ellipse*, and $k$ is positive. The plane through $O$ parallel to the cutting plane meets the cone only in $O$.

When $\beta<\alpha$, the plane cuts both sheets (Fig. 25*b*); the section is a curve, with two open branches, called the *hyperbola*, and $k$ is negative. The plane through $O$ parallel to the cutting plane cuts the cone in two generating lines

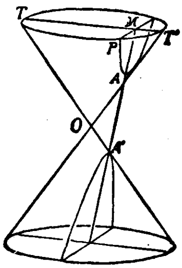



FIG. 25*b*.

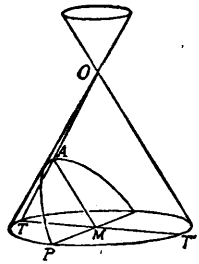



FIG. 25*c*.

When $\beta=\alpha$, the plane is parallel to a tangent-plane of the cone; the section is an open curve called the *parabola* (Fig. 25*c*). In this case $MT'$ is constant $=2OA \sin\alpha$; also $TM=2AM\sin\alpha$, hence

$$PM^2=4OA\sin^2\alpha \,.\, AM,$$

*i.e.* $PM^2 : AM$ is constant.

**1a. Focal properties** (Dandelin's construction).

(1) When the plane of section is not parallel to a tangent-plane, there are two spheres inscribed in the cone and touching the plane of section in points $F$, $F'$. These spheres have *ring-contact* with the cone, each touching the cone along a circle. Let $P$ be any point on the curve of section and let the generating line through $P$ touch the two spheres in $E$, $E'$. Then since $PF$ and $PE$ are both tangents to the same sphere,

$$PF=PE;$$

similarly $PF'=PE'$.

Hence if the section is an ellipse, so that $E$, $E'$ lie on the same side of $O$,

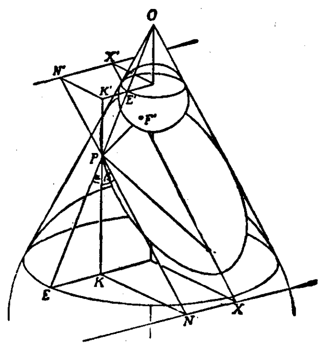



FIG. 25*d*.

$$PF+PF'=PE+PE'=\text{constant}.$$

If the section is a hyperbola, it may be shown similarly that the difference

$$PF\sim PF'=\text{const}.$$

$F$, $F'$ are called *foci*. *For the ellipse the sum, and for the hyperbola the difference, of the focal distances is constant.*

(2) Let the plane of section cut the plane of contact of one of the spheres in $NX$, and draw $PN \perp NX$, and $PK \perp$ the plane of contact. Then

$$PK = PN \cos\beta,$$

also $$PK = PE \cos\alpha = PF \cos\alpha.$$

Hence $$\frac{PF}{PN} = \frac{\cos\beta}{\cos\alpha} = \text{const.}$$

The fixed line $NX$ is called the *directrix* corresponding to the focus $F$. Hence we have the "focus-directrix property" of a conic: *the distance of any point on a conic from a focus bears a constant ratio to its distance from the corresponding directrix.* For an ellipse the ratio is less than unity, for a hyperbola it is greater, and for a parabola it is equal to unity.

There is one property which every section of a cone possesses, viz. it is cut by an arbitrary line in its plane in two points, real, coincident, or imaginary. This property is expressed by saying that a conic is a *curve of the second order*, and a consequence of this is that the equation of a conic in cartesian coordinates is of the second degree. The converse of this will be dealt with when we consider the general equation of the second degree.

**1b.** To define the conics as plane curves, without having recourse to three dimensions, we shall use the property proved in § 1,

$$PM^2 = kAM \,.\, MA',$$

which is a modification of a well-known property of the circle.

**2. To find the equation of the ellipse in its simplest form.** Take $A'A$ as axis of $x$ and the mid-point of $A'A$ as origin. Let $A'O = OA = a$. Then $MP = y$, $OM = x$, $A'M = a + x$, $MA = a - x$.

Hence from the defining property

$$PM^2 = kA'M \,.\, MA,$$

we have $y^2 = k(a+x)(a-x) = k(a^2 - x^2)$.

This equation, which is of the second degree, contains only squares of $x$ and $y$; hence the curve is symmetrical about both axes.

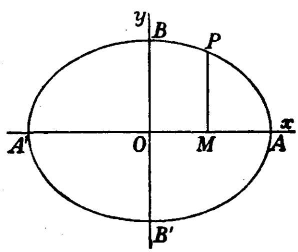


FIG. 26.

Putting $x = 0$, we find where it cuts the axis of $y$, viz. where $y^2 = ka^2$. Since, for the ellipse, $k$ is positive, this gives real values of $y$. Let $ka^2 = b^2$. Then the equation takes the form

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

and the curve cuts the axis of $y$ in two points $B$, $B'$, such that $B'O = OB = b$.

*Cor.* Writing the equation in the forms

$$\frac{x}{a}=\sqrt{1-\frac{y^2}{b^2}}, \quad \frac{y}{b}=\sqrt{1-\frac{x^2}{a^2}},$$

we see that there are no real values of $x$ for values of $y$ greater than $b$ and no real values of $y$ for values of $x$ greater than $a$. Hence the ellipse lies everywhere in the finite part of the plane, in fact within a rectangle the equations of whose sides are $x=\pm a$, $y=\pm b$.

If $a>b$, $A'A$ is called the *major axis* and $B'B$ the *minor axis.* The points $A$, $A'$, $B$, $B'$ are called *vertices.*

If $a=b$, the curve is a circle.

3. From the close resemblance of the equation of an ellipse to that of a circle the curve possesses many properties similar to those of a circle.

Thus, by exactly the same method as in the case of the circle, we find that the equation of the *tangent* at $(x', y')$ is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

When $(x', y')$ lies outside the curve, this equation is the equation of the chord of contact of tangents from the point to the curve, and in all cases represents the *polar* of $(x', y')$ with regard to the ellipse. The polar of $P$ is the locus of harmonic conjugates of $P$ with respect to the curve. The same reciprocal properties of pole and polar hold as in the case of the circle. If the polar of $P$ passes through $Q$, the polar of $Q$ passes through $P$; $P$ and $Q$ are conjugate points, their polars are conjugate lines. A pair of conjugate points are harmonically separated by the points in which their join cuts the curve; a pair of conjugate lines are harmonically separated by the tangents from their point of intersection to the curve. The relation between the coordinates of two conjugate points $(x_1, y_1)$ and $(x_2, y_2)$ is

$$\frac{x_1x_2}{a^2}+\frac{y_1y_2}{b^2}=1.$$

The relation between two conjugate lines

$$l_1x+m_1y+n_1=0 \quad \text{and} \quad l_2x+m_2y+n_2=0,$$

is

$$a^2l_1l_2+b^2m_1m_2=n_1n_2.$$

4. If $(x, y)$ is a point on the curve, so is the point $(-x, -y)$, and the join of these points is bisected at $O$. Thus every chord of the curve which passes through $O$ is bisected there. $O$ is therefore called the *centre* of the ellipse. A chord through the centre is called a *diameter.*

The polar of the centre $(0, 0)$ is the line at infinity, and the polar of a point at infinity passes through $O$ and is a diameter.

5. For all points on the ellipse, $x^2/a^2+y^2/b^2-1=0$. We may show further that, according as $(x, y)$ is inside or outside the ellipse, the expression $x^2/a^2+y^2/b^2-1$ is $<$ or $>0$. Join $OP$ cutting the ellipse in $Q$, and

let $OP = k \cdot OQ$. Then the coordinates of $Q$ are $(x/k, y/k)$. But $Q$ lies on the ellipse; therefore $x^2/a^2 + y^2/b^2 = k^2$. Now $P$ is outside or inside the ellipse according as $k >$ or $< 1$, *i.e.* according as $x^2/a^2 + y^2/b^2 >$ or $< 1$.

**6.** The circle whose diameter is the major axis $A'A$ is called the major auxiliary circle or simply the *auxiliary circle*. If $MP$ produced cuts the auxiliary circle in $Q$, $P$ and $Q$ are called corresponding points. Let

$$P \equiv (x', y') \quad \text{and} \quad Q \equiv (x', y_1);$$

then
$$\frac{x'^2}{a^2} + \frac{y'^2}{b^2} = 1 \quad \text{and} \quad x'^2 + y_1^2 = a^2;$$

therefore $y_1 = \frac{a}{b} y'$, *i.e.* $QM : PM = a : b$.

Hence *the ordinate of every point on the ellipse bears a constant ratio to the ordinate of the corresponding point on the auxiliary circle.*

The tangents at $P$ and $Q$ are respectively

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1,$$

and
$$xx' + yy_1 = a^2, \quad i.e. \ \frac{xx'}{a^2} + \frac{yy'}{ab} = 1.$$

These meet the major axis at the same point $(a^2/x', 0)$.

Join $OQ$ and draw through $P$ a line $\parallel QO$ cutting the axes in $H$, $K$. Then

$$PK = OQ = a,$$

and
$$PH : QO = PM : QM = b : a;$$

therefore
$$PH = b \quad \text{and} \quad HK = a - b.$$

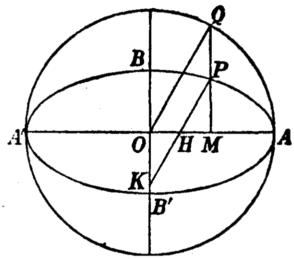


FIG. 27.

This result affords a means of describing the ellipse mechanically. If two pegs $H$, $K$ are fixed on a rod at a distance apart equal to $c$, and slide in grooves fixed at right angles, a point $P$ on the rod which divides $KH$ externally in the ratio $l : m$ will describe an ellipse whose axes lie along the grooves; the lengths of the semi-axes being

$$KP = cl/(l - m) \quad \text{and} \quad HP = cm/(l - m).$$

This is the principle of the carpenter's trammel, or elliptic compasses.

It may be proved also directly that if $P$ divides $KH$, either internally or externally, into two fixed parts $KP = a$ and $PH = b$, the locus of $P$ is an ellipse with semi-axes $a$ and $b$. More generally, if a parallelogram with diagonals $2c$, $2c'$ moves so that two opposite vertices slide along fixed rectangular axes, the other two vertices describe ellipses with axes $c \pm c'$.

If the rod is kept fixed while the grooves are moved, a fixed point on the rod will trace an ellipse on the moving surface. This arrangement enables an elliptic cylinder to be turned on a lathe. The fixed point is the cutting tool, and the wood to be turned is fixed to the surface which carries the grooves. The whole arrangement is called an "oval chuck."

**Examples.**

1. Prove directly that the equation of the tangent to the ellipse $x^2/a^2+y^2/b^2=1$ at the point $(x', y')$ is $xx'/a^2+yy'/b^2=1$.

2. Prove that the locus of harmonic conjugates of the point $(x', y')$ with regard to the ellipse $x^2/a^2+y^2/b^2=1$ is $xx'/a^2+yy'/b^2=1$.

3. Show that the tangents at the ends of a diameter are parallel.

4. Find whether the points (0·9, 0·9), (1·3, 0·4) lie inside or outside the ellipse $2x^2+3y^2=4$.

5. Find within what values of $t$ the points on the line $x=1+t$, $y=2t$ will lie within the ellipse $x^2/9+y^2/4=1$.

6. Show that $4x+3y=11$ is a tangent to the ellipse $2x^2+3y^2=11$, and find the coordinates of the point of contact.

7. If a circle is drawn on the minor axis of an ellipse as diameter (the *minor auxiliary circle*), and the perpendicular $PN$ on the minor axis from any point $P$ on the ellipse cuts this circle in $Q$, prove that $NP:NQ=a:b$. ($Q$ is called the *corresponding point* on the minor auxiliary circle.)

8. If $Q$ and $Q'$ are the points which correspond to $P$ on the major and minor auxiliary circles respectively, show that $QQ'$ passes through the centre.

9. Two circles have a common centre $O$. A variable radius cuts them in $Q$ and $Q'$. Through $Q$ and $Q'$ are drawn lines parallel respectively to two fixed rectangular diameters. Show that the locus of their intersection is an ellipse.

**7. Condition that the line $lx+my+n=0$ should be a tangent to the ellipse $x^2/a^2+y^2/b^2=1$.** We find the points of intersection of the straight line with the ellipse by solving the two equations simultaneously. Eliminating $y$, we have $\quad b^2m^2x^2+a^2(lx+n)^2-a^2b^2m^2=0,$

giving a quadratic for $x$:

$$(a^2l^2+b^2m^2)x^2+2a^2nlx+a^2(n^2-b^2m^2)=0.$$

Hence we get two values for $x$, and corresponding to each of these we get one value for $y$ from the equation of the straight line. These are the coordinates of the two points of intersection of the straight line with the ellipse. If the line is a tangent, the two points of intersection will coincide, and the quadratic equation must have equal roots. The condition for this is

$$a^4n^2l^2=a^2(a^2l^2+b^2m^2)(n^2-b^2m^2),$$

which reduces to

$$a^2l^2+b^2m^2=n^2.$$

This is therefore the *tangential equation* of the ellipse.

**8.** The equation of the two tangents from $(x', y')$ to the ellipse is

$$\left(\frac{xx'}{a^2}+\frac{yy'}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right).$$

The two tangents from $(x', y')$ to the ellipse are at right angles if the sum of the coefficients of $x^2$ and $y^2$ in this equation is zero, *i.e.* if

$$\frac{1}{a^2}\left(\frac{y'^2}{b^2}-1\right)+\frac{1}{b^2}\left(\frac{x'^2}{a^2}-1\right)=0, \quad \textit{i.e.} \text{ if } x'^2+y'^2=a^2+b^2.$$

Hence the locus of points, the tangents from which are at right angles, is a circle concentric with the ellipse. This circle is called the *orthoptic circle*.

This circle is very often called the director circle, since in the case of a parabola it reduces to the directrix, but the name is much more applicable to the circles which are noted in § 24. The first use of the term director circle, or simply director, appears to be found in a work by THOMAS GASKIN, *The geometrical construction of a conic section subject to five conditions* (Cambridge, 1852), but the property of this circle was proved by DE LA HIRE (*Sectiones conicae*, Paris, 1685), and the analogy with the directrix of a parabola was pointed out by R. J. BOSCOVICH (*Sectionum conicarum elementa*, Venice, 1757). Its properties, and those of an analogous sphere in solid geometry, were developed by MONGE, and in France the circle is often called the circle of Monge. The term orthoptic circle was used by H. PICQUET, *Étude géométrique des systèmes ponctuels et tangentiels de sections coniques* (Paris, 1872). CHARLES TAYLOR suggested the term *orthocycle*; see his *Ancient and modern geometry of conics* (Cambridge, 1881, p. 280), which contains many valuable historical references.

## 9. Conjugate diameters.

*Def.* Two diameters which are also conjugate lines with regard to the ellipse are called *conjugate diameters*.

*Each of two conjugate diameters bisects all chords parallel to the other.*

Let $PP'$ and $DD'$ be two conjugate diameters, and let $ST$, any chord parallel to $DD'$, cut $PP'$ in $M$. Since $DD'$ is conjugate to $PP'$ it passes through $Q$, the pole of $PP'$, which is the point at infinity on $DD'$. Since $ST$ is parallel to $DD'$ it also passes through $Q$. $(ST, MQ)$ is then a harmonic range, and therefore $M$ is the mid-point of $ST$.

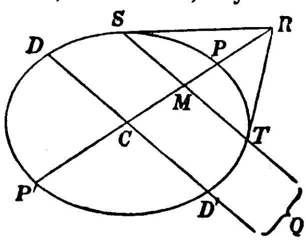


FIG. 28.

When the chord $ST$ becomes a tangent, its mid-point $M$ coincides with the point of contact, and lies on $PP'$; hence *the tangents at the ends of a diameter are parallel to the conjugate diameter.*

*To find the relation between the gradients of two conjugate diameters.*

Let $y=\mu_1 x$ and $y=\mu_2 x$ be the equations of two conjugate diameters. Let $P\equiv(x_1, y_1)$ be a point on the first. Its polar with regard to the ellipse $x^2/a^2+y^2/b^2=1$ is $xx_1/a^2+yy_1/b^2=1$. Since $y_1=\mu_1 x_1$, we can write this

$$\frac{x}{a^2}+\mu_1\frac{y}{b^2}=\frac{1}{x_1}.$$

For all values of $x_1$ this represents a line parallel to the diameter $b^2x+\mu_1 a^2 y=0$, and as $x_1\to\infty$, it tends to coincidence with this diameter. Identifying this equation with the equation $y=\mu_2 x$, we have

$$\mu_2=-b^2/\mu_1 a^2, \quad i.e. \quad \mu_1\mu_2=-b^2/a^2.$$

**Examples.**

1. Show that the line $y=\mu x+\sqrt{a^2\mu^2+b^2}$ will touch the ellipse $x^2/a^2+y^2/b^2=1$ for all values of $\mu$.

2. Find the equations of the tangents to the ellipse $x^2/16+y^2=1$ which make with the axis of $x$ an angle of 60°.

3. Show that the equation of the tangents to the ellipse $x^2/a^2+y^2/b^2=1$ at its points of intersection with the line $lx+my+n=0$ is
$$(lx+my+n)^2=(x^2/a^2+y^2/b^2-1)(a^2l^2+b^2m^2-n^2).$$

4. An ellipse slides between two straight lines which are at right angles; show that the locus of its centre is an arc of a circle.

5. Find the coordinates of the mid-points of the following chords of the ellipse $x^2/9+y^2/4=25$:
(i) $14x-3y+150=0$, (ii) $2x+3y+6=0$, (iii) $2x-y+10=0$.

6. Find the coordinates of the mid-point of the chord $x+2y=1$ of the ellipse $4x^2+5y^2=20$.

7. Find the equation of the chord of the ellipse $2x^2+3y^2=12$ which has its mid-point at (1, 1).

8. Find the locus of mid-points of chords of the ellipse $3x^2+4y^2=8$ which pass through the point (2, 3).

9. Prove that the equation of the chord of the ellipse $x^2/a^2+y^2/b^2=1$ which is bisected at $(\alpha, \beta)$ is $\alpha(x-\alpha)/a^2+\beta(y-\beta)/b^2=0$.

10. Show that the locus of the mid-points of chords of the ellipse
$$x^2/a^2+y^2/b^2=1$$
which pass through $(\alpha, \beta)$ is $x(x-\alpha)/a^2+y(y-\beta)/b^2=0$.

11. Prove that conjugate diameters of a circle are at right angles.

**10. Eccentric angle.** The freedom-equations of the auxiliary circle are
$$x=a\cos\varphi,\quad y=a\sin\varphi,$$
where $\varphi$ is the angle $ACQ$. If $P$ is the corresponding point on the ellipse, its coordinates are (see § 6),
$$x=a\cos\varphi,\quad y=b\sin\varphi.$$
These are therefore freedom-equations of the ellipse. The angle $\varphi$ is called the *eccentric angle* of the point $P$. We may conveniently speak of *the point* $\varphi$ on the ellipse.

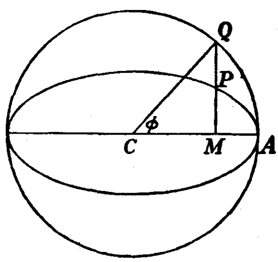


FIG. 29.

Q. Could the eccentric angle be defined by means of the circle on $BB'$ as diameter, the minor auxiliary circle?

The eccentric angles of the extremities of any diameter differ by $\pi$, or more generally by an odd multiple of $\pi$. For if $(x_1, y_1)$ and $(x_2, y_2)$ are the two ends of a diameter,
$$x_1=a\cos\varphi,\qquad x_2=-x_1=a\cos(\varphi+\pi),$$
$$y_1=b\sin\varphi,\qquad y_2=-y_1=b\sin(\varphi+\pi).$$

The eccentric angles of the extremities of two conjugate diameters differ by $\frac{\pi}{2}$, or an odd multiple of $\frac{\pi}{2}$. For if $\varphi, \varphi'$ are the eccentric angles, we have from the relation $xx'/a^2+yy'/b^2=0$,
$$\cos\varphi\cos\varphi'+\sin\varphi\sin\varphi'=0,$$
*i.e.* $\cos(\varphi-\varphi')=0$; therefore $\varphi-\varphi'=(2n+1)\frac{\pi}{2}$.

The diameters of the auxiliary circle which correspond to conjugate diameters of the ellipse are therefore at right angles, and since conjugate diameters of a circle are at right angles we have the result that to conjugate diameters of an ellipse correspond conjugate diameters of the auxiliary circle.

**11. Angle between conjugate diameters.** The gradient of the diameter through the point $\varphi$ is $\mu = \frac{b}{a}\tan\varphi$, and the gradient of the conjugate diameter is $\mu' = -\frac{b}{a}\cot\varphi$. Let $\theta$ be the angle between the two diameters. Then

$$\tan\theta = \frac{\mu' - \mu}{1 + \mu\mu'} = \frac{-\frac{b}{a}(\tan\varphi + \cot\varphi)}{1 - \frac{b^2}{a^2}} = \frac{-2ab}{a^2 - b^2}\operatorname{cosec} 2\varphi.$$

The maximum value of the smaller of the two angles is $\frac{\pi}{2}$ when $\varphi = 0$ or $\frac{\pi}{2}$, and the principal axes are the only pair of conjugate diameters which are at right angles. The minimum value of the angle occurs when $2\varphi = \frac{\pi}{2}$, *i.e.* when $\varphi = \frac{\pi}{4}$, and then $\tan\theta = 2ab/(a^2 - b^2)$. The gradients of the two diameters are then $\pm b/a$, *i.e.* they are equally inclined to the major axis, and, from the symmetry of the ellipse, are therefore equal. They are called the *equi-conjugate diameters*. They are easily constructed as the diagonals of the rectangle formed by the tangents at the ends of the axes.

**12. Supplemental chords.** If $PCP'$ is a diameter and $Q$ is any point on the ellipse, $QP$ and $QP'$ are called supplemental chords.

*The diameters parallel to a pair of supplemental chords are conjugate.*

Draw the diameter $SCS' \parallel QP$, and let it cut $QP'$ in $M$. Then, since $C$ is the mid-point of $PP'$, $M$ is the mid-point of $QP'$. Hence the diameter $SS'$ bisects the chord $QP'$, and is therefore conjugate to the diameter which is parallel to $QP'$. This affords a simple way of constructing a pair of conjugate diameters making any given angle with one another. On any diameter $PP'$ construct an arc of a circle containing the given angle. If this cuts the ellipse in a point $Q$, then $QP$ and $QP'$ are parallel to the conjugate diameters. If the given (acute) angle is less than the angle between the equi-conjugate diameters the circle will not cut the ellipse in any points besides $P$ and $P'$, and the problem is insoluble.

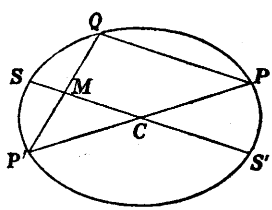


FIG. 30.

**13.** Let $P \equiv (x_1, y_1)$ and $D \equiv (x_2, y_2)$ be extremities of two conjugate diameters, so that

$$x_1 = a\cos\varphi, \quad x_2 = -a\sin\varphi,$$
$$y_1 = b\sin\varphi, \quad y_2 = b\cos\varphi.$$

Then $$CP^2 = x_1^2 + y_1^2 = a^2\cos^2\varphi + b^2\sin^2\varphi,$$
$$CD^2 = x_2^2 + y_2^2 = a^2\sin^2\varphi + b^2\cos^2\varphi.$$
Hence $$CP^2 + CD^2 = a^2 + b^2,$$
*i.e. the sum of the squares of two conjugate diameters is constant.*

**14.** The equation of the tangent at the point $\varphi$ is found by putting $x_1 = a\cos\varphi$, $y_1 = b\sin\varphi$, and is thus
$$\frac{x}{a}\cos\varphi + \frac{y}{b}\sin\varphi = 1,$$
or $bx\cos\varphi + ay\sin\varphi = ab$.

Let $p$ be the length of the perpendicular from the centre upon the tangent; then
$$p^2 = \frac{a^2b^2}{a^2\sin^2\varphi + b^2\cos^2\varphi} = \frac{a^2b^2}{CD^2},$$
*i.e.* $$p \,.\, CD = ab.$$
Hence *the area of the parallelogram formed by the tangents at the ends of two conjugate diameters is constant and equal to* $4ab$.

**15. To find the equation of the chord joining the points $\varphi$, $\varphi'$.** The coordinates of the two points are $(a\cos\varphi, b\sin\varphi)$ and $(a\cos\varphi', b\sin\varphi')$; hence the equation of the chord is
$$\frac{x - a\cos\varphi}{y - b\sin\varphi} = \frac{a(\cos\varphi' - \cos\varphi)}{b(\sin\varphi' - \sin\varphi)} = -\frac{a\sin\frac{1}{2}(\varphi + \varphi')}{b\cos\frac{1}{2}(\varphi + \varphi')},$$
which reduces to
$$\frac{x}{a}\cos\tfrac{1}{2}(\varphi + \varphi') + \frac{y}{b}\sin\tfrac{1}{2}(\varphi + \varphi') = \cos\varphi\cos\tfrac{1}{2}(\varphi + \varphi') + \sin\varphi\sin\tfrac{1}{2}(\varphi + \varphi')$$
$$= \cos\tfrac{1}{2}(\varphi - \varphi').$$
By putting $\varphi = \varphi' = \alpha$, this becomes the equation of the tangent at $\alpha$
$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1.$$

If $\varphi + \varphi'$ is kept constant $= 2\alpha$, we have a system of chords
$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = \cos\tfrac{1}{2}(\varphi - \varphi'),$$
all parallel to the tangent at $\alpha$. Hence *for a system of parallel chords the sum of the eccentric angles of their extremities is constant, and is equal to double the eccentric angle of the point of contact of a tangent parallel to the chord.*

*Note.* If the eccentric angle is not measured always the same way round, it may be necessary to add or subtract $2\pi$ from the sum.

**16.** Let $SPQ$ be a secant of an ellipse drawn through the point $S \equiv (x', y')$. In the case of a circle the product $SP \,.\, SQ$ depends only on the position of the point $S$, and is independent of the direction of the secant. We shall investigate the corresponding property for the ellipse.

The equation of a line through $S$ may be written

$$\frac{x-x'}{\cos\theta}=\frac{y-y'}{\sin\theta}=r,$$

where $\theta$ is the angle which the line makes with the major axis, and $r$ is the distance of the variable point $(x, y)$ from the fixed point $(x', y')$. Substituting the values of $x$, $y$ in the equation of the ellipse, we have

$$b^2(x'+r\cos\theta)^2+a^2(y'+r\sin\theta)^2=a^2b^2,$$

*i.e.* $r^2(a^2\sin^2\theta+b^2\cos^2\theta)+2r(a^2y'\sin\theta+b^2x'\cos\theta)+b^2x'^2+a^2y'^2-a^2b^2=0.$

The roots of this quadratic in $r$ are $SP$ and $SQ$; hence

$$SP \,.\, SQ=\frac{b^2x'^2+a^2y'^2-a^2b^2}{a^2\sin^2\theta+b^2\cos^2\theta}.$$

But if $d$ is the length of the semi-diameter parallel to $SP$, and $(x_1, y_1)$ are the coordinates of one extremity,

$$x_1=d\cos\theta,\quad y_1=d\sin\theta,\quad \text{and}\quad x_1^2/a^2+y_1^2/b^2=1.$$

Therefore $$a^2\sin^2\theta+b^2\cos^2\theta=\frac{a^2b^2}{d^2}.$$

Hence $$SP \,.\, SQ=\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)d^2.$$

The product therefore depends upon two factors, one depending only on the position of the point $S$. the other depending only on the direction of the secant.

If $SP'Q'$ is another secant through $S$, and $d'$ is the length of the parallel semi-diameter,

$$SP'\cdot SQ'=\left(\frac{x'^2}{a^2}+\frac{y'^2}{b^2}-1\right)d'^2.$$

Hence $$SP \,.\, SQ : SP' \,.\, SQ'=d^2 : d'^2.$$

*Cor.* If $SU$ and $SV$ are tangents to the ellipse and $d$, $d'$ are the lengths of the semi-diameters parallel to these tangents,

$$SU : SV=d : d'.$$

**17. Area of an ellipse.** The area bounded by an arc of a curve $PQ$, the axis of $x$, and the ordinates $PM$, $QN$ at the ends of the arc, may be found by the following general method. Divide $MN$ into any number $n$ of equal parts, and draw ordinates at the points of division. Then, completing the rectangles externally and internally as in the figure (Fig. 31), we get a series of inscribed rectangles whose sum is less than the given area, and a series of projecting rectangles whose sum exceeds the given area. But these two sums differ by the sum of the small rectangles which border the curve, and their sum is equal to the rectangle $QR$. Now, as $n$ is increased, the breadth of each rectangle becomes smaller and smaller, and the rectangle $QR$ tends to zero. Hence the two sums of rectangles tend

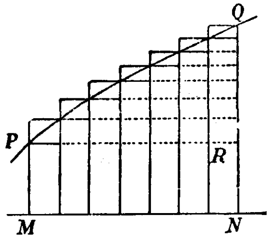


FIG. 31.

each to the same limit, and since the given area lies always intermediate between them, this limit is the area required.

In the case of the ellipse we can compare the areas of the small strips with the areas of corresponding strips of the auxiliary circle. Since $PM : QM = b : a$, the ratio of corresponding strips is always $= b : a$. Hence the whole area of the ellipse is to the whole area of the auxiliary circle as $b : a$; hence the area of the ellipse $= \pi ab$.

The area of any sector bounded by two radii through the centre can be expressed very simply in terms of the eccentric angles. The area of the sector $ACP$

$$= AMP + MCP$$

$$= \frac{b}{a}(AMQ + MCQ) = \frac{b}{a} ACQ$$

$$= \tfrac{1}{2} ab\varphi.$$

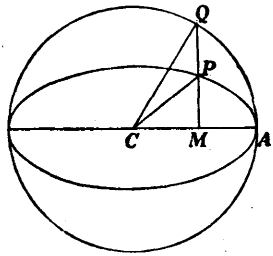


FIG. 32.

Hence the area of a sector $PCP' = \tfrac{1}{2} ab(\varphi' - \varphi)$.

Ex. Two conjugate diameters divide an ellipse into four equal areas.

**18. Application of the integral calculus.** The most powerful analytical instrument for calculating areas is the integral calculus. The method, which is fully explained in any text-book on the calculus, is essentially the method of dividing the area into elements and summing these. Thus, if the breadth of each strip in Fig. 31 is $dx$, the area of a strip of height $y$ is $y\,dx$, and the whole area is represented by the integral

$$\int y\,dx,$$

taken within the required limits. Expressing $y$ in terms of $x$ from the equation of the ellipse, we have $y = \frac{b}{a}\sqrt{a^2 - x^2}$, and, for the area of a quadrant, the limits of integration are from $x = 0$ to $x = a$. Hence the whole area is

$$4\int_0^a \frac{b}{a}\sqrt{a^2 - x^2}\,dx = 2\frac{b}{a}\left[x\sqrt{a^2 - x^2} + a^2 \sin^{-1}\frac{x}{a}\right]_0^a$$

$$= 2\frac{b}{a}\frac{\pi}{2}a^2 = \pi ab.$$

**19. Area of a triangle inscribed in an ellipse.** If $\varphi_1, \varphi_2, \varphi_3$ are the eccentric angles of the three vertices of a triangle inscribed in an ellipse, the area of the triangle is

$$\tfrac{1}{2}ab\begin{vmatrix} \cos\varphi_1 & \sin\varphi_1 & 1 \\ \cos\varphi_2 & \sin\varphi_2 & 1 \\ \cos\varphi_3 & \sin\varphi_3 & 1 \end{vmatrix}$$

$$= \tfrac{1}{2}ab\{\sin(\varphi_3 - \varphi_2) + \sin(\varphi_1 - \varphi_3) + \sin(\varphi_2 - \varphi_1)\}$$

$$= 2ab \sin\tfrac{1}{2}(\varphi_2 - \varphi_3) \sin\tfrac{1}{2}(\varphi_3 - \varphi_1) \sin\tfrac{1}{2}(\varphi_1 - \varphi_2).$$

which is $b/a$ times the area of the corresponding triangle inscribed in the auxiliary circle.

The maximum triangle which can be inscribed in a circle is equilateral; hence the maximum triangle which can be inscribed in an ellipse is such that the eccentric angles of its vertices are $\varphi$, $\varphi+\frac{2\pi}{3}$, $\varphi+\frac{4\pi}{3}$, and its area is $\frac{3}{4}\sqrt{3}ab$.

Similarly the minimum triangle which can be circumscribed about an ellipse touches the ellipse at the vertices of a maximum inscribed triangle, and its area is $3\sqrt{3}ab$.

**Examples.**

1. Show that the chord of an ellipse joining the points whose eccentric angles have a constant difference touches a fixed concentric ellipse, and that the point of contact lies on the line joining the centre to the point of contact of the tangent to the original ellipse which is parallel to the chord.

2. If a rhombus is circumscribed about an ellipse, show that its vertices lie on the axes.

3. Show that the largest rectangle that can be circumscribed about an ellipse is a square, and that its area is $2(a^2+b^2)$.

4. If a parallelogram is inscribed in an ellipse, show that its sides are parallel to a pair of conjugate diameters.

5. Show that only one square can be inscribed in an ellipse, and that its area is $4a^2b^2/(a^2+b^2)$.

6. If $\varphi$ and $\psi$ are the eccentric angles of the points of contact of two adjacent sides of a parallelogram circumscribed about an ellipse, prove that its area $=4ab$ cosec $(\psi-\varphi)$. Hence show that the parallelograms formed by the tangents at the ends of two conjugate diameters have a minimum area.

7. Show that all inscribed parallelograms of an ellipse whose vertices are at the ends of a pair of conjugate diameters have the same area $2ab$, and are maximum inscribed parallelograms.

8. Prove that the area of a triangle whose sides touch the ellipse $x^2/a^2+y^2/b^2=1$ at the points $\varphi_1$, $\varphi_2$, $\varphi_3$ is

$$ab\tan\tfrac{1}{2}(\varphi_2-\varphi_3)\tan\tfrac{1}{2}(\varphi_3-\varphi_1)\tan\tfrac{1}{2}(\varphi_1-\varphi_2).$$

**20. The foci.** On the major axis of an ellipse there are two remarkable points $F$, $F'$, called the *foci*, which have the property that if $P$ is any point on the curve, the sum of the focal distances $PF+PF'$ is constant. We have to prove the existence of these points, determine their positions, and find the value of the constant sum.

Let us provisionally assume the existence of the two points, situated as in Fig. 33, then, taking $P$ first at $A$ and then at $A'$, we have

$$AF+AF'=A'F+A'F';$$

therefore

$$AF+2a-F'A'=2a-AF+F'A'.$$

Hence $AF=F'A'$, *i.e.* the foci are equidistant from the centre, and the value of the constant $AF+AF'=2a$. Next, taking $P$ at $B$, we find $BF=BF'=a$. The foci may therefore be constructed by drawing a circle with centre $B$ and radius equal to $a$ cutting the major axis in $F$ and $F'$.

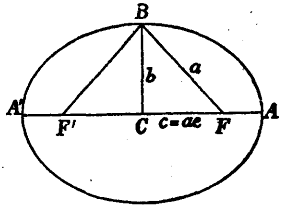


FIG. 33.

Hence $CF=\sqrt{a^2-b^2}=c$. It is convenient to represent this distance as a fraction of the major semi-axis, $ea$, and the number $e$ is called the *eccentricity*, since it indicates how far the foci are removed from the centre. We have then

$$b^2=a^2(1-e^2),$$

and

$$e=\cos CFB=\sqrt{a^2-b^2}/a.$$

We can now prove that if $F$ and $F'$ are the points thus determined, and if $P$ is any point on the ellipse, $PF+PF'=2a$.

We have $$PF^2=(x-ae)^2+y^2,$$

and $$y^2=\frac{b^2}{a^2}(a^2-x^2)=(1-e^2)(a^2-x^2).$$

Therefore $$PF^2=e^2x^2-2aex+a^2=(ex-a)^2.$$

Hence, since $x<a$ and $e<1$,

$$PF=a-ex. \qquad \text{.......(1)}$$

Similarly $$PF'=a+ex. \qquad \text{.......(2)}$$

Hence $$PF+PF'=2a.$$

Equation (1), written in the form

$$PF=e\left(\frac{a}{e}-x\right),$$

expresses that the distance of $P$ from the focus is equal to $e$ times its distance from the straight line $a/e-x=0$. This line is the polar of the focus $F\equiv(ae, 0)$, and is called the *directrix* corresponding to this focus. A similar result is obtained from equation (2).

Hence *the ratio of the distances of any point on an ellipse from a focus and the corresponding directrix is constant and equal to the eccentricity.*

*Cor.* If the directrix cuts the major axis produced in $X$, $CX=a/e$.

The chord $RFR'$ through the focus $F$ perpendicular to the axis is called the *latus rectum*. Let $FR$, the semi-latus rectum, $=l$; then

$$l=eFX=e(CX-CF)=a(1-e^2)=b^2/a.$$

Q. What becomes of the foci and the directrices in the case of a circle?

**21. Mechanical description of an ellipse.** Let a thread of length $2a$ have its ends attached to the points $F$, $F'$, or better let an endless thread of length $2(a+c)$, where $FF'=2c$, be passed round pins fixed at $F$, $F'$; then, if a pencil moves so as to keep the string always stretched, it will describe an ellipse with foci $F$, $F'$, major axis $=2a$, and eccentricity $=c/a$. This is popularly called the "gardener's method" of describing an ellipse.

**22. Imaginary foci.** If the investigation of § 20 is carried out on the supposition that the foci lie on the minor axis, it will be found that the constant sum of the focal distances is $2b$, and that the foci $G$ and $G'$ are the points of intersection of the minor axis with a circle of radius $b$ and centre $A$. These foci are evidently imaginary. Continuing as in the previous investigation, it can be shown that if $OG=c'=e'b$, $a^2=b^2(1-e'^2)$ and $PG=b\pm e'y$. Hence the directrix property holds also for these imaginary foci.

Q. Has the circle any imaginary foci?

**23. Polar equation of an ellipse referred to a focus as pole.** Taking $FA$ as initial line, $\angle AFP=\theta$, $FP=r=ePN$.

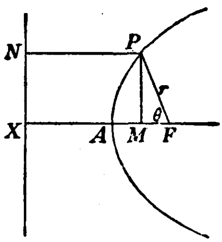


FIG. 34.

But $PN=XF-r\cos\theta$;

hence $r=l-er\cos\theta$,

*i.e.* $$\frac{l}{r}=1+e\cos\theta.$$

If $FA'$ is taken as initial line the equation is

$$\frac{l}{r}=1-e\cos\theta.$$

*Note.* The equations $\frac{l}{r}=\pm1+e\cos\theta$ represent the same curve, for the one is changed into the other by changing $\theta$ into $\theta+\pi$ and $r$ into $-r$, and the polar coordinates $(r, \theta)$ and $(-r, \theta+\pi)$ represent the same point.

**24.** Let the tangent at $P$ meet the major axis in $T$. The equation of the tangent at $(x', y')$ is

$$\frac{xx'}{a^2}+\frac{yy'}{b^2}=1.$$

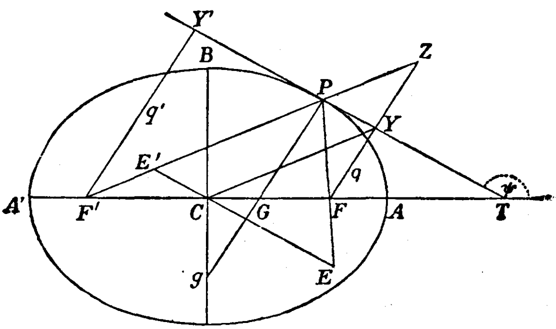


FIG. 35.

Putting $y=0$, we find $CT=a^2/x'$.

Hence $$FT=\frac{a^2}{x'}-ae,\quad F'T=\frac{a^2}{x'}+ae.$$

But if $r$, $r'$ are the focal distances $FP$ and $F'P$,

$$r = e\left(\frac{a}{e} - x'\right) = a - ex' \quad \text{and} \quad r' = a + ex'.$$

Hence $$FT : F'T = a - ex' : a + ex' = FP : F'P.$$

Therefore *the tangent at $P$ is equally inclined to the two focal lines through $P$.* The tangent and normal are the bisectors of the angles between the focal lines.

Draw $FY \perp$ the tangent and produce it to meet $F'P$ in $Z$. Then $PZ = PF$, and $F'Z = F'P + PF = 2a$. Also, since $C$, $Y$ are the mid-points of $FF'$ and $FZ$, $CY \parallel F'P$ and $= a$.

Hence *the locus of $Y$, the foot of the perpendicular from a focus upon the tangent, is the auxiliary circle.*

Since the locus of $Z$ is a circle with centre $F'$ and radius $2a$, and $PZ = PF$, the ellipse may be described by a point which moves so that its distance from the focus $F$ is equal to its distance from the fixed circle with centre $F'$ and radius $2a$. Hence this circle is called a *director circle.* There is an equal director circle with centre $F$. (Compare § 8.)

**25. The normal.** To find the equation of the normal at any point $(x', y')$ we have simply to find the equation of the line through $(x', y')$ perpendicular to the tangent at this point. The equation of the tangent at $(x', y')$ being

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 1$$

the equation of the normal is

$$(x - x')\frac{y'}{b^2} = (y - y')\frac{x'}{a^2}.$$

**26.** *Through a given point there can be drawn four normals to an ellipse* for if the normal at $(x, y)$ passes through $(x_1, y_1)$, we have

$$(a^2 - b^2)xy + b^2y_1x - a^2x_1y = 0.$$

This quadratic equation, together with the equation of the ellipse, determines the coordinates of four points.

The eccentric angle of the foot of a normal from $(x, y)$ to the ellipse is given by the equation

$$(a^2 - b^2)\cos\varphi\sin\varphi - ax\sin\varphi + by\cos\varphi = 0.$$

Let $\tan\frac{1}{2}\varphi = t$, so that $\sin\varphi = 2t/(1 + t^2)$, $\cos\varphi = (1 - t^2)/(1 + t^2)$. Then we get an equation of the fourth degree in $t$,

$$(a^2 - b^2)2t(1 - t^2) - 2axt(1 + t^2) + by(1 - t^4) = 0,$$

*i.e.* $$byt^4 + 2(ax + a^2 - b^2)t^3 + 2(ax - a^2 + b^2)t - by = 0.$$

If $t_1$, $t_2$, $t_3$, $t_4$ are the roots of this equation,

$$t_1t_2t_3t_4 = -1 \quad \text{and} \quad \Sigma t_1t_2 = 0.$$

Now $$\tan\tfrac{1}{2}(\varphi_1 + \varphi_2 + \varphi_3 + \varphi_4) = \frac{\Sigma t_1 - \Sigma t_1t_2t_3}{1 - \Sigma t_1t_2 + t_1t_2t_3t_4};$$

hence $\frac{1}{2}(\varphi_1+\varphi_2+\varphi_3+\varphi_4)=(2n+1)\frac{\pi}{2}$. Hence, if the normals at the four points $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ are concurrent,

$$\varphi_1+\varphi_2+\varphi_3+\varphi_4=(2n+1)\pi.$$

This condition is necessary but not sufficient, for two conditions are required in order that four lines may be concurrent. These are the conditions

$$t_1t_2t_3t_4=-1 \quad \text{and} \quad \Sigma t_1t_2=0.$$

**27. Intersection of a circle and an ellipse.** A circle cuts an ellipse in four points, $P$, $Q$, $R$, $S$, which may be coincident or imaginary in pairs. Let the equation of the circle be

$$x^2+y^2+2gx+2fy+c=0.$$

Then the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1+\lambda(x^2+y^2+2gx+2fy+c)=0$$

represents a curve of the second degree passing through the points of intersection of the circle and the ellipse. If this breaks up into two straight lines they are a pair of common chords, $PQ$ and $RS$, or $PR$ and $QS$, or $PS$ and $QR$. There should therefore be three values of $\lambda$ for which this happens.

If the equation represents two straight lines, by Chap. II. § 17, they are parallel to the lines represented by the homogeneous equation

$$\left(\lambda+\frac{1}{a^2}\right)x^2+\left(\lambda+\frac{1}{b^2}\right)y^2=0,$$

and are therefore equally inclined to the $x$-axis. Hence *the pairs of common chords of an ellipse and a circle are equally inclined to the major axis of the ellipse.*

Let $\varphi_1$, $\varphi_2$, $\varphi_3$, $\varphi_4$ be the eccentric angles of the four points of intersection, and put $\tan\frac{1}{2}\varphi=t$. Then $\sin\varphi=2t/(1+t^2)$, $\cos\varphi=(1-t^2)/(1+t^2)$. Putting $x=a\cos\varphi$, $y=b\sin\varphi$ in the equation of the circle, we get, after reduction,

$$(a^2-2ga+c)t^4+4fbt^3+2(-a^2+2b^2+c)t^2+4fbt+(a^2+2ga+c)=0.$$

Hence, $t_1$, $t_2$, $t_3$, $t_4$ being the roots of this equation,

$$\Sigma t_1=\Sigma t_1t_2t_3.$$

But $\quad \tan\frac{1}{2}(\varphi_1+\varphi_2+\varphi_3+\varphi_4)=(\Sigma t_1-\Sigma t_1t_2t_3)/(1-\Sigma t_1t_2+t_1t_2t_3t_4).$

Hence, if the four points are concyclic,

$$\varphi_1+\varphi_2+\varphi_3+\varphi_4=2n\pi;$$

and, conversely, if this condition is satisfied the four points are concyclic, for one condition is sufficient in order that four points should be concyclic.

**28.** If two of the points of intersection $R$, $S$ coincide, the circle touches the ellipse at $R$, and cuts it in the two other points $P$, $Q$. The relation between the eccentric angles is then

$$\varphi_1+\varphi_2+2\varphi_3=2n\pi.$$

If $P$ and $Q$ are given there are two circles through these points which will touch the ellipse, and the points of contact are at ends of a diameter; for this equation gives two values for $\varphi_3$, viz. $\pi - \frac{1}{2}(\varphi_1 + \varphi_2)$ and $2\pi - \frac{1}{2}(\varphi_1 + \varphi_2)$, and these differ by $\pi$.

If $R$ coincides with $S$, and $P$ with $Q$, the circle has *double contact* with the ellipse. The relation between the eccentric angles is then $\varphi_1 + \varphi_3 = n\pi$. The common chord is thus perpendicular to one of the axes, and the centre of the circle lies on one of the axes.

If three of the points of intersection, $Q$, $R$, $S$, coincide, the circle is said to *osculate* the ellipse, or to have *contact of the second order* with the ellipse, at $Q$, and it cuts the ellipse in one other point $P$, whose eccentric angle is determined by $\varphi_1 = 2n\pi - 3\varphi_2$.

In general a circle cannot have contact of higher order than the second at a given point, since four points do not in general lie on a circle. If the four points $P$, $Q$, $R$, $S$ all coincide the eccentric angle $\varphi = \frac{1}{2}n\pi$, and the point must be the extremity of one of the axes.

The circle which has highest order of contact with the ellipse at a given point is used to determine the *curvature* of the ellipse. This circle is called the *circle of curvature* or *osculating circle*, its centre the *centre of curvature*, and its radius the *radius of curvature*. The curvature is defined to be the reciprocal of the radius of curvature.

**29. Coordinates of the centre of curvature.** If $U$ is a centre of curvature two of the normals from $U$ to the ellipse are coincident with the radius to the point of contact of the circle of curvature.

For let $A$, $B$, $C$ be three points close together on the ellipse, and let the perpendiculars at the mid-points $M$ and $N$ of the chords $AB$, $BC$ meet in $U$. Then, as $A$ approaches $B$, $AB$ becomes the tangent at $B$ and $UM$ becomes the normal. Also, as $C$ approaches $B$, independently of $A$, $UN$ also becomes the normal at $B$. $U$ is thus the ultimate intersection of two normals which approach coincidence.

This leads to a method of finding the coordinates of the centre of curvature. The equation of the normal at the point $\varphi$ is

$$xa \sin\varphi - yb \cos\varphi = (a^2 - b^2)\sin\varphi\cos\varphi$$
$$= \tfrac{1}{2}c^2 \sin 2\varphi.$$

To find the equation of the normal at a near point, put $\varphi + d\varphi$ for $\varphi$. Then, retaining only first powers of $d\varphi$, we get

$$xa(\sin\varphi + \cos\varphi\, d\varphi) - yb(\cos\varphi - \sin\varphi\, d\varphi) = \tfrac{1}{2}c^2(\sin 2\varphi + 2\cos 2\varphi\, d\varphi).$$

The point of intersection of these two normals is the point of intersection of

$$xa \sin\varphi - yb\cos\varphi = c^2 \sin\varphi\cos\varphi, \quad \text{.............................(1)}$$

and

$$xa\cos\varphi + yb\sin\varphi = c^2(\cos^2\varphi - \sin^2\varphi). \quad \text{.........................(2)}$$

Hence

$$xa = c^2\cos^3\varphi \quad \text{and} \quad yb = -c^2\sin^3\varphi.$$

It will be noticed that equation (2) could have been obtained from (1) by differentiating with regard to $\varphi$.

**30. The evolute.** The locus of the centre of curvature has the freedom equations

$$x = \frac{c^2}{a}\cos^3\varphi,$$

$$y = -\frac{c^2}{b}\sin^3\varphi,$$

and the constraint equation is therefore

$$(ax)^{\frac{2}{3}} + (by)^{\frac{2}{3}} = c^{\frac{4}{3}}.$$

From any point on this curve two of the normals to the ellipse are coincident, and the curve divides the plane into two regions such that from any point in one region the four normals to the ellipse are all real and distinct, and from any point in the other region two of the normals are imaginary.

This curve is called the *evolute* of the ellipse. If a string is wound on the curve along the arc $RS$ and extended to $A$, then as the string is unwound the extremity $A$ will describe the ellipse. For at any moment the end of the string $P$ is describing a circle with centre $U$ (the centre of curvature), and radius $PU$ (the radius of curvature at $P$). All the tangents to the evolute are normals to the ellipse, and the evolute is the envelope of normals to the ellipse.

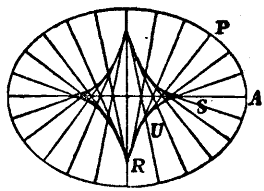


FIG. 36.

Q. What becomes of the four normals from any point in the case of the circle? (This may be answered after reading Chap. XIV.)

**Examples.**

1. Show that the gradient of the tangent at the end of the latus rectum is equal to the eccentricity.

2. If the minor axis cuts the auxiliary circle at $b$ and $b'$, show that the tangents from $b$ and $b'$ touch the ellipse at the ends of the latus rectum.

3. If the major axis cuts the minor auxiliary circle at $a$ and $a'$, and the tangents to the circle at these points cut the ellipse in $P, Q, P', Q'$, show that the diameters $PP'$, $QQ'$ are parallel to the tangents at the ends of the latus rectum.

4. Prove that the angle subtended by the latus rectum at its pole is equal to the angle subtended at the end of the latus rectum by the segment joining the foci.

5. If $q, q'$ are the perpendiculars from the foci upon a tangent, show that $qq' = b^2$.

6. If $p$ is the perpendicular from the focus $F$ upon the tangent at $P$, and $FP = r$, prove that $l/p^2 = 2/r - 1/a$ (the *pedal equation* of the ellipse referred to a focus).

7. If $r$ is the focal distance of a point whose eccentric angle is $\varphi$, prove that

$$r = a(1 - e\cos\varphi).$$

8. If $Q$ is the point on the auxiliary circle corresponding to $P$, and $FK$ is perpendicular to $CQ$, prove that $KQ = FP$.

9. If $BF$ cuts the ellipse again in $P$, and $R$ is the extremity of the latus rectum through $F$, prove that $FP : FB = FR : F'R$.

10. If the circle described on $FF'$ as diameter cuts the minor axis in $K$ and $K'$, prove that the sum of the squares of the perpendiculars from $K$ and $K'$ on any tangent is constant, $=2a^2$.

11. Prove that the semi-latus rectum is a harmonic mean between the two segments of any focal chord.

12. Prove that at all points on the ellipse the normal passes between the foci.

13. Show that the radius of curvature at the end of the major axis is equal to the semi-latus rectum, $=b^2/a$.

14. Show that the radius of curvature at the end of the minor axis $=a^2/b$.

15. Show that the radius of curvature at the end of the latus rectum $=l(1+e^2)^{\frac{3}{2}}$.

**31. Historical Note.** The discovery of the curves which we call conics is attributed to the Greek geometer Menaechmus, a pupil of Eudoxus and contemporary of Plato (fourth century B.C.). He applied these curves as plane loci to the solution of the famous problem of the "duplication of the cube." The problem propounded by the Delian oracle was to construct a cubical altar of double the volume of the existing one. Arithmetic was not applied to geometry then as we do it, so the solution had to be geometrical. Using algebraic notation, the problem is to solve the equation $x^3=2$. Doubtless on the analogy with the corresponding equation of the second degree, whose solution by Euclidean methods involves finding a mean proportional, it was shown that the Delian problem could be reduced to that of finding two mean proportionals between 1 and 2. For if

$$1 : x = x : y = y : 2,$$

we get $$(1)\ x^2=y,\quad (2)\ y^2=2x,\quad (3)\ xy=2;$$

hence $x^3=2$. Now the equations (1), (2), (3) represent in our modern notation two parabolas and a rectangular hyperbola, and it was by the intersection of these curves that Menaechmus solved the problem.

The conics were first studied as sections of a cone; and there is extant an extensive treatise on the subject by Apollonius of Perga (about 250 B.C. English edition by Heath, Cambridge, 1896). One of the first properties obtained was that which we have chosen to define the conics, viz. for the ellipse and hyperbola $PM^2=kAM \cdot MA'$, and for the parabola $PM^2=kAM$. If the curves are referred to axes through the vertex their equations are easily found to be $y^2=px$ for the parabola and $y^2=kx(d \pm x)$ for the hyperbola and ellipse, or, putting $kd=p$, we have the three equations

$$\text{Parabola:}\quad y^2=px,$$
$$\text{Ellipse:}\quad y^2=px-x^2p/d,$$
$$\text{Hyperbola:}\quad y^2=px+x^2p/d.$$

$p$ is the length of the latus rectum, or parameter as it was also called. The names parabola, ellipse and hyperbola, which were brought into use by Apollonius, have their origin in the properties expressed by these equations. Thus in the parabola a rectangle of area equal to the square on the ordinate and breadth equal to the abscissa can be "applied" (παραβάλλειν) to the parameter $p$. This process of application which was called παραβολή is frequently found in Euclid, *e.g.* in Book i. Prop. 44 a parallelogram of given area is applied to a given segment. In the ellipse the rectangle equal to $y^2$ "falls short" (ἐλλείπειν, ἔλλειψις) of the rectangle $px$, and in the hyperbola it "overshoots" it (ὑπερβάλλειν, ὑπερβολή). In constructing the rectangle $px$ the parameter $p$ was erected at the end of the axis; hence the term *latus rectum* or "side erected."

The foci of the ellipse and hyperbola were obtained by Apollonius as points dividing the transverse axis into segments whose rectangle equals $\frac{1}{4}pd$. The discoveries of the focus-directrix property and the existence of the focus of a parabola are due to Pappus (fourth century A.D.). The term focus was introduced by Kepler (1604) from the optical property of an ellipse that rays proceeding from a focus are reflected at the curve to the other focus.

## EXAMPLES IV. A.

1. Find the eccentric angles of the extremities of the latera recta of an ellipse.

2. Obtain the equation, in polar coordinates, of the line joining two points $(r_1, \theta_1)$ and $(r_2, \theta_2)$ on the ellipse $l/r = 1 + e\cos\theta$, and deduce that the equation of the tangent at $(r_1, \theta_1)$ is $l/r = e\cos\theta + \cos(\theta_1 - \theta)$.

3. If $F$ is a focus of an ellipse and the normal at $P$ meets the major and minor axes respectively in $G$, $g$, and $CD$ is conjugate to $CP$, prove that

$$FG : CF = FP : AC \quad \text{and} \quad Fg : CF = CD : BC.$$

(Peterhouse, etc., 1914.)

4. $CP$, $CD$ are conjugate semi-diameters of an ellipse; the tangent at $P$ meets the major axis in $T$, and $N$ is the foot of the ordinate of $P$. If $PD$ meets the major axis in $K$, prove that $KD^2 : KP^2 = CN : NT$. (Selwyn, 1914.)

5. The ordinate $MP$ of a point $P$ on an ellipse is produced to meet the tangent at the end of the latus rectum through the focus $F$ in $R$. Prove that $MR = FP$.

6. Tangents are drawn at pairs of points of an ellipse whose eccentric angles differ by a constant angle $2\alpha$; find the locus of their intersection.

7. The normal at any point $P$ on an ellipse, whose corresponding point on the auxiliary circle is $p$, is produced to meet the radius $Cp$ in $R$. Find the locus of $R$.

8. From Ex. 7 show how to construct the minor axis, given the major axis, a point on the curve, and the tangent at that point.

9. A circumscribed parallelogram of an ellipse is formed by drawing the tangents at the vertices of an inscribed parallelogram; prove that the sides of the latter are parallel to the diagonals of the former.

10. Find the locus of the centre of a circle which touches two fixed circles, one of which lies entirely within the other.

11. $SP$, $S'P'$ are focal radii of an ellipse drawn in the same direction, and the tangents at $P$ and $P'$ meet $S'P'$ and $SP$ in $Q'$ and $Q$ respectively. Prove that $QQ'$ is parallel to $PP'$. (Math Tripos I., 1913.)

12. Prove that the tangents from any point to an ellipse are equally inclined respectively to the two focal lines through the point.

13. If $r$, $r'$ are the focal radii of a point $P$, and $CD$ the semi-diameter conjugate to $CP$, prove that $rr' = CD^2$.

14. If $PG$ is the normal at $P$, and $r$, $r'$ the focal radii, prove that

$$PG^2 : rr' = b^2 : a^2.$$

15. Prove that the sum of the reciprocals of the squares of any two diameters of an ellipse which are at right angles is constant.

16. A series of ellipses is constructed with a common axis $BB'$, and the other axes lie along the line $AA'$ and form an arithmetical progression. Show that the areas enclosed between consecutive arcs are all equal. Further, if any line is drawn parallel to $AA'$, the areas enclosed between this line, the line $AA'$, and the arcs of consecutive ellipses are all equal.

17. If $\phi$, $\phi'$ are the eccentric angles of the ends of a focal chord of an ellipse, prove that $\tan \frac{1}{2}\phi \tan \frac{1}{2}\phi' = -(1-e)/(1+e)$ or $-(1+e)/(1-e)$. Distinguish the two cases.

18. $PFQ$ and $PF'R$ are two focal chords of an ellipse, and the eccentric angles of $Q$ and $R$ are $\phi$ and $\phi'$; show that the ratio $\tan \frac{1}{2}\phi : \tan \frac{1}{2}\phi'$ is constant for all positions of $P$. (Corpus, etc., 1900.)

19. Prove that the length of a focal chord of an ellipse is $2d^2/a$, where $d$ is the semi-diameter parallel to the shord and $a$ is the major semi-axis.

20. If $q$ is the perpendicular from the focus $F$ upon the tangent at $P$, $r=FP$, and $d$ is the semi-diameter parallel to the tangent at $P$, prove that $qd=br$.

21. If $\angle F'FP=\theta$ and $\angle FF'P=\theta'$, prove that $\tan \frac{1}{2}\theta \tan \frac{1}{2}\theta' = (1-e)/(1+e)$.

22. If $\theta$ is the vectorial angle, referred to a focus, and $\phi$ the eccentric angle of a point on an ellipse, prove that $\tan \dfrac{\theta}{2} = \sqrt{\dfrac{1+e}{1-e}} \tan \dfrac{\phi}{2}$. Show also that the maximum value of $\theta - \phi$ is $2 \sin^{-1} \sqrt{\dfrac{a-b}{a+b}}$, and that it occurs when $\theta + \phi = \pi$, $r=b$. (Pembroke, etc., 1913.)

23. From a point $R$ on the directrix for the focus $F$ a secant $RPP'$ is drawn to the ellipse; show that $RF$ bisects the exterior angle between $FP$ and $FP'$.

24. Prove that every pair of conjugate lines through a focus are at right angles.

25. Prove that through any point $P$ there are two lines which are conjugate with regard to a given ellipse and also at right angles, and show that they are the bisectors of the angles between the lines joining $P$ to the foci.

26. $P$, $P'$ are two fixed points on an ellipse, and $Q$ is a variable point on the curve. $PQ$, $P'Q$ cut the directrix in $R$, $R'$. Prove that $RR'$ subtends a constant angle at the corresponding focus.

27. If the directrix cuts the tangents at two points $P$, $P'$ in $T$, $T'$, and the chord $PP'$ in $Q$, prove that $QF$ bisects the angle $TFT'$.

28. If $A$, $B$, $C$, $D$ are fixed points on an ellipse, and $P$ a variable point on the curve, prove that the cross-ratio of the pencil $P(AB, CD)$ is constant and equal to

$$(\sin \tfrac{1}{2}AFC/\sin \tfrac{1}{2}AFD)/(\sin \tfrac{1}{2}BFC/\sin \tfrac{1}{2}BFD),$$

$F$ being either focus.

29. If the tangents at four fixed points $A$, $B$, $C$, $D$ on an ellipse cut a variable tangent in $P$, $Q$, $R$, $S$, prove that the cross-ratio of the range $(PQ, RS)$ is constant and equal to

$$(\sin \tfrac{1}{2}AFC/\sin \tfrac{1}{2}AFD)/(\sin \tfrac{1}{2}BFC/\sin \tfrac{1}{2}BFD),$$

$F$ being either focus.

30. If the tangent at $P$ to an ellipse is cut in $T$, $T'$ by two parallel tangents, prove that $TP.PT'=CD^2$, and that $FT.FT' : F'T.F'T'=FP : F'P$, where $F$, $F'$ are the foci and $CD$ the semi-diameter conjugate to $CP$.

(St. Catharine's, 1907.)

31. If from a point $P$ on one of the equi-conjugate diameters of an ellipse with centre $C$ tangents $PA$, $PB$ are drawn, prove that $PABC$ are concyclic.

(Magdalene, 1907.)

32. $F$, $F'$ are the foci of an ellipse, and $P$ is a point on it. Find the loci of the centres of the in- and e-scribed circles of the triangle $FPF'$.

33. Find the condition that $lx+my+n=0$ should be a normal to the ellipse $x^2/a^2+y^2/b^2=1$.

34. If $P_1\equiv(x_1, y_1)$ and $P_2\equiv(x_2, y_2)$ are two adjacent vertices of a parallelogram circumscribed about the ellipse $S\equiv x^2/a^2+y^2/b^2-1=0$, and $S_1$, $S_2$ are the values of $S$ at the points $P_1$, $P_2$, prove that $S_1S_2=1$.

35. Prove that an indefinite number of parallelograms can be circumscribed about the ellipse $x^2/a^2+y^2/b^2=1$ and inscribed in the ellipse $x^2/a^2+y^2/b^2=2$.

36. Prove that the envelope of the chord of contact of two perpendicular tangents to an ellipse is another ellipse. (Selwyn, 1907.)

37. Investigate the general *isoptic locus* for an ellipse, *i.e.* the locus of points the tangents from which contain a given angle $\alpha$.

38. The points $A$, $A'$ are the ends of the major axis of an ellipse, and $PAQ$, $P'A'Q'$ are tangents to the conic there; if $PP'$, $QQ'$ are two other tangents to the conic, prove that $AP.A'P'=AQ.A'Q'$, and that the lines $PQ'$, $P'Q$ intersect on $AA'$.

39. If $QFq$ is a chord of an ellipse parallel to the tangent at a point $P$, and $PFp$ is a chord of the circle of curvature at $P$, prove that $Qq=Pp$, $F$ being a focus. (Corpus, 1910.)

40. $PCP'$ and $DCD'$ are conjugate diameters of an ellipse, and $\varphi$ is the eccentric angle of $P$. Prove that $\frac{1}{2}\pi-3\varphi$ is the eccentric angle of the point where the circle $PP'D$ again cuts the ellipse. (Math. Tripos I., 1910.)

41. Show that through a given point on an ellipse there pass three osculating circles, and that their points of contact are the vertices of a maximum inscribed triangle.

42. Prove that the radius of curvature at any point $P$ of an ellipse is $d^3/ab$, where $d$ is the semi-diameter parallel to the tangent at $P$.

43. Prove that the length of the common chord of an ellipse and its circle of curvature (the *chord of curvature*) at the point whose eccentric angle is $\alpha$ is $d \sin 2\alpha$, where $d$ is the diameter conjugate to $CP$.

44. Prove that the chord of curvature at $P$ which passes through either focus is equal to $2d^2/a$, where $d$ is the semi-diameter parallel to the tangent at $P$ and $a$ is the major semi-axis.

45. A circle rolls within another circle of double the diameter. Prove that any point on the circumference of the rolling circle describes a straight line, while any other point fixed to the rolling circle describes an ellipse.

46. An ellipse rolls on an equal ellipse so that the two ellipses are always symmetrical with respect to the tangent at the point of contact. Prove that the foci of the moving ellipse describe circles.

47. Assuming that the earth describes an ellipse round the sun as focus, and that the sun's distance is inversely proportional to the sun's apparent diameter or angle which its diameter subtends at the earth, calculate the eccentricity of the orbit when it is found that the maximum and minimum values of the sun's apparent diameter are 32′ 35″ and 31′ 31″.

48. In a certain month it is found that the moon's apparent diameter varies from a maximum of 33′ 05″ to a minimum of 29′ 33″. Find the eccentricity of its elliptic orbit round the earth as focus.

49. On three different dates the sun's apparent diameter is found to be 31′ 38″, 31′ 32″ and 31′ 57″. At these dates the sun's longitude (*i.e.* the vectorial angle measured from a certain initial line) is 60° 24′, 120° 24′, 180° 24′. Determine the eccentricity of the elliptic orbit and the longitude of perihelion (*i.e.* the vectorial angle of the nearer extremity of the major axis).

50. Assuming that any section of the Earth by a plane through the polar axis is an ellipse whose minor axis is the polar axis, major semi-axis $a$, and eccentricity $e$, and defining the "geographical latitude" of any place as the angle which the normal makes with the major axis, (i) prove that the distance from the earth's centre of a place at latitude $\phi$ is approximately equal to $a(1-\frac{1}{2}e^2\sin^2\phi)$, where powers of $e$ above the second are neglected; (ii) show also that if the "geocentric latitude" $\phi'$ is the angle which the line joining the point to the earth's centre makes with the major axis, then approximately

$$\phi' = \phi - \tan^{-1}(\tfrac{1}{2}e^2 \sin 2\phi).$$

## EXAMPLES IV. B.

1. $PFQ$, $PF'R$ are focal chords of an ellipse; show that the tangents at $Q$ and $R$ intersect on the normal at $P$. (Corpus, 1911.)

2. $PQ$ is a focal chord of an ellipse. The tangents at $P$ and $Q$ intersect in $T$, and the normals at $P$ and $Q$ intersect in $R$. Show that $TR$ passes through the other focus.

3. If in Ex. 2 $TR$ cuts the ellipse in $P'$, $Q'$, show that the normals at $P'$, $Q'$ intersect on $PQ$.

4. At a point $P$ on the circumference of the auxiliary circle of an ellipse whose major axis is $AA'$ the tangent drawn to the circle meets the major axis in $T$, and the lines $PA$, $PA'$ meet the ellipse in $Q$ and $R$ respectively. Show that $TQR$ are collinear. (King's, etc., 1913.)

5. From the foci $F$, $F'$ parallel lines $FP$, $F'P'$ are drawn to an ellipse; the tangents at $P$, $P'$ meet in $T$, and $FP'$, $F'P$ meet in $Q$. Prove that $T$ is on the auxiliary circle, that $TQ$ bisects the angle $PQP'$, and that the projection of $TQ$ on $QP$ or $QP' = \frac{1}{2}$ latus rectum. (Corpus, etc., 1914.)

6. Lines are drawn through the origin perpendicular to the tangents from a point $P$ to the ellipse $x^2/a^2 + y^2/b^2 = 1$. Find the locus of $P$ if the lines are conjugate diameters of the ellipse. (Selwyn, 1912.)

7. Prove that the locus of points from which tangents to an ellipse and its auxiliary circle form a harmonic pencil is a concentric ellipse.

(Corpus, etc., 1902.)

8. On two concentric and coaxial ellipses two points are taken of which the eccentric angles are equal; prove that the line joining them is normal to a third concentric ellipse at a point with the same eccentric angle. (Corpus, etc., 1902.)

9. If from any point $P$ on an ellipse perpendiculars are drawn to the axes, show that the line joining the feet of these perpendiculars is always normal to a fixed concentric ellipse.

10. A pair of conjugate diameters of an ellipse, whose centre is $O$, are cut by a fixed straight line in points $P$ and $P'$. Prove that the locus of the centre of the circle circumscribing the triangle $OPP'$ is a straight line. (Selwyn, 1913.)

11. Tangents are drawn to the ellipse $x^2/a^2+y^2/b^2=1$ from a point $P$, and with the chord of contact form a triangle whose centroid lies on the curve. Find the locus of $P$. (Queens', 1907.)

12. Two conjugate diameters of an ellipse meet the polar of a fixed point $P$ in $Q$ and $Q'$, and the perpendiculars to these diameters at $Q$ and $Q'$ meet in $S$. Prove that the locus of $S$ is the polar of $P$ with regard to the orthoptic circle.

13. Through a point $K$ on the major axis of an ellipse a chord $PQ$ is drawn. Prove that the tangents at $P$ and $Q$ intersect the line through $K$ perpendicular to the major axis in points equidistant from $K$. (Selwyn, 1914.)

14. The chord $PQ$ of the ellipse $x^2/a^2+y^2/b^2=1$ is such that the tangents at $P$ and $Q$ each pass through the pole of the other with respect to the ellipse $x^2/a'^2+y^2/b'^2=1$. Prove that the envelope of $PQ$ is the ellipse

$$x^2(a^2+a'^2)/a^4+y^2(b^2+b'^2)/b^4=(a^2+a'^2)(b^2+b'^2)/(a^2b'^2+a'^2b^2).$$

(Pembroke, etc., 1900.)

15. Show that the locus of a point from which two tangents to the ellipse $x^2/a^2+y^2/b^2=1$ make equal angles with the line $y=x\tan\varphi$ is the conic (rectangular hyperbola)

$$x^2-2xy\cot 2\varphi-y^2=a^2-b^2. \quad \text{(Pembroke, etc., 1911.)}$$

16. Show that the tangents at the extremities of all chords of the ellipse $x^2/a^2+y^2/b^2=1$ which subtend a right angle at the centre intersect on the ellipse $x^2/a^4+y^2/b^4=1/a^2+1/b^2$. (Corpus, etc., 1901.)

17. $O$ is a given point within a circle, and a line through $O$ meets the circle in $P$. $PT$ is drawn perpendicular to $OP$. Show that the envelope of $PT$ is an ellipse. Explain what happens when $O$ lies on the circumference of the circle.

18. Prove that if the normals at four points of an ellipse $x^2/a^2+y^2/b^2=1$ are concurrent, and two of the points lie on the line $lx/a+my/b+1=0$, the other two will lie on the line $x/al+y/bm=1$.

Prove that if two lines drawn through the point $(\frac{2}{3}a, \frac{2}{3}b)$ meet the ellipse at four points the normals at which are concurrent, one of the lines will be $4x/a-y/b=2$, and find the equation of the other. (Peterhouse, etc., 1901.)

19. $PQ$ is a normal chord of an ellipse at $P$, and $p, q$ are the corresponding points on the auxiliary circle; $R$ is the mid-point of the arc $pq$. Prove that $Cp$ and $CR$ are conjugate diameters.

20. If $p$ is the perpendicular from the centre on the normal at the point whose eccentric angle is $\varphi$ on the ellipse $x^2/a^2+y^2/b^2=1$, prove that

$$p^2=(a-b)^2\{1-(a\sin^2\varphi-b\cos^2\varphi)^2(a^2\sin^2\varphi+b^2\cos^2\varphi)^{-1}\}.$$

Hence show that the greatest value of $p$ is $a-b$, when $\tan^2\varphi=b/a$.

21. Prove that the maximum ellipse inscribed in a given triangle touches the sides at their mid-points, and the minimum ellipse circumscribed about the same triangle has its axes twice the length of those of the inscribed ellipse and coincident with them. (The minimum circumscribed ellipse is called the *Steiner ellipse* of the triangle.)

22. Two circles with fixed centres $B$, $B'$ cut at a constant angle $\alpha$; prove that the envelope of their common tangents is an ellipse with $BB'$ as minor axis and eccentricity $\cos \frac{1}{2}\alpha$.

23. Two circles pass through two fixed points $F$, $F'$ and cut at a fixed angle $\alpha$; prove that the envelope of their common tangents is an ellipse with foci $F$, $F'$ and eccentricity $\cos \frac{1}{2}\alpha$.

24. Prove that every circle which has its centre on the major axis of a given ellipse of eccentricity $\frac{1}{2}\sqrt{2}$ and which has double contact with the ellipse is cut at the ends of a diameter by every circle which passes through the foci.

25. Through the focus $F$ of an ellipse is drawn a line $FY$ cutting in $Y$ the tangent at $P$, and making the angle $FYP$ equal to that subtended at $F$ by either half of the minor axis. Show that the locus of $Y$ consists of the osculating circles at the two ends of the minor axis. (Math. Tripos II., 1913.)

26. The tangents to an ellipse at $A$ and $B$, extremities of the major and minor axes, meet in $K$. Show that the osculating circles at $A$ and $B$ have $K$ as an external limiting point, and subtend supplementary angles at $K$.

(Pembroke, etc., 1910.)

27. Prove that the equation of the circle of curvature at the point $\varphi$ on the ellipse $x^2/a^2+y^2/b^2=1$ is

$$x^2+y^2-2\frac{a^2-b^2}{a}x\cos^3\varphi+2\frac{a^2-b^2}{b}y\sin^3\varphi=(2b^2-a^2)\cos^2\varphi+(2a^2-b^2)\sin^2\varphi.$$

Show also that through any fixed point $(f, g)$ there pass six circles of curvature, and verify that the six centres of curvature lie on the conic

$$\{2(f^2+g^2-2fx-2gy)-a^2-b^2\}^2=12(a^2x^2+b^2y^2)-3(a^2-b^2)^2.$$

(Pembroke, etc., 1899.)

28. $P$ is any point and $F$ is one focus of an ellipse. The circle on $PF$ as diameter cuts the auxiliary circle in $Y$, $Z$. Prove that $PY$, $PZ$ are tangents to the ellipse.

29. If $P$ is any point on an ellipse, and $F$ a focus, show that the circle on $FP$ as diameter touches the auxiliary circle.

30. If $QP$ and $QP'$ are tangents to an ellipse, and $F$ is a focus, prove that $FQ$ bisects the angle $PFP'$ (*i.e.* the tangents subtend equal angles at the focus).

# CHAPTER V.

## THE HYPERBOLA.

**1.** THE hyperbola is defined in a similar way to the ellipse. Let $A$ and $A'$ be two fixed points, and $P$ a variable point, $PM$ perpendicular to $AA'$. Then the locus of $P$, such that

$$PM^2 = kAM \,.\, MA',$$

where $k$ is a negative constant, is a hyperbola.

Since $PM^2$ is positive the segments $AM$ and $MA'$ are of opposite sign, and therefore $M$ lies outside the segment $AA'$.

**2.** To find the equation of the hyperbola in its simplest form. Take $A'A$ as axis of $x$, and the mid-point of $A'A$ as origin. Then

$$MP = y, \quad OM = x, \quad A'M = x + a, \quad AM = x - a.$$

Hence, from the defining property, we have

$$y^2 = -k(x+a)(x-a) = -k(x^2 - a^2).$$

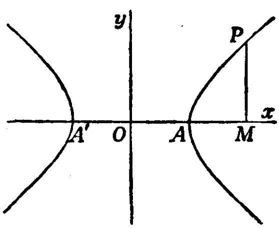


FIG. 37.

This equation, which is of the second degree, contains only squares of $x$ and $y$; hence the curve is symmetrical about both axes.

Since $k$ is negative, $x$ cannot be numerically less than $a$. Hence no part of the curve lies between the lines through $A$ and $A'$ parallel to the $y$-axis, and in particular the curve does not cut the $y$-axis. Otherwise there is no limit to the value of $x$, and as $x$ increases $y$ also increases numerically without limit. The curve consists of two open branches extending to infinity.

It is convenient to write $-ka^2 = b^2$; then the equation assumes the standard form

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

$A'A$ is called the *transverse axis*. There is no minor axis, but the other axis of symmetry, the $y$-axis, is called the *conjugate axis*. $A$ and $A'$ are called *vertices*. $O$ is the *centre*, and chords through the centre are *diameters*.

**3. Asymptotes.** The equation of the hyperbola in polar coordinates with $O$ as pole is

$$\frac{\cos^2\theta}{a^2} - \frac{\sin^2\theta}{b^2} = \frac{1}{r^2}.$$

When $\theta=0$, $r=a$, and as $\theta$ increases $\cos^2\theta$ diminishes and $\sin^2\theta$ increases; therefore $1/r^2$ diminishes and $r$ increases. This continues until

$$\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}=0,$$

*i.e.*

$$\tan^2\theta=\frac{b^2}{a^2},$$

when $r$ becomes indefinitely large. When $\tan\theta>b/a$, $r$ becomes imaginary, so that all the diameters drawn between the lines $ay=bx$ and $ay=-bx$ fail to cut the curve. These two lines only meet the curve at infinity, and separate the diameters into two classes, intersecting and non-intersecting. They are called the *asymptotes*. The diameter $y=\mu x$ is intersecting or non-intersecting according as $\mu$ is numerically less or greater than $b/a$.

The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ may be regarded as having imaginary asymptotes represented by the equation $\frac{x^2}{a^2}+\frac{y^2}{b^2}=0$. The asymptotes of the circle $x^2+y^2=a^2$ are the imaginary lines $x^2+y^2=0$.

*An asymptote may be regarded as the limiting case of a tangent when the point of contact goes to infinity.* Writing the equation of the hyperbola homogeneously

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z^2,$$

we see that the line at infinity $z=0$ cuts the curve where $x^2/a^2-y^2/b^2=0$; hence the coordinates of the points at infinity on the curve are $(a, \pm b, 0)$. The equation of the tangent at $(x', y', z')$ is

$$\frac{xx'}{a^2}-\frac{yy'}{b^2}=zz';$$

hence the equation of the tangent at $(a, b, 0)$ is $\frac{x}{a}-\frac{y}{b}=0$, which represents one of the asymptotes.

Unlike the ellipse, which consists of a single closed curve, the hyperbola consists of two branches which run to infinity along the two asymptotes.

If $2\alpha$ is the angle between the asymptotes, $\tan\alpha=b/a$. When $a=b$, $2\alpha=\frac{\pi}{2}$, and the asymptotes are at right angles. In this case the curve is called a *rectangular hyperbola*, also sometimes an *equilateral hyperbola*.

**4.** Many of the properties of the hyperbola can be obtained at once from the corresponding property of the ellipse by changing $b^2$ into $-b^2$.

The condition that $lx+my+n=0$ should touch the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ is

$$a^2l^2-b^2m^2=n^2,$$

which is therefore the *tangential equation* of the curve.

The equation of the tangents from $(x', y')$ is

$$\left(\frac{xx'}{a^2}-\frac{yy'}{b^2}-1\right)^2=\left(\frac{x^2}{a^2}-\frac{y^2}{b^2}-1\right)\left(\frac{x'^2}{a^2}-\frac{y'^2}{b^2}-1\right).$$

The locus of points from which tangents to the hyperbola are at right angles, *i.e.* the *orthoptic locus*, is

$$x^2+y^2=a^2-b^2.$$

The orthoptic locus is therefore a circle concentric with the hyperbola. It is real only if $a>b$, and for a rectangular hyperbola reduces to a point-circle.

**5. Conjugate diameters.** The theory of conjugate diameters of a hyperbola is somewhat different from the case of an ellipse, since we have to distinguish between diameters which cut and those which do not cut the curve.

The procedure of Chap. IV. § 9 applies equally to the hyperbola, but it is instructive to use a different method.

Consider a system of parallel chords $y=\mu x+c$, all parallel to the diameter $y=\mu x$. The line $y=\mu x+c$ cuts the curve in two points whose abscissae are determined by the equation

$$b^2x^2-a^2(\mu x+c)^2=a^2b^2,$$

*i.e.*
$$(a^2\mu^2-b^2)x^2+2a^2\mu cx+a^2(b^2+c^2)=0.$$

These points are real, if

$$a^2\mu^2c>(b^2+c^2)(a^2\mu^2-b^2),$$

*i.e.* if
$$c^2>b^2(a^2\mu^2-b^2).$$

If $\mu^2<b^2/a^2$, this condition is always satisfied; hence all chords parallel to a diameter which cuts the curve cut the curve in real points. But if $\mu^2>b^2/a^2$, *i.e.* if the diameter does not cut the curve, the parallel chords will only cut the curve if they are sufficiently remote from the centre.

Let $(x, y)$ be the coordinates of the mid-point of the chord; then, if $x_1$, $x_2$ are the roots of the above quadratic in $x$,

$$x=\tfrac{1}{2}(x_1+x_2)=-\frac{a^2\mu c}{a^2\mu^2-b^2}. \qquad \text{.........(1)}$$

This is always real whether the chord cuts the curve in real points or not. Then substituting in the equation of the chord $y=\mu x+c$, we have

$$y=-\frac{b^2c}{a^2\mu^2-b^2}. \qquad \text{.........(2)}$$

Eliminating $c$ between (1) and (2), we have

$$y=\frac{b^2}{a^2\mu}x.$$

Hence the locus of the mid-points of chords parallel to the diameter $y=\mu x$ is the diameter $y=\mu' x$, where

$$\mu\mu'=\frac{b^2}{a^2}.$$

These two diameters are called *conjugate diameters*. From the symmetry of the relationship connecting $\mu$ and $\mu'$, we see that each diameter bisects all chords parallel to the other. If $\mu<b/a$, then $\mu'>b/a$, so that of two conjugate diameters one is intersecting and the other is non-intersecting

Since a tangent is the limiting case of a chord when the end points come to coincide at the mid-point, we see that the tangents at the extremities of the intersecting diameter are parallel to the conjugate diameter.

The pole of a diameter is thus the point at infinity on the conjugate diameter, and the polars of all points on a fixed diameter are parallel to the conjugate diameter.

**6. The conjugate hyperbola.** Consider the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=-1.$$

This hyperbola has the same asymptotes and axes as the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1,$$

but its transverse axis is the conjugate axis of the given hyperbola, and *vice versa*. The two hyperbolas are called *conjugate hyperbolas*. Since the ratio $b^2/a^2$ is the same for both, a pair of diameters which are conjugate with regard to the one hyperbola are conjugate with regard to the other, and the tangents at the points where a non-intersecting diameter of the first hyperbola cuts the conjugate hyperbola are parallel to the conjugate diameter.

Writing the polar equations of the two hyperbolas referred to the centre as pole, we have

$$\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}, \qquad -\frac{1}{r^2}=\frac{\cos^2\theta}{a^2}-\frac{\sin^2\theta}{b^2}.$$

Hence the lengths of corresponding semi-diameters $CP$ and $Cp$ drawn in the same direction are connected by the relation

$$CP^2=-Cp^2.$$

If one is real the other is imaginary.

7. The circle on $AA'$ as diameter is called the *auxiliary circle*, but there is no simple relation between corresponding ordinates of this circle and the hyperbola. Nor is there an eccentric angle in terms of which we may express the freedom equations of the hyperbola. Analogous freedom equations may, however, be expressed by means of hyperbolic functions. Since $\cosh^2\varphi-\sinh^2\varphi=1$, we may express the coordinates of any point on the hyperbola by means of the equations

$$x=a\cosh\varphi,$$
$$y=b\sinh\varphi.$$

As $\varphi$ varies from $-\infty$ to $+\infty$, the point describes the whole of one branch of the curve. The other branch is expressed by the equations

$$x=-a\cosh\varphi,$$
$$y=-b\sinh\varphi.$$

We may pass from one branch to the other by adding $i\pi$ to $\varphi$, since

$$\cosh(\varphi+i\pi)=\cosh\varphi\cos\pi+\sinh\varphi . i\sin\pi=-\cosh\varphi,$$

and

$$\sinh(\varphi+i\pi)=\sinh\varphi\cos\pi+\cosh\varphi . i\sin\pi=-\sinh\varphi.$$

Thus the parameters corresponding to opposite ends of a diameter differ by $i\pi$.

The freedom equations of the conjugate hyperbola are then

$$x = a \sinh \varphi,$$
$$y = b \cosh \varphi.$$

Let $\varphi_1$ be the parameter of the point $P \equiv (x_1, y_1)$, so that

$$x_1 = a \cosh \varphi_1,$$
$$y_1 = b \sinh \varphi_1;$$

and $\varphi_2$ that of the point $d \equiv (x_2, y_2)$ on the conjugate hyperbola, so that

$$x_2 = a \sinh \varphi_2,$$
$$y_2 = b \cosh \varphi_2.$$

Then, $CP$ and $Cd$ being conjugate diameters, $\frac{x_1 x_2}{a^2} = \frac{y_1 y_2}{b^2}$; therefore

$$\cosh \varphi_1 \sinh \varphi_2 = \sinh \varphi_1 \cosh \varphi_2,$$

*i.e.* $\tanh \varphi_1 = \tanh \varphi_2$, and hence $\varphi_1 = \varphi_2$.

We have therefore, for the ends of two conjugate diameters,

$$\left.\begin{array}{l} x = a \cosh \varphi, \\ y = b \sinh \varphi, \end{array}\right\} \qquad \left.\begin{array}{l} x = a \sinh \varphi, \\ y = b \cosh \varphi, \end{array}\right\}$$

and $$CP^2 = x_1^2 + y_1^2 = a^2 \cosh^2 \varphi + b^2 \sinh^2 \varphi,$$
$$Cd^2 = x_2^2 + y_2^2 = a^2 \sinh^2 \varphi + b^2 \cosh^2 \varphi;$$

therefore $$CP^2 - Cd^2 = (x_1^2 - x_2^2) - (y_2^2 - y_1^2)$$
$$= a^2 - b^2.$$

Hence *the difference of the squares of two conjugate diameters of a hyperbola and the conjugate hyperbola is constant.*

*Note.* Since the actual length of the conjugate diameter (which is imaginary) is equal to $\sqrt{-Cd^2}$, the theorem for a single hyperbola is that the *sum* of the squares of two conjugate diameters is constant, just as in the case of the ellipse.

**8.** The equation of the tangent at $P(\varphi)$ is

$$\frac{x}{a} \cosh \varphi - \frac{y}{b} \sinh \varphi = 1,$$

and the equation of the tangent at the point $d$, where the conjugate diameter meets the conjugate hyperbola is

$$\frac{x}{a} \sinh \varphi - \frac{y}{b} \cosh \varphi = -1.$$

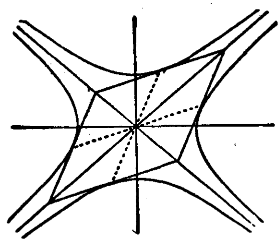


FIG. 38.

Adding these equations we find that the two tangents intersect on the line $x/a - y/b = 0$, *i.e.* on one of the asymptotes; similarly for the tangents at the other ends of the diameters. Hence *the tangents at the points where two conjugate diameters cut one or other of the two conjugate hyperbolas form a parallelogram whose vertices lie on the asymptotes.*

**9. Area of a sector of a hyperbola.** The area of the sector $AOP$ is

$$\int_0^\theta \tfrac{1}{2}r^2\,d\theta.$$

Now $\qquad r\cos\theta = x = a\cosh\varphi,$

$\qquad\qquad r\sin\theta = y = b\sinh\varphi;$

therefore $\qquad a\tan\theta = b\tanh\varphi.$

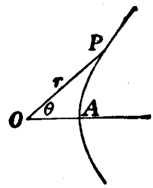



FIG. 39

Differentiating, we have

$$a\sec^2\theta\,d\theta = b\,\mathrm{sech}^2\varphi\,d\varphi.$$

But $\qquad a^2\sec^2\theta = r^2\,\mathrm{sech}^2\varphi;$

therefore $\qquad r^2\,d\theta = ab\,d\varphi,$

and the area of the sector is

$$\tfrac{1}{2}\int_0^\theta r^2\,d\theta = \tfrac{1}{2}ab\int_0^\phi d\varphi = \tfrac{1}{2}ab\varphi.$$

We have thus found a geometrical meaning for the parameter $\varphi$ of a point $P$, viz. it is proportional to the area of the sector $AOP$.

**10.** Freedom equations for the hyperbola may also be found in terms of circular functions, viz.

$$x = a\sec v,$$
$$y = b\tan v.$$

The geometrical meaning of the angle $v$ can be expressed with reference to the auxiliary circle. Let $ACQ = v$, and draw the tangent $QM$ at $Q$ meeting the major axis in $M$. Erect $MP \perp CA$. Then

$$x = a\sec v = CM,$$

$$y = \frac{b}{a}a\tan v = \frac{b}{a}QM.$$

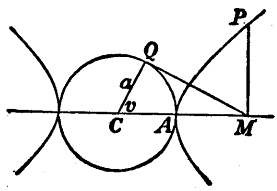



FIG. 40.

As $v$ increases from $-\frac{\pi}{2}$ to $+\frac{\pi}{2}$, one branch of the hyperbola is described, and as $v$ continues to increase from $\frac{\pi}{2}$ to $\frac{3\pi}{2}$, the other branch is described.

**11.** In dealing with a rectangular hyperbola the simplest equation is obtained by taking the asymptotes as coordinate axes.

Referring the curve first to its axes, so that its equation is $x^2 - y^2 = a^2$, the equations of the asymptotes are $x \pm y = 0$. The perpendiculars from a point $P \equiv (x, y)$ on the asymptotes are $(x \pm y)/\sqrt{2}$. These are now to be the coordinates of $P$, so that we may write

$$\sqrt{2}\xi = x + y, \quad \sqrt{2}\eta = x - y,$$

and therefore $\quad 2\xi\eta = x^2 - y^2 = a^2.$

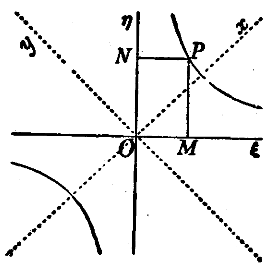



FIG. 41

The equation of the rectangular hyperbola referred to its asymptotes is therefore

$$2xy = a^2.$$

We shall see later (Chap. VIII.) how this can be applied to any hyperbola.

Freedom equations for this form are now easily written down, viz.

$$x = \tfrac{1}{2}at, \quad y = a/t.$$

**12. The foci.** Like the ellipse the hyperbola has two foci, $F$, $F'$ on its transverse axis, but in this case the difference and not the sum of the focal distances is constant,

$$F'P \sim FP = \text{constant}.$$

Provisionally assuming the existence of these two points, situated as in Fig. 42, their positions can be determined as follows.

Taking $P$ first at $A$, and then at $A'$, we find

$$F'A - AF = A'F - F'A';$$

therefore

$$F'A' + 2a - AF = 2a + AF - F'A'.$$

Hence $F'A' = AF$, *i.e.* the foci are equidistant from the centre, and the value of the constant

$$F'A - AF = 2a.$$

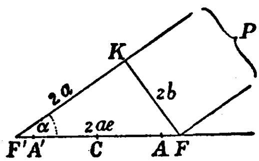


FIG 42.

Let $CF = c = ae$, $e$ being the eccentricity. To find the value of $e$, and the distance $CF$, let $P$ go to infinity along the asymptote. Then $FP$ and $F'P$ become parallel to the asymptote. Draw $FK \perp F'P$, then $F'K = F'P - FP = 2a$, and since $\tan\alpha = b/a$, $FK = 2b$. Hence

$$c^2 = a^2 + b^2, \quad b^2 = a^2(e^2 - 1), \quad e = \sec\alpha = \sqrt{a^2 + b^2}/a.$$

We can now prove that, if $F$ and $F'$ are the points thus determined, and $P$ is any point on the hyperbola, $F'P \sim FP = 2a$.

We have $$PF^2 = (x - ae)^2 + y^2,$$

and from the equation of the hyperbola

$$y^2 = b^2\left(\frac{x^2}{a^2} - 1\right).$$

Hence $$PF^2 = (x - ae)^2 + (e^2 - 1)(x^2 - a^2)$$
$$= e^2x^2 - 2aex + a^2 = (ex - a)^2.$$

Hence, since when $P$ is on the right-hand branch, $x > a$, and $e > 1$,

$$PF = ex - a.$$

Similarly $$PF' = ex + a.$$

Hence $$PF' - PF = 2a.$$

If $P$ is on the left-hand branch we find similarly that $PF - PF' = 2a$.

Also the distance of $P$ from the focus $F$ is $e$ times its distance from the line $x - a/e = 0$. This line, which is perpendicular to the transverse axis, is the corresponding *directrix*, and is the polar of the focus $F \equiv (ae, 0)$.

**13. Mechanical description of the hyperbola.** From the focal property $PF' - PF = 2a$ a mechanical method of describing the curve can be derived.

Knot a thread to a pencil $P$ and pass the thread round the foci as in Fig. 43, the two ends being drawn tight at $T$. Then $P$ will describe one branch of a hyperbola. For the two parts of the string $TF'P$ and $TF'FP$ are of constant lengths $l'$ and $l$ say. Therefore

$$F'P - FP = (l' - TF') - (l - 2c - TF')$$
$$= 2c - (l - l').$$

Hence if $l - l' = 2(c - a)$, the transverse axis of the hyperbola will be $2a$.

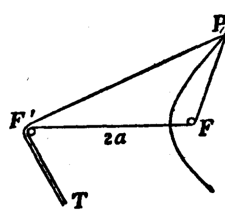


FIG. 43.

**14.** The reader will find it a useful exercise to work out the focal properties corresponding to those of the ellipse (Chap. IV. §§ 23, 24); and also the investigation of the normals from a given point, the coordinates of the centre of curvature, and the equation of the evolute (cf. Chap. IV. §§ 29, 30).

## EXAMPLES V. A.

1. Prove that the normal at $(x', y')$ to the rectangular hyperbola $x^2 - y^2 = a^2$ is $xy' + x'y = 2x'y'$.

2. If $K, L, M, N$ are the feet of the normals from any point $P$ to a rectangular hyperbola with centre $C$, show that the centroid of $KLMN$ divides $CP$ in the ratio 1 : 3.

3. Prove that in a rectangular hyperbola the lines joining any point on the curve to the ends of a diameter are equally inclined to one of the asymptotes.

4. If $PP'$ is a fixed diameter of a rectangular hyperbola, and $Q$ any point on the curve, show that the difference of the angles $QPP'$ and $QP'P$ is constant.

5. Prove that in a rectangular hyperbola a chord subtends equal or supplementary angles at the ends of a diameter.

6. If $P$ is any point on a rectangular hyperbola with centre $C$, prove that $CP$ and the tangent at $P$ are equally inclined to one of the asymptotes.

7. The tangent at a point $P$ of a rectangular hyperbola meets a diameter $BCB'$ in $Q$. Prove that the angles $BPC$ and $B'PQ$ are equal.

8. If the coordinates of a point on the hyperbola $xy = c^2$ are represented by $x = ct$, $y = c/t$, prove that the normals at the four points $t_1, t_2, t_3, t_4$ will be concurrent if $\Sigma t_1 t_2 = 0$ and $t_1 t_2 t_3 t_4 = -1$.

9. From any point on the normal to a rectangular hyperbola at a given point $P$ the other three normals are drawn to the curve. Show that the locus of the centroid of their feet is the diameter of the hyperbola parallel to the normal at $P$. (King's, etc., 1913.)

10. Prove that the locus of a point such that the lines joining it to two fixed points make an isosceles triangle with a fixed straight line is a rectangular hyperbola which has one asymptote parallel to the fixed straight line.

11. A line moves so that the sum of the squares of its distances from two fixed points is equal to the square of half the distance between the points. Show that the envelope of the line is a rectangular hyperbola. (Pembroke, etc., 1911.)

12. If from a point $P$ on a hyperbola $PM$ is drawn parallel to an asymptote meeting the directrix in $M$, prove that $PM = FP$.

Hence establish the following mechanical method of describing a hyperbola. A right-angled triangle slides with one edge $MN$ along a fixed line $OY$; a string of length $AM$ is attached to a point $A$ in the hypotenuse, the other end being attached to the fixed point $F$. A pencil which keeps the string pressed against the edge $AM$ will describe a hyperbola with focus $F$, directrix $OY$, and an asymptote parallel to $AM$.

13. Show that confocal conics of reciprocal eccentricities intersect at the ends of their latera recta. (St. Catharine's, 1907.)

14. A circle which passes through the foci of a hyperbola cuts the asymptotes in $P, P'$ and $Q, Q'$. Prove that $PQ$ and $P'Q'$ are tangents to the hyperbola.

15. A hyperbola is drawn having two sides of a given triangle for asymptotes and touching the other side. Determine the foci. (St. Catharine's, 1907.)

16. A variable circle always touches two fixed circles. Discuss the locus of its centre in the various cases when the fixed circles do or do not intersect.

17. The coordinates of points on a hyperbola being expressed in terms of the parameter $\phi$ by the equations $x = a \sec\phi$, $y = b\tan\phi$, prove that the lines joining the points whose parameters have a given sum all pass through the same point. (Math. Tripos II., 1911.)

18. If the four points $t_1, t_2, t_3, t_4$ on the hyperbola $x = a(1+t^2)/(1-t^2)$, $y = 2bt/(1-t^2)$ are concyclic, prove that $\Sigma t_1 + \Sigma t_1 t_2 t_3 = 0$.

19. Show that a pair of conjugate diameters of a hyperbola revolve in opposite directions, while in the case of an ellipse they revolve in the same direction.

20. Two rectangular hyperbolas are such that the asymptotes of one coincide with the axes of the other. Prove that they cut at right angles.

21. If $x + iy = \sqrt{\phi + i\psi}$, where $\phi$ and $\psi$ are real parameters, prove that $\phi = \text{const.}$ and $\psi = \text{const.}$ represent two systems of rectangular hyperbolas cutting at right angles.

22. Show that from any point on one side of the transverse axis of a hyperbola one and only one real normal can be drawn to each of the portions of the curve which lie in that region.

23. $P, Q, R, S$ are four points on a rectangular hyperbola such that the chord $PQ$ is perpendicular to the chord $RS$; prove that each of the four points is the orthocentre of the triangle formed by the other three.

## EXAMPLES V. B.

1. Prove that the angle which any chord of a hyperbola (or an ellipse) subtends at a focus is bisected by the line joining the focus to the pole of the chord.

2. If $P, P'$ are any two points on a hyperbola (or an ellipse) with foci $F, F'$, prove that the four focal lines $FP, FP', F'P, F'P'$ touch a circle whose centre is the pole of the chord $PP'$. (M. Chasles, 1830.)

3. The coordinates of a point on a hyperbola being expressed by $a\sec\phi$, $b\tan\phi$, prove that the normal at the point $\phi$ is $ax\cos\phi + by\cot\phi = a^2 + b^2$.

4. If the hyperbola $xy = c^2$ is represented by the freedom equations $x = c\tan\phi$, $y = c\cot\phi$, show that $\phi = \frac{1}{4}\pi - \frac{1}{2}v$, where $v$ is the angle in §10.

5. Find the equation of the normal to the hyperbola $x^2/a^2 - y^2/b^2 = 1$, drawn in a given direction, in the form

$$x \cos \alpha + y \sin \alpha = (a^2 + b^2) \sin \alpha \cos \alpha (a^2 \sin^2 \alpha - b^2 \cos^2 \alpha)^{-\frac{1}{2}}.$$

Discuss the cases in which $\tan \alpha =$ or $> b/a$. Show that the perpendicular from the centre on any normal is a non-intersecting diameter of the hyperbola.

6. A circle cuts a hyperbola in the points $P_1, P_2, P_3, P_4$; perpendiculars $P_1M_1$ and $P_2M_2$ are drawn to one asymptote, $P_3N_3$ and $P_4N_4$ to the other; prove that $P_1M_1 . P_2M_2 = P_3N_3 . P_4N_4$.

7. A circle cuts a rectangular hyperbola in four points. Show that the mean point of the four points of intersection is midway between the centres of the two curves.

8. $P_1, P_2, P_3, P_4$ are the feet of the four normals from $P$ to the rectangular hyperbola $xy = c^2$; prove that, if $O$ is the centre, $OP^2 = \Sigma OP_1^2$.
(Corpus, etc., 1912.)

9. A rectangular hyperbola is cut by any circle in four points. Prove that the sum of the squares of the distances of these four points from the centre of the hyperbola is equal to the square on the diameter of the circle.
(Trinity, etc., 1906.)

10. If a triangle circumscribes a rectangular hyperbola, and straight lines are drawn from the centre at right angles to the diameters through the angular points to meet the opposite sides, prove that the points of intersection lie on a tangent. (Trinity, 1900.)

11. Show that if the sum of the squares of the normals from a point to the hyperbola $xy = a^2$ is constant, the point must lie on a circle. (Magdalene, 1901.)

12. Find (as a determinant) the relation which connects the parameters of three points of the hyperbola $x = a \sec \phi$, $y = b \tan \phi$, at which the normals are concurrent, and show that it is identical with the relation which connects the eccentric angles of three points of an ellipse at which the normals are concurrent.

13. Prove that the lines joining points with the same parameter on the ellipse $x = a \cos \phi$, $y = b \sin \phi$, and the hyperbola $x = a \sec \phi$, $y = b \tan \phi$, pass through a fixed point. Prove also that the tangents at corresponding points intersect on a fixed line.

14. A straight line is drawn so as to be divided harmonically by a circle and two fixed diameters of the circle. Show that it touches a fixed hyperbola whose asymptotes are the diameters. (Math. Tripos I., 1909.)

15. From a point $T$ on the orthoptic circle of a hyperbola, whose centre is $C$, two tangents are drawn to the curve. Show that, if the chord of contact meets one of the asymptotes in $R$, and $CT$ produced in $V$, $RV^2 = CV . TV$.
(Pembroke, etc., 1899.)

16. Find the locus of the foci of hyperbolas which pass through two given points and whose asymptotes are parallel to two given straight lines.
(Trinity, etc., 1906.)

17. If one of the common chords of a circle and a rectangular hyperbola is a diameter of the hyperbola, show that another of the common chords is a diameter of the circle.

# CHAPTER VI.

## THE PARABOLA.

**1.** THE parabola was defined in Chap. IV. as follows. $A$ is a fixed point and $AX$ a fixed straight line; $P$ is a variable point, $PM$ perpendicular to $AX$. Then the locus of $P$, such that

$$PM^2 = p \,.\, AM,$$

where $p$ is a constant, is a parabola.

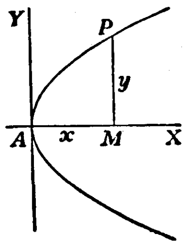


FIG. 44.

**2.** To find the equation of the parabola in its simplest form, take $A$ as origin, $AX$ as axis of $x$. Then $AM = x$, $MP = y$, and the equation is

$$y^2 = px.$$

It is more convenient to write this

$$y^2 = 4ax.$$

$a$ is evidently the length of some line. By taking different values of $a$ we get parabolas of different sizes. It is just as if we draw the same curve on a different scale. Hence all parabolas have the same shape.

Assuming $a$ to be positive, we see that $x$ must be positive; hence the curve lies entirely to the right of the $y$-axis. Otherwise $x$ is unlimited in magnitude, and the curve extends to infinity. Since $y$ only occurs squared, the axis of $x$ is an axis of symmetry, and there is no other axis of symmetry.

The parabola is a very specialized type of conic, and it will be found that while it has numerous properties similar to those of the ellipse and the hyperbola it differs from them in many ways and requires special treatment.

**3. The tangent.** To find the equation of the tangent at a point $P \equiv (x_1, y_1)$ on the curve, we shall find the equation of the chord joining $P$ to a point $Q \equiv (x_2, y_2)$ on the curve near to $P$, and then let $Q$ approach $P$. The equation of the chord is

$$\frac{y - y_1}{x - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Since $P$ and $Q$ both lie on the curve

$$y_1^2 = 4ax_1 \quad \text{and} \quad y_2^2 = 4ax_2,$$

therefore $$y_2^2 - y_1^2 = 4a(x_2 - x_1).$$

Hence the gradient of the chord is

$$\frac{y_2-y_1}{x_2-x_1}=\frac{4a}{y_1+y_2}.$$

As $Q$ approaches $P$, $y_2 \to y_1$ and the gradient $\to 2a/y_1$.

Hence the equation of the tangent at $(x_1, y_1)$ becomes

$$y-y_1=\frac{2a}{y_1}(x-x_1).$$

This result may also be obtained by means of the calculus. Differentiating $y^2=4ax$, we obtain $y\frac{dy}{dx}=2a$, *i.e.* $\frac{dy}{dx}=2a/y$. Hence, as before, the equation of the tangent at $(x_1, y_1)$ is

$$y-y_1=\left(\frac{dy}{dx}\right)_1(x-x_1)=\frac{2a}{y_1}(x-x_1).$$

This equation may also be written

$$yy_1=2a(x+x_1).$$

*Cor.* The condition that $lx+my+n=0$ should be a tangent is found by comparing this equation with the equation of the tangent at the point $(x_1, y_1)$, viz.

$$2ax-y_1y+2ax_1=0.$$

Hence

$$\begin{aligned} l:m:n&=2a:-y_1:2ax_1 \\ &=4a:-2y_1:y_1^2. \end{aligned}$$

Eliminating $y_1$, we have $$am^2=nl,$$
which is the *tangential equation* of the parabola.

**4. Pole and polar.** As in the case of the other conics, if $(x_1, y_1)$ does not lie on the curve the equation

$$yy_1=2a(x+x_1)$$

represents the polar of the point $(x_1, y_1)$.

Using Joachimsthal's method, let $k$ be the position-ratio with respect to the points $P\equiv(x_1, y_1)$ and $Q\equiv(x, y)$ of one of the points $X$, $Y$ in which the join of these two points cuts the curve, then the coordinates of $X$ or $Y$ are

$$\frac{kx+x_1}{k+1},\quad \frac{ky+y_1}{k+1}.$$

Substituting in the equation of the parabola, since $X$ lies on the curve, we have $$(ky+y_1)^2=4a(k+1)(kx+x_1).$$
Hence we have the quadratic equation in $k$,

$$k^2(y^2-4ax)+2k\{yy_1-2a(x+x_1)\}+(y_1^2-4ax_1)=0,$$

whose roots $k_1$ and $k_2$ are the position-ratios of $X$ and $Y$. Now, if

$$yy_1=2a(x+x_1),$$

we have $k_2=-k_1$. Hence $X$ and $Y$ divide $PQ$ internally and externally in the same ratio and $(XY, PQ)$ is a harmonic range. Hence the polar of $P$ is the locus of harmonic conjugates of $P$ with regard to the parabola.

*Cor.* The relation which connects two conjugate points $P \equiv (x_1, y_1)$ and $Q \equiv (x_2, y_2)$ is

$$y_1 y_2 = 2a(x_1 + x_2).$$

This equation expresses that the polar of $P$ passes through $Q$, and that the polar of $Q$ passes through $P$.

**5.** *To find the coordinates of the pole of a given line.* Let the equation of the given line be

$$lx + my + n = 0,$$

and let $P \equiv (x_1, y_1)$ be its pole. The equation of the polar of $P$ is

$$2ax - y_1 y + 2ax_1 = 0.$$

Since these two equations represent the same straight line, we have

$$l : m : n = 2a : -y_1 : 2ax_1;$$

hence

$$x_1 = \frac{n}{l}, \quad y_1 = -\frac{2am}{l}.$$

*Cor.* If the line $l_1 x + m_1 y + n_1 = 0$ passes through the pole of the line $l_2 x + m_2 y + n_2 = 0$, we have

$$l_1 \frac{n_2}{l_2} - m_1 \frac{2am_2}{l_2} + n_2 = 0,$$

*i.e.*

$$l_1 n_2 + l_2 n_1 = 2am_1 m_2.$$

This is the relation between two conjugate lines, and it is related to the tangential equation $ln = am^2$ in exactly the same way as the relation between the coordinates of two conjugate points is related to the equation $y^2 = 4ax$. In fact, the tangential equation may be obtained from this relation by putting $(l_2, m_2, n_2) = (l_1, m_1, n_1)$, since two conjugate lines can only coincide if they are tangents to the curve.

**6. Diameters.** *To find the locus of the mid-points of a system of parallel chords.*

Let $y = \mu x + c$ be the equation of a variable chord in a fixed direction, so that $\mu$ is fixed while $c$ is variable. Substituting the value of $x$ in terms of $y$ in the equation $y^2 = 4ax$, we have

$$\mu y^2 - 4ay - 4ac = 0,$$

a quadratic giving the values of $y$ for the two points of intersection of the chord. If $y_1, y_2$ are the roots of this equation, and $y$ the ordinate of the mid-point of the chord,

$$y = \tfrac{1}{2}(y_1 + y_2) = 2a/\mu.$$

Hence $y$ is constant, and the locus of the mid-points of chords parallel to the direction $y = \mu x$ is the straight line $\mu y = 2a$. As the corresponding locus for the ellipse or hyperbola is a diameter or line through the centre, we shall define a *diameter* of a parabola as a straight line parallel to the axis.

There is no such thing as conjugate diameters of a parabola, but there is a direction or gradient conjugate to every diameter, such that all chords drawn in this direction are bisected by the diameter. The gradiant conjugate to the diameter $y = b$ is $2a/b$. Since the tangent at the end of a diameter is the limiting position of the parallel chords which cut the parabola, the direction which is conjugate to a diameter is that of the tangent at the end of the diameter.

Since the diameters of a parabola are all parallel lines we can conceive of a parabola as the limiting form of an ellipse whose centre is at infinity.

In the case of the ellipse or hyperbola a diameter is the polar of a point at infinity; this holds also for the parabola. The point at infinity on the line $y=\mu x$ is $(1, \mu, 0)$; substituting in the equation $yy_1=2a(xz_1+x_1z)$ these values for $x_1$, $y_1$, $z_1$ and putting $z=1$, we get $\mu y=2a$. The point at infinity on the axis is $X\equiv(1, 0, 0)$, and its polar is $z=0$, *i.e.* the line at infinity. Hence, since the polar of $X$ passes through $X$ itself, $X$ lies on the curve and the line at infinity is a tangent. This appears also from the equation of the curve written in the form $y^2=4axz$, for $z=0$ gives equal roots, and is therefore a tangent.

**7. The focus.** The focal property of the ellipse or hyperbola,

$$FP \pm F'P = \text{constant},$$

breaks down for the parabola, but there still exists a focus-directrix property, *i.e.* there is a point $F$ and a corresponding line such that the distance from the focus $F$ is in a constant ratio to the distance from the corresponding line, the directrix.

The equation $y^2=4ax$ may be written

$$y^2=(x+a)^2-(x-a)^2,$$

*i.e.*
$$(x-a)^2+y^2=(x+a)^2.$$

But this expresses that the distance from the point $(a, 0)$ is equal to the distance from the line $x+a=0$. Hence a focus exists at the point $(a, 0)$ and the directrix is $x=-a$, which is the polar of the focus. Since the constant ratio is unity, we say that the eccentricity of a parabola $=1$.

**Examples.**

1. Prove that the equation of a parabola referred to its axis and directrix as axes of $x$ and $y$ is
$$y^2=4a(x-a).$$

2. Prove that the equation of a parabola referred to axes through the focus, parallel and perpendicular to the axis, is
$$y^2=4a(x+a).$$

3. Show that the equation $y^2=ax+b$ represents a parabola. Find its latus rectum, and the coordinates of the focus.

4. Show that the equation $y=ax^2+2bx+c$ represents a parabola. Find its latus rectum, and the coordinates of the vertex.

5. Draw the following parabolas, and find the coordinates of the vertex, and the angle which the tangent makes with the axis of $x$ at the point where the curve cuts the $y$-axis: (i) $y=x^2+4x+9$, (ii) $4y=x^2-2x-11$, (iii) $10y=3x^2+5x+20$.

6. Draw the following parabolas; find the coordinates of the focus, and the angle which the tangent makes with the axis of $x$ at the point where the curve cuts the positive $y$-axis: (i) $y^2=8x+9$, (ii) $y^2+12x=16$, (iii) $y^2=5x+3$.

7. Prove that the diameter which bisects the chord $lx+my+n=0$ of the parabola $y^2=4ax$ is $y=-2am/l$, and that the distance between the curve and this chord, measured along the diameter, is $am^2/l^2-n/l$.

8. We shall now prove some geometrical properties of the parabola. Let $F$ be the focus, $XM$ the directrix, $PT$ the tangent and $PG$ the normal at $P$. Then $XA=AF=a$.

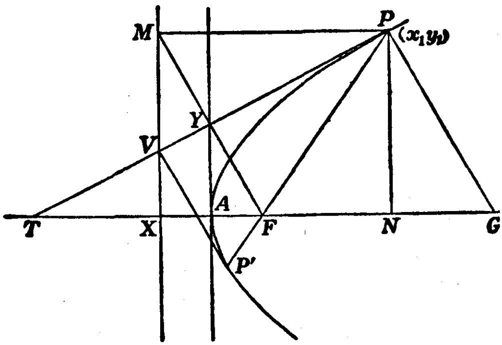



FIG. 45.

The tangent at $P$ is $yy_1=2a(x+x_1)$.

This cuts the axis at $T$, where $y=0$, $x=-x_1$. Therefore $TA=AN$. Hence $TF=XN=MP=FP$, and therefore $\angle FTP=\angle FPT$, *i.e. the tangent is equally inclined to the focal line and the axis.* This property is the principle of the parabolic reflector. Light from a source at $F$ would be reflected along lines parallel to the axis and produce a beam of parallel rays which could be projected to a great distance. Conversely, parallel rays, from the sun for example, would be reflected and come to a focus at $F$; hence the term focus.

9. Join $FM$ cutting the tangent at $P$ in $Y$. Then, since $FP=MP$, $\angle FPY=\angle MPY$, the triangles $FPY$ and $MPY$ are congruent, and $\angle FYP=\angle MYP=$ a right angle. Hence $Y$ is the foot of the perpendicular from the focus upon the tangent at $P$.

Then, since $TF=FP$, $Y$ is the mid-point of $TP$. But $A$ is the mid-point of $TN$. Therefore $AY \parallel NP$, and $AY$ is therefore the tangent at the vertex.

Hence *the locus of the foot of the perpendicular from the focus on a tangent is the tangent at the vertex.* This straight line is for the parabola the degenerate form of the auxiliary circle.

10. Draw the normal $PG$. Then $PG \parallel YF$; hence $F$ is the mid-point of $TG$. Therefore $FG=TF=FP$.

Hence $FG=XN$, and therefore $NG=XF=2a$.

$NG$ is called the *subnormal.* We have, therefore, proved that *the subnormal is constant and equal to the semi-latus rectum.*

**11. Polar equation referred to the focus.** Let the angle $\theta$ be $\angle AFP$. We have
$$r = FP = PM = XL = 2a - r\cos\theta,$$
or, putting $2a = l$, the semi-latus rectum,
$$\frac{l}{r} = 1 + \cos\theta.$$

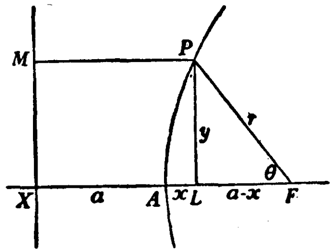


FIG. 46.

**12. Freedom equations of the parabola.** The parabola admits of a very simple system of freedom equations. Taking the equation $y^2 = 4ax$, put $y = 2at$; then $x = at^2$, and we have the freedom equations
$$x = at^2,\ y = 2at.$$

**13. Equation of the chord joining two points.** The gradient of the chord joining the two points $t_1$ and $t_2$ is $2a(t_1 - t_2)/(at_1^2 - at_2^2) = 2/(t_1 + t_2)$. Hence the equation of the chord is $y - 2at_1 = 2(x - at_1^2)/(t_1 + t_2)$, which reduces to $x - \frac{1}{2}(t_1 + t_2)y + at_1t_2 = 0$.

Putting $t_1 = t_2 = t$, we get the equation of the tangent at $t$, viz.
$$x - ty + at^2 = 0.$$

*Cor.* 1. $1/t$ is the gradient of the tangent at $t$, *i.e. the parameter $t$ is the cotangent of the angle which the tangent makes with the axis.*

*Cor.* 2. If the chord $t_1t_2$ passes through the focus $(a, 0)$, we have
$$t_1t_2 + 1 = 0.$$
Hence the gradients of the tangents at the ends of a focal chord are reciprocals with opposite signs, and therefore *the tangents at the ends of a focal chord are at right angles.*

*Cor.* 3. Since the directrix is the polar of the focus, tangents at the ends of a focal chord intersect on the directrix. Hence *the orthoptic locus of the parabola is the directrix.*

**Examples.**

1. Prove that the semi-latus rectum of a parabola is the harmonic mean between the two segments of any focal chord.

2. Show that the length of a focal chord of a parabola which makes an angle $\theta$ with the axis is $2l \operatorname{cosec}^2\theta$.

3. Prove the following mechanical method of drawing a parabola. A set-square $KMQ$, with right angle at $M$, slides with one edge $MK$ along a fixed line; a string of length $MQ$ has one end fixed to $Q$ and the other to a fixed point $F$. Then a pencil which keeps the string tight and presses against the edge $MQ$ will describe a parabola.

4. Show that the parameters of the ends of the latus rectum of the parabola $x = at^2$, $y = 2at$ are $\pm 1$.

5. Show that the parameters of the ends of a focal chord are $t$ and $-1/t$.

**14. The normal.** The gradient of the tangent at $(x_1, y_1)$ is $2a/y_1$. Hence the equation of the normal at $(x_1, y_1)$ is

$$y - y_1 = -\frac{y_1}{2a}(x - x_1),$$

*i.e.*

$$y_1(x - x_1) + 2a(y - y_1) = 0.$$

*Cor.* The equation of the normal at $t$ is

$$2at(x - at^2) + 2a(y - 2at) = 0,$$

*i.e.*

$$tx + y - (at^3 + 2at) = 0.$$

**15.** To find the normals which pass through a given point $(x_1, y_1)$, substitute $x_1, y_1$ for $x, y$ in the equation of the normal at $t$, and we get

$$at^3 + (2a - x_1)t - y_1 = 0.$$

This is a cubic in $t$, and its roots are the values of the parameters of the feet of the normals from $(x_1, y_1)$. Hence *three normals can in general be drawn to a parabola from a given point.*

Q. Why is it that only three normals can be drawn from a given point to a parabola, while four can be drawn to an ellipse or hyperbola? What becomes of the fourth normal?

*Cor.* 1. If $t_1, t_2, t_3$ are the roots of the above cubic in $t$, we have

$$t_1 + t_2 + t_3 = 0.$$

This is therefore the relation which must hold between the parameters of three points in order that the normals at these points may be concurrent.

*Cor.* 2. Consider a system of parallel chords. Since the locus of their mid-points is a straight line parallel to the axis, the coordinates of the extremities of any chord are connected by the relation $y_1 + y_2 = \text{constant}$, and hence if $t_1$ and $t_2$ are the parameters of the two ends of any chord of a parallel system,

$$t_1 + t_2 = \text{constant}.$$

Let the normals at the extremities of one chord meet in $P$, and let the parameter of the foot of the third normal from $P$ to the curve be $t_3$. Then $t_3 = -(t_1 + t_2) = \text{constant}$; hence the third normal is a fixed line, *i.e. the normals at the extremities of a system of parallel chords intersect upon a fixed line which is a normal to the parabola.*

**16. The evolute.** Consider two normals at points $Q$, $Q'$ very near to one another, and let them meet at $P$. As the normals tend to coincidence, $P$ tends to a limiting position which is the centre of curvature at $Q$, and the locus of the centre of curvature is the evolute.

From any point on the evolute two of the normals which can be drawn to the parabola are coincident; we may therefore write the parameters of the feet of the three normals $t, t, t'$, and $2t + t' = 0$.

The normal at the point $t$ is

$$tx + y - at(t^2 + 2) = 0,$$

and the normal at $t' = -2t$ is

$$-2tx + y + 2at(4t^2 + 2) = 0.$$

The coordinates of the point of intersection of these two lines are

$$x = a(3t^2 + 2),$$
$$y = -2at^3.$$

These are therefore the coordinates of the centre of curvature at the point $t$. Together these two equations represent freedom equations of the evolute. Eliminating $t$, we have

$$4(x - 2a)^3 = 27ay^2.$$

The evolute is therefore a curve of the third degree, symmetrical about the axis of $x$. It cuts the parabola where $x = 8a$, and it has a cusp where $x = 2a$. This curve is called the "semi-cubical parabola." A curve whose equation is of the form $y = cx^n$ is called a generalized parabola. When $n = 2$ or $\frac{1}{2}$, it is the ordinary parabola. For the semi-cubical parabola $n = \frac{3}{2}$.

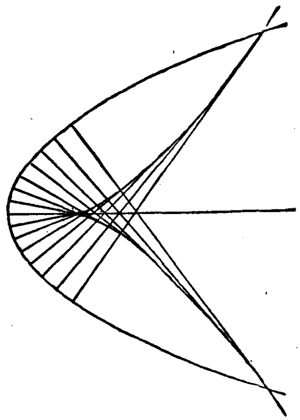


FIG. 47.

**17. Area of a parabolic segment or sector.** (1) To find the area of the segment cut off by the chord $PP'$.

Take any point $Q$ on the arc and draw a chord $PP'$. Let $L$, $M$ be the mid-points of these chords and join $LM$, cutting the curve in $A$. Then $AL$ is the diameter corresponding to the chord $PP'$, and the tangent at $A$ is parallel to $PP'$.

Take another point $Q'$ very near to $Q$ on the curve, and draw the chord $QQ'$ meeting the diameter in $T$. Draw the ordinate $Q'M'$, and complete the parallelograms $MTKQ$, $M'TK'Q'$, $K'Q$, $QM'$ and $PP'R'R$.

Ultimately when $Q'$ comes to coincide with $Q$, $QQ'$ becomes the tangent at $Q$, and $TA = AM$.

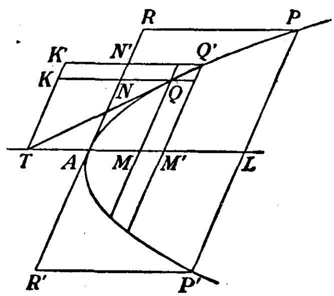



FIG. 48.

Now the parallelogram

$$QM' = K'Q = 2N'Q.$$

The areas $LAQP$ and $AQPR$ are therefore divided into small strips which are ultimately in the ratio $2 : 1$. Hence the area $LAQP = 2AQPR$, and therefore the area of the segment $P'AQP$ is $\frac{2}{3}$ of the parallelogram $P'R'RP$.

(2) To find the area of the sector bounded by a radius vector $FP$ and the radius vector $FA$.

Assuming the preceding result, we have (Fig. 46)

$$\begin{aligned}\text{Area } FAP &= LAP + FLP = \tfrac{2}{3}AL \,.\, LP + \tfrac{1}{2}LF \,.\, LP \\ &= y\{\tfrac{2}{3}x + \tfrac{1}{2}(a - x)\} = y(\tfrac{1}{6}x + \tfrac{1}{2}a) \\ &= \tfrac{1}{2} \,.\, 2at(\tfrac{1}{3}at^2 + a) \\ &= a^2(t + \tfrac{1}{3}t^3),\end{aligned}$$

where $t = \cot\psi = \tan\frac{1}{2}\theta$.

This could be obtained also directly by integrating $\frac{1}{2}\int r^2 d\theta$.

We have $$r = a\sec^2\tfrac{1}{2}\theta = a(1+t^2),$$
and $$dt = \tfrac{1}{2}\sec^2\tfrac{1}{2}\theta\, d\theta = \tfrac{1}{2}(1+t^2)\,d\theta.$$
Therefore $$\tfrac{1}{2}r^2 d\theta = a^2(1+t^2)\,dt.$$
Hence $$\text{the area} = \tfrac{1}{2}\int r^2 d\theta$$
$$= a^2\int(1+t^2)\,dt = a^2(t+\tfrac{1}{3}t^3).$$

## EXAMPLES VI. A.

1. $OP$, $OQ$ are tangents to a parabola, and $PQ$ cuts the axis in $K$; prove that the distance of $K$ from the focus is equal to the distance of $O$ from the directrix. (Corpus, 1911.)

2. Find the equation of the chord of the parabola $y^2=8x$ which is bisected at the point $(2, -3)$.

3. Two perpendicular focal chords of a parabola meet the directrix in $T$ and $T'$ respectively; show that the tangents to the parabola, which are parallel to these chords, intersect in the middle point of $TT'$. (Math. Tripos I., 1915.)

4. Prove that the locus of the mid-points of all chords of the parabola $y^2=4ax$ which pass through the point $(x', y')$ is the parabola $2a(x-x')=y^2-yy'$.

5. Two parabolas have the same directrix and the same axis, intersecting at $X$. The vertex of one is $A$ and the vertex of the other is at the focus of the first. The parabolas intersect at $P$, and $PM$ is drawn perpendicular to the directrix. Show that $AP^2=4AX\,.\,AM$. (St. Catharine's, 1907.)

6. The chain of a suspension bridge hangs in the form of a parabola whose axis is vertical and vertex downwards. In the Menai Suspension Bridge the chain hangs symmetrically with a span of 570 feet, and the dip is 43 feet. Find the latus rectum of the parabola, and the inclination to the horizon at each end of the chain.

7. In the Tower Bridge of London each of the side spans is a suspension bridge whose span is 285 feet; the height at the shore end is 34 feet and at the river end 136 feet, measured from the lowest point. Find the position of the vertex of the parabola, the distance of the directrix below the vertex, and the inclinations to the horizon at each end of the chain.

8. When a stone is thrown it describes a parabola whose axis is vertical and vertex upwards. A stone is thrown from the ground so as just to clear two fences, the first at a horizontal distance of 10 feet and 5 feet high, the second 8 feet high and at a distance of 24 feet. Find the inclination to the ground at which the stone must be thrown, and the point at which it will reach the ground.

9. An airplane when moving horizontally at an altitude of 2000 feet drops a bomb which reaches the ground at a point 500 feet away from the point which was vertically beneath the airplane when the bomb was dropped. Find at what angle it strikes the ground.

10. Show that any circle whose diameter is a focal chord of a parabola touches the directrix.

11. Prove that if the chord of a parabola $x=at^2$, $y=2at$, whose ends have parameters $n$, $n'$, passes through the fixed point $\{a(m^2+4), -2am\}$, the chord joining the points whose parameters are $m$, $n$ is perpendicular to the chord joining the points $m$, $n'$. What conclusion can you draw by making the point $n$ move up to $m$? (Peterhouse, etc., 1914.)

12. Prove that the lines joining any point on a parabola to the two ends of any double ordinate cut the axis at points equidistant from the vertex.

13. If four concurrent lines cut a parabola in $P, P'$; $Q, Q'$; $R, R'$; $S, S'$; prove that the cross-ratio of the parameters of $PQRS$ is equal to that of $P'Q'R'S'$.

14. Show that the chord joining the points $t$ and $-1/t$ of the parabola $x=at^2$, $y=2at$ passes through the focus.

15. Two equal parabolas have the same vertex, and their axes at right angles. Show that they cut again at an angle $\tan^{-1}\frac{3}{4}$.

16. Two parabolas have the same focus and axis, but their vertices in opposite directions. Prove that they cut at right angles.

17. Two tangents to the parabola $y^2=4ax$ make with each other an angle 60°; show that the locus of their intersection is one branch of a hyperbola. What locus does the other branch represent?

18. A parabola rolls symmetrically on an equal parabola. Find the locus of the focus.

19. Prove that a circle cuts a parabola in four points whose centroid lies on the axis.

20. Prove that the centroid of the feet of the three normals which can be drawn from any point to a parabola lies on the axis.

21. Show that the feet of the three normals that can be drawn from any point to a parabola lie on a circle which passes through the vertex.

22. $P$ is any point on a parabola whose vertex is $A$, and $Q$, $R$ are the feet of the normals from $P$ to the curve. Show that $QR$ passes through a fixed point and that $AP$ and $QR$ meet on a fixed line. (Math. Tripos I., 1909.)

23. Normals to the parabola $y^2=4ax$ at the points $P_1, P_2, P_3$ meet in the point $(h, k)$; find the coordinates of the centroid of the triangle $P_1P_2P_3$.

24. If two equal parabolas have a common focus, show that their common chord passes through the focus and bisects the angle between the axes.

25. Prove that the angles of intersection of two equal parabolas having a common focus are supplementary, and that one of them is equal to half the angle between the axes.

26. Prove that the area of a segment of a parabola cut off by a focal chord which makes an angle $\alpha$ with the axis is $\frac{8}{3}a^2 \operatorname{cosec}^3 \alpha$.

27. Two equal parabolas of latus rectum $4a$ have a common focus. Show that if $\alpha$ is the inclination of their axes, the area common to both is $\frac{16}{3}a^2 \operatorname{cosec}^3 \frac{1}{2}\alpha$.

28. A circle whose centre is at the focus of a parabola whose latus rectum is $4a$ cuts the parabola at an acute angle $\alpha$. Show that the area of the crescent-shaped portion between the curves is $2a^2(\alpha \sec^4 \alpha - \tan \alpha - \frac{1}{3}\tan^3 \alpha)$.

29. A variable chord of a given conic subtends a right angle at a focus. Find the position of the point of contact of the chord with its envelope, and show that the envelope is a parabola.

30. At the point of intersection of the rectangular hyperbola $xy=k^2$ and the parabola $y^2=4ax$ the tangents to the hyperbola and the parabola make angles $\theta$ and $\phi$ respectively with the axis of $x$. Prove that $\tan\theta=-2\tan\phi$.

(Downing, 1911.)

31. The polar of the point $P$ with respect to the parabola $y^2=4ax$ meets the curve in $Q$, $R$. Show that, if $P$ lies on the straight line $lx+my+n=0$, then the middle point of $QR$ lies on the parabola $l(y^2-4ax)+2a(lx+my+n)=0$.

(Math. Tripos I., 1911.)

32. If the line $lx+my+na=0$ meets $y^2=4ax$ in $P$ and $Q$, and if the lines joining $P, Q$ to the focus meet the parabola in $T$, $U$, show that the equation of $TU$ is $nx-my+la=0$. (Trinity, etc., 1900.)

33. The tangents drawn from a point $P$ to a parabola whose focus is $F$ touch it at $Q$ and $Q'$; prove that $FP^2=FQ\,.\,FQ'$.

34. The three points $(x_1, y_1)$, $(x_2, y_2)$, $(x_3, y_3)$ form a triangle self-conjugate for the parabola $y^2=4ax$. Prove that the area of the triangle is

$$\tfrac{1}{4}(y_2-y_3)(y_3-y_1)(y_1-y_2)/a.$$

(Corpus, 1907.)

## EXAMPLES VI. B.

1. Triangles are inscribed in the parabola $y^2=4ax$, each having its centroid on the line $x=c$. Prove that the tangents at the angular points form triangles whose centroids lie on a parabola of latus rectum $\frac{8}{3}a$. (Peterhouse, etc., 1900.)

2. From a point $M$ on the axis of a parabola normals $MP$, $MP'$ are drawn to the curve. Show that the circle circumscribed to the triangle formed by the tangents at $P$ and $P'$ and the tangent at the vertex subtends at $M$ an angle $2\sin^{-1}\frac{1}{3}$. (Trinity, etc., 1901.)

3. Four fixed tangents $T_1$, $T_2$, $T_3$, $T_4$ to a parabola intersect in the six points $P_{12}$, $P_{13}$, .... From these points perpendiculars $p_{12}$, $p_{13}$, ... are drawn to any other tangent. Prove that

$$p_{12}\cdot p_{34}=p_{13}\cdot p_{24}=p_{14}\cdot p_{23}.$$ (Trinity, etc., 1901.)

4. Prove that a circle whose diameter is a chord of a parabola, such that the distance between the diameters through its extremities is double the length of the latus rectum, will touch the parabola. (Trinity, etc., 1906.)

5. Prove that the length of the chord through a point $P$ on the parabola $y^2=4ax$ drawn in a direction making an angle $\theta$ with the axis of $x$ is equal to $4a\sin(\alpha-\theta)\operatorname{cosec}^2\theta\operatorname{cosec}\alpha$, where $\alpha$ is the inclination of the tangent at $P$ to the axis of $x$. (Math. Tripos I., 1913.)

6. $PFQ$ is a focal chord of a parabola; circles are described through the focus $F$ to touch the parabola at $P$ and $Q$ respectively; prove that these circles cut orthogonally. (Corpus, etc., 1912.)

7. Any circle is described through the focus of a parabola so as to touch the curve at one point and meet it again in $P$ and $Q$. Prove that the locus of the intersection of the normals at $P$ and $Q$ is a parabola. (St. Catharine's, 1912.)

8. $CP$, $CP'$ are two fixed tangents to a parabola drawn from a point $C$ on the axis. A variable tangent cuts them in $Q$ and $Q'$. Prove that either the sum or the difference of the segments $CQ$, $CQ'$ is constant.

9. The tangent at any point $P$ of the parabola $y^2=4ax$ is met in $Q$ by a line through the vertex $A$ at right angles to $AP$, and $Z$ is the foot of the perpendicular from $A$ on the tangent at $P$. Show that there are three positions of the point $P$ on the parabola for which the point $Z$ lies on the straight line $lx+my+na=0$, and that the corresponding points $Q$ lie on the line $(2l-n)x+4my+2na=0$.
(Pembroke, etc., 1910.)

10. A point $P$ moves in a plane in such a way that $VP-SP=SK$, where $S$ is a fixed point in the plane, $V$ a fixed point not in the plane, and $K$ the foot of the perpendicular from $V$ on the plane. Prove that the locus of $P$ is a parabola with focus at $S$; and that, if the vertex of this parabola is $A$, the point $V$ lies on a second equal parabola with vertex at $S$ and focus at $A$.
(Pembroke, etc., 1910.)

11. A sphere rolls on a parabolic wire, with which it is in contact at two points; show that the locus of the centre of the sphere is an equal parabola.
(King's, etc., 1913.)

12. $PQ$ is a chord of a parabola normal at $P$; a circle described on $PQ$ as diameter cuts the parabola again in $R$. Prove that the projection of $QR$ on the axis is twice the latus rectum. (Queens', 1911.)

13. Prove that if $a > b > 0$ and $c > 2(a-b)$, the two parabolas $y^2=4a(x+c)$, $y^2=4bx$ have a pair of common normals, inclined to the common axis, and that the distance $d$ between the curves, measured along one of these common normals, is given by $d^2=4(a-b)(c-a+b)$. (Pembroke, etc., 1909.)

14. If from a given point $P$ three normals be drawn to any parabola having the same focus and its axis in the same direction as that of a given parabola, prove that the sum of the angles which the normals make with the axis is constant.
(Queens', 1900.)

15. A fixed line meets the directrix of a given parabola in $F$ and the polar of $F$ in $F'$; pairs of points are taken on the line such that the pairs of tangents from them contain equal or supplementary angles; prove that they divide $FF'$ harmonically. (Pembroke, etc., 1900.)

16. A parabola whose axis is along the axis of $x$ intersects the ellipse $x^2/a^2+y^2/b^2=1$ orthogonally at the point whose eccentric angle is $\varphi$. Show that the latus rectum of the parabola is $2a \sin\varphi \tan\varphi$. (Pembroke, etc., 1901.)

17. $CP$ and $CD$ are two conjugate semi-diameters of an ellipse, lying on the same side of the major axis, and a parabola is drawn through $P$ and $D$ with its axis along the major axis of the ellipse. Show that the ratio of the latera recta of the parabola and the ellipse is equal to $FP \sim F'D : FF'$, $F$ and $F'$ being the foci of the ellipse. (Selwyn, 1907.)

18. Prove that the point of intersection of the normals to the parabola $y^2=4ax$ at its points of intersection with the line $lx+my+n=0$ are given by $l^2x=2al^2+4am^2-nl$, $l^2y=2mn$.

19. A chord of a parabola passes through a fixed point; prove that the normals at its extremities meet on another fixed parabola.

20. Prove that the normals at the ends of a focal chord of the parabola $y^2=4ax$ intersect on the parabola $y^2=ax-3a^2$.

21. Show that $a+2a\cos\theta$, $2a\sin\theta$ are the coordinates of a point on the circle $x^2+y^2-2ax-3a^2=0$, and deduce that the polar of any point on this circle, with respect to the circle $x^2+y^2+2ax-3a^2=0$, touches the parabola $y^2+4ax=0$.

22. Show that the polar of any point on the parabola $y^2=4ax$, with respect to the ellipse $x^2/\alpha+y^2/\beta=1$, touches a parabola whose latus rectum is $\beta^2/(a\alpha)$.

23. Prove that the polar, with respect to the parabola $y^2-x=0$, of any point on the hyperbola $2y^2-x^2=1$, touches this hyperbola.

24. $PQ$ is a focal chord of an ellipse, of which $H$ is the other focus, and the circle $PQH$ cuts the ellipse again in $U$, $V$. Show that, as $PQ$ varies, $UV$ touches a fixed parabola. (Math. Tripos I., 1909.)

25. Show that if the middle point of a chord of a parabola lies on a fixed straight line, then the chord is either in a fixed direction or touches another fixed parabola. (Math. Tripos II., 1914.)

26. Show that the common chord of a parabola and its circle of curvature at any point constantly touches another parabola having the same vertex. (Pembroke, etc., 1901.)

27. A dot is made on a sheet of paper, which has a straight edge. The paper is folded so that this edge always passes through the dot. Show that the crease touches a fixed parabola.

28. If in the last example the sheet of paper is circular, show that the crease will envelop an ellipse, whose foci are the fixed point and the centre of the circle, and major axis half the diameter of the circle. (Ferguson Schol., 1901.)

29. The chords $PQ$, $PR$ of the parabola $y^2=4ax$ pass respectively through the points $(0, 4a)$, $(0, -4a)$. Prove that $QR$ envelops the circle $x^2+y^2=4ax$. (Trinity, etc., 1911.)

30. Through a fixed point on the axis of a parabola any line is drawn cutting the curve in the points $P, Q$, and the circle through $P, Q$ and the focus $F$ cuts the parabola again in the points $P', Q'$. Prove that $P'Q'$ envelops another parabola whose focus is $F$. (Trinity, etc., 1909.)

31. Show that the pole of a chord of a given conic which subtends a constant angle at a focus lies on another conic having the same focus and directrix. (Selwyn, 1914.)

32. Prove that a chord of a conic which subtends a constant angle at one focus envelops another conic with the same focus.

33. Conics are described having the same focus, and their major axes equal $(2a)$ and in the same straight line. Prove that the tangents at the ends of the latera recta through the common focus touch a parabola of latus rectum $4a$. (Corpus, 1907.)

34. Two parabolas have a common focus and axes inclined at an angle $\alpha$. Prove that the locus of the intersection of two perpendicular tangents, one to each of the parabolas, is a conic. (St. Catharine's, 1899.)

35. Normals are drawn from any point $(x, y)$ to the parabola $y^2=4ax$. Prove that $\rho_1$, $\rho_2$, $\rho_3$, the radii of curvature at the feet of the normals, satisfy the equation

$$\rho_1^{\frac{2}{3}}+\rho_2^{\frac{2}{3}}+\rho_3^{\frac{2}{3}}=4^{\frac{1}{3}}a^{-\frac{1}{3}}(2x-a).$$ (Trinity, 1902.)

36. Prove that if $\rho_1$, $\rho_2$, $\rho_3$ are the radii of curvature of a parabola, with focus $F$, at the feet of the normals from a point $P$, $\rho_1\rho_2\rho_3=8FP^3$. (Corpus, etc., 1912.)

37. Through the vertex $A$ of a parabola a line $AY$ is drawn perpendicular to the tangent at $P$ meeting the ordinate $PN$ in $K$. Show that $KN$ is half the distance of the centre of curvature at $P$ from the axis of the parabola. (St. Catharine's, 1914.)

**38.** The normal at a point $P$ of a parabola touches the evolute at $Q$, and $R$ is the centre of curvature of the evolute at $Q$. Prove that the straight line $PR$ makes with the axis an angle $\cot^{-1}(\cot\psi + \cot 2\psi)$, where $\psi$ is the inclination of the tangent at $P$ to the axis. (Selwyn, 1914.)

**39.** Find the coordinates of the point of intersection of the parabola $y^2 = 4ax$ with a circle of radius $b$ which touches it at the vertex, the centre of the circle being within the parabola. What happens when $b$ is equal to, and less than, $2a$ respectively ?

Prove that if $a=1$ inch and $b=2{\cdot}2$ inches, the maximum distance (measured parallel to the axis of the parabola) between the nearly coincident portions of the two curves between their points of intersection is $\cdot 01$ inch. (Downing, 1911.

**40.** Tangents $OP$, $OP'$ are drawn from the point $O(x', y')$ to the parabola $y^2 = 4ax$, and $F$ is the focus. Prove that $OP.\,OP'/FO = (y'^2 - 4ax')/a$. (Selwyn, 1907.)

**41.** Through the extremities $Q$, $Q'$ of a focal chord of a parabola the double ordinates $QR$, $Q'R'$ are drawn to cut the axis in $N$ and $N'$. Chords are drawn through $R$ and $R'$ parallel to $QQ'$ to meet the diameter which bisects $QQ'$ in $M$ and $M'$; prove that

$$RM^2 - R'M'^2 = 2QQ'.\,NN'.$$ (Selwyn, 1913.)

# CHAPTER VII.

## SYSTEMS OF CIRCLES.

**1. Radical axis of two circles.** Consider two circles

$$S \equiv x^2 + y^2 - 2\alpha x \ - 2\beta y \ + c \ = 0,$$

$$S' \equiv x^2 + y^2 - 2\alpha' x - 2\beta' y + c' = 0.$$

The expression denoted by $S$ represents the power of the point $(x, y)$ with respect to the circle, and is equal to the square of the tangent from $(x, y)$ to the circle.

The locus of a point which has the same power with respect to the two circles is therefore
$$S = S',$$

*i.e.*
$$2(\alpha - \alpha')x + 2(\beta - \beta')y + c' - c = 0.$$

As this equation is of the first degree and is satisfied by values of $x$ and $y$ which satisfy both $S=0$ and $S'=0$, it represents a straight line passing through the points of intersection of the two circles. This line exists whether the circles cut in real points or in imaginary points, and is called the *radical axis* of the two circles. When the two circles intersect in real points, the radical axis is the line of their common chord ; when the points of intersection are imaginary, the radical axis lies entirely outside both circles. In either case there are points of the radical axis which lie outside both circles, and from these points the tangents to the two circles are equal. The radical axis could therefore be constructed by drawing any two common tangents and bisecting the segments between the points of contact. The join of these mid-points is the radical axis.

*The radical axis is perpendicular to the line of centres*, for the gradient of the line joining the centres is $(\beta - \beta')/(\alpha - \alpha')$, while the gradient of the radical axis is $-(\alpha - \alpha')/(\beta - \beta')$.

**2. The radical axes of three circles taken in pairs are concurrent.** If $S_1$, $S_2$, $S_3$ are the three circles, the equations of the radical axes are

$$S_2 - S_3 = 0, \quad S_3 - S_1 = 0, \quad S_1 - S_2 = 0.$$

Since the sum of the three expressions on the left vanishes identically, these three lines are concurrent. The point of concurrence is called the *radical centre* of the three circles. The radical centre has the property that the tangents drawn from it to each of the three circles are all equal. If the three circles overlap and enclose a common area, the radical centre lies within this common area, and real tangents cannot be drawn from the radical centre to the three circles.

**Examples.**

1. If two of the radical axes of three circles taken in pairs are coincident, prove that all three radical axes coincide.

2. If two of the radical centres of four circles, taken in threes, coincide, and no three of the circles have a common radical axis, prove that all four radical centres coincide.

**3. Pencil of circles.** *If $S$ and $S'$ are two given circles the equation $S+\lambda S'=0$ represents a circle passing through the points of intersection of the two circles.*

Writing the equation in full, we have

$$x^2+y^2-2\alpha x-2\beta y+c+\lambda(x^2+y^2-2\alpha' x-2\beta' y+c')=0,$$

or

$$(1+\lambda)(x^2+y^2)-2(\alpha+\lambda\alpha')x-2(\beta+\lambda\beta')y+c+\lambda c'=0,$$

which represents a circle with centre

$$\frac{\alpha+\lambda\alpha'}{1+\lambda},\quad \frac{\beta+\lambda\beta'}{1+\lambda}.$$

Further, the coordinates of any point which satisfy the two equations $S=0$ and $S'=0$ simultaneously will also satisfy the equation $S+\lambda S'=0$. Hence the circle passes through the common points of the two given circles.

By giving different values to the parameter $\lambda$, we obtain then a system of circles all passing through two fixed points. This is analogous to the equation $u+\lambda v=0$, in which $u$ and $v$ represent straight lines, and which represents a pencil of lines through a fixed point. The system of circles represented by the equation $S+\lambda S'=0$ is analogously called a pencil of circles.

**4. Coaxal circles.** The radical axis of the two circles $S$ and $S'$ is represented by the equation

$$S-S'=0.$$

This is just a particular case of the general equation $S+\lambda S'=0$ for $\lambda=-1$, and therefore we must regard the radical axis as a degenerate circle of the pencil.

Now take any two circles of the pencil

$$S+\lambda S'=0 \quad\text{and}\quad S+\mu S'=0,$$

or, written in full,

$$S+\lambda S'\equiv(1+\lambda)(x^2+y^2)-2(\alpha+\lambda\alpha')x-2(\beta+\lambda\beta')y+(c+\lambda c')=0,$$

$$S+\mu S'\equiv(1+\mu)(x^2+y^2)-2(\alpha+\mu\alpha')x-2(\beta+\mu\beta')y+(c+\mu c')=0.$$

To find the radical axis of these we have to eliminate $x^2+y^2$. Multiply the first equation by $1+\mu$ and the second by $1+\lambda$, and subtract. We get then

$$(1+\mu)(S+\lambda S')-(1+\lambda)(S+\mu S')=(\mu-\lambda)(S-S')=0.$$

The equation of the radical axis is therefore again $S-S'=0$.

Hence *every pair of circles of a pencil have the same radical axis.* For this reason a pencil of circles is called a *system of coaxal circles* or a *coaxal system of circles.*

**5. Reduction of the equation of a coaxal system to the simplest form.** To find the equation of a system of coaxal circles in the simplest form take the axis of $y$ as the common radical axis; then, if

$$S \equiv x^2 + y^2 - 2\alpha x - 2\beta y + c = 0,$$
$$S' \equiv x^2 + y^2 - 2\alpha' x - 2\beta' y + c' = 0,$$

the equation of the radical axis is

$$2(\alpha - \alpha')x + 2(\beta - \beta')y = c - c'.$$

But this is to reduce to $x = 0$; hence

$$\beta = \beta' \quad \text{and} \quad c = c'.$$

The first condition shows that the centres of all the circles lie on the straight line $y = \beta$ perpendicular to the radical axis. Take this line as the axis of $x$; then $\beta = 0 = \beta'$. The second equation of condition shows that the constant term is the same for all the circles. Hence only the coefficient of $x$ can vary, and the equation of the system reduces to

$$x^2 + y^2 - 2\lambda x + c = 0,$$

where $\lambda$ is a variable parameter, and $c$ is a constant.

**Examples.**

1. Prove that through any point there passes one, and only one, circle of a given coaxal system.

2. Prove that in any coaxal system there are two circles, real, coincident, or imaginary, which touch a given straight line.

3. Interpret geometrically the analytical condition that the two circles of Ex. 2 should be coincident.

4. If the common points of a coaxal system are imaginary, prove that the two circles which touch a given straight line are always real.

**6. Types of coaxal systems.** Every circle of the system

$$x^2 + y^2 - 2\lambda x + c = 0$$

cuts the radical axis, $x = 0$, where

$$y^2 + c = 0.$$

There are two distinct cases according as $c$ is positive or negative.

(1) Let $c$ be negative and equal to $-k^2$. Then every circle of the system cuts the radical axis in the fixed points $(0, k)$, $(0, -k)$. These are common points of the system. Writing the equation

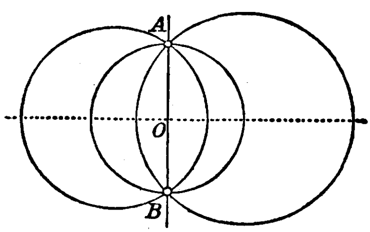


FIG. 49

$$(x - \lambda)^2 + y^2 = \lambda^2 + k^2,$$

we see that the least circle of the system has its centre at the origin and its radius equal to $k$.

(2) Let $c$ be positive and equal to $k^2$. Then no circle of the system cuts the axis of $y$. The equation becomes

$$(x-\lambda)^2+y^2=\lambda^2-k^2.$$

$\lambda$ must lie between $-k$ and $+k$, and as $\lambda$ approaches either of these values the radius tends to zero. We therefore get two point-circles belonging to the system, the points $(\pm k, 0)$. These are called *limiting-points.* In this case the common points are imaginary. In case (1) the common points are real and the limiting-points are imaginary.

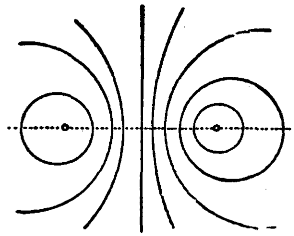


FIG. 50.

(3) The case where $c=0$ is intermediate between these two cases, and is a limiting case of each. The common points coincide and the limiting points coincide, and all the circles touch the axis of $y$ at the origin. The least circle of the system is a point circle at the common point of contact.

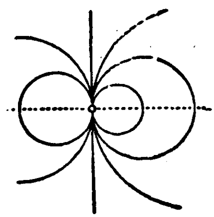


FIG. 51.

A coaxal system assumes special forms when one or both of the common points or limiting points becomes a point at infinity.

(4) Let one of the common points $B$ become a point at infinity; then the line of centres becomes the line at infinity, and the circles reduce to a pencil of lines through $A$. The radical axis becomes indeterminate, coinciding with any one of the lines.

(5) Let one of the limiting points become a point at infinity; then the radical axis becomes the line at infinity, and the circles become concentric, with the remaining point as common centre.

(6) Let the second common point also become a point at infinity; then the circles reduce to a system of parallel straight lines. The common points must be regarded as coincident points at infinity, as the system is the limiting case of a common-tangent system when the common point of contact is a point at infinity. The limiting points also coincide with the same point at infinity.

7. *If a circle cuts two circles of a coaxal system orthogonally it will cut them all orthogonally.* Let the circle

$$x^2+y^2-2Ax-2By+C=0$$

cut orthogonally each of the circles

$$S\equiv x^2+y^2-2\alpha x-2\beta y+c=0,$$

$$S'\equiv x^2+y^2-2\alpha' x-2\beta' y+c'=0,$$

so that
$$2A\alpha+2B\beta=C+c,$$

and
$$2A\alpha'+2B\beta'=C+c'.$$

Then the condition that it should cut orthogonally the circle $S+\lambda S'=0$ is

$$2A\frac{\alpha+\lambda\alpha'}{1+\lambda}+2B\frac{\beta+\lambda\beta'}{1+\lambda}=C+\frac{c+\lambda c'}{1+\lambda},$$

*i.e.* $$(2A\alpha+2B\beta-C-c)+\lambda(2A\alpha'+2B\beta'-C-c')=0.$$

But each of these terms vanishes by the given conditions; hence the condition is satisfied for all values of $\lambda$.

**8. Conjugate systems of coaxal circles.** Consider the coaxal system

$$x^2+y^2-2\lambda x+c=0.$$

Then if the circle $$x^2+y^2-2Ax-2By+C=0$$

cuts every circle of this system orthogonally, we have for all values of $\lambda$,

$$\lambda A-c-C\equiv 0;$$

hence $$A=0 \quad\text{and}\quad C=-c.$$

$B$ is then quite undetermined, and we find that every circle of the system

$$x^2+y^2-2\mu y-c=0,$$

where $\mu$ is variable and $c$ is constant, cuts orthogonally every circle of the system $$x^2+y^2-2\lambda x+c=0.$$

But the former equation is of exactly the same type as the latter, with $x$ and $y$ interchanged. It therefore represents a system of coaxal circles whose axis is the axis of $x$.

Let us assume that the given system has real common points, so that $c$ is negative, $=-k^2$, and the common points are $(0, \pm k)$. Then the other system cuts its axis where $x^2+k^2=0$, *i.e.* in imaginary points; its limiting points, the point-circles, are found by writing the equation in the form

$$x^2+(y-\mu)^2=\mu^2-k^2,$$

giving $\mu=\pm k$. Hence the limiting points are $(0, \pm k)$, and coincide with the common points of the other system. In the same way the imaginary limiting points $(\pm ik, 0)$ of the given system coincide with the imaginary common points of the other system.

Two such systems of coaxal circles are called *conjugate systems*.

If $k=0$, we have two conjugate systems, one touching the axis of $x$ at the origin, and the other touching the axis of $y$ at the origin, their equations being

$$x^2+y^2-2\lambda x=0,$$
$$x^2+y^2-2\mu y=0.$$

It is obvious in this case that every circle of the one system cuts every circle of the other system orthogonally, since they cut at right angles at the origin.

It may be noticed also that a system of concentric circles and a pencil of lines through their common centre are conjugate systems, and also, as a still more special case, two systems of parallel lines cutting at right angles.

**9. Orthotomic circle of three given circles.** Given three circles, to construct a circle cutting each of them orthogonally. Let $O$ be the radical centre of the three circles. Then, since the tangents from $O$ to the three circles are equal, a circle with centre $O$ and radius equal to the tangent will cut all three circles orthogonally. If the circles overlap so as to have an area in common, the radius of the orthotomic circle is imaginary; if they pass through one point the radius is zero, and the orthotomic circle is a point-circle.

**10.** It is easily proved geometrically that the ends of a diameter of any circle are conjugate points with regard to a circle which cuts the first circle orthogonally. Hence, if $P$ is any point on the orthotomic circle of three given circles, its polars with regard to the three circles will all pass through the other end of the diameter through $P$ of the orthotomic circle, and are therefore concurrent.

This property can be used to find the equation of the orthotomic circle of three given circles.

The equation of the polar of the point $(x', y')$ with regard to the circle

$$x^2+y^2-2\alpha_1 x-2\beta_1 y+c_1=0$$

is

$$x(x'-\alpha_1)+y(y'-\beta_1)-\alpha_1 x'-\beta_1 y'+c_1=0.$$

Writing down the equations of the polars with regard to the other two circles and eliminating $x$ and $y$, we have as the equation of the locus of $(x', y')$,

$$\begin{vmatrix} x-\alpha_1 & y-\beta_1 & \alpha_1 x+\beta_1 y-c_1 \\ x-\alpha_2 & y-\beta_2 & \alpha_2 x+\beta_2 y-c_2 \\ x-\alpha_3 & y-\beta_3 & \alpha_3 x+\beta_3 y-c_3 \end{vmatrix}=0$$

This may be written in another and more general form. Make the equations of the circles homogeneous, so that

$$S_1\equiv x^2+y^2-2\alpha_1 xz-2\beta_1 yz+c_1 z^2;$$

then $\quad x-\alpha_1 z=\frac{1}{2}\frac{\partial S_1}{\partial x},\quad y-\beta_1 z=\frac{1}{2}\frac{\partial S_1}{\partial y},\quad -\alpha_1 x-\beta_1 y+c_1 z=\frac{1}{2}\frac{\partial S_1}{\partial z}.$

Hence the equation of the orthotomic circle can be written

$$\begin{vmatrix} \frac{\partial S_1}{\partial x} & \frac{\partial S_1}{\partial y} & \frac{\partial S_1}{\partial z} \\ \frac{\partial S_2}{\partial x} & \frac{\partial S_2}{\partial y} & \frac{\partial S_2}{\partial z} \\ \frac{\partial S_3}{\partial x} & \frac{\partial S_3}{\partial y} & \frac{\partial S_3}{\partial z} \end{vmatrix}=0.$$

This determinant is called the *Jacobian* of the three functions $S_1$, $S_2$, $S_3$, and is usually written

$$\frac{\partial(S_1, S_2, S_3)}{\partial(x, y, z)}.$$

Q. Since the Jacobian is of the third degree, the locus should be a curve of the third degree. Explain this.

**11. Net of circles.** If $S_1$, $S_2$, $S_3$ represent any three circles, whose centres are not collinear, the equation

$$S_1 + \lambda S_2 + \mu S_3 = 0, \qquad \text{......................................(1)}$$

where $\lambda$, $\mu$ are variable parameters, represents a circle which belongs to a system with two degrees of freedom. This is called a *net* of circles (sometimes also a *bundle*).

Let the origin be chosen as the radical centre of the three circles

$$S_1 \equiv x^2 + y^2 - 2\alpha_1 x - 2\beta_1 y + c_1 = 0,$$

etc. Then, since the radical axis of $S_1$ and $S_2$ passes through $O$, therefore $c_1 = c_2$; and similarly $c_1 = c_3 = k$, say. Then the equation (1) becomes

$$x^2 + y^2 - 2\,\frac{\alpha_1 + \lambda\alpha_2 + \mu\alpha_3}{1 + \lambda + \mu}\,x - 2\,\frac{\beta_1 + \lambda\beta_2 + \mu\beta_3}{1 + \lambda + \mu}\,y + k = 0. \qquad \text{...... .....(2)}$$

Hence the radical axis of every pair of circles of the system passes through the origin, and therefore the origin is the radical centre of every set of three circles, *i.e. a net of circles is a system of circles with a common radical centre.*

Equation (2), which represents any circle of the system, is of the form

$$x^2 + y^2 - 2\alpha x - 2\beta y + k = 0,$$

where $k$ is constant, and $\alpha$, $\beta$ can be regarded as the variable parameters. The square of the radius of this circle is $\alpha^2 + \beta^2 - k$; hence *the locus of the centres of point-circles of the net is the circle*

$$x^2 + y^2 = k,$$

and it is easily verified that *this circle cuts orthogonally every circle of the net.*

If $k > 0$, the orthotomic circle is real.

If $k = 0$, the orthotomic circle is a point-circle, and all the circles pass through this common point.

If $k < 0$, the orthotomic circle is virtual. In this case there is a real circle $x^2 + y^2 + k = 0$, which is cut by every circle of the system in points which lie on a line $\alpha x + \beta y = 0$, passing through its centre, *i.e. every circle of the system cuts this fixed circle at ends of a diameter.*

**Examples.**

1. Prove that all the circles of a net which have their centres on a given straight line form a pencil of circles.

2. Prove that all circles which cut a given circle orthogonally form a net.

**12.** The pencil and the net of circles are particular cases of systems of circles. A circle depends upon three constants or parameters, $\alpha$, $\beta$ and $c$, and if these are quite independent it has three degrees of freedom. One condition, which leaves it with two degrees of freedom, can be represented by an equation involving $\alpha$, $\beta$ and $c$. The simplest equation is one of the first degree or a linear equation, say

$$A\alpha + B\beta + Cc + D = 0.$$

This then represents a *linear system* of circles with two degrees of freedom. Now this equation expresses that the given circle cuts orthogonally the circle

$$C(x^2 + y^2) + Ax + By + D = 0.$$

Hence the linear system is a net, or system with a common radical centre.

Again, two linear equations in $\alpha$, $\beta$, $c$ represent a linear system with one degree of freedom. This is a system cutting two fixed circles orthogonally, and is therefore a coaxal system or pencil.

*Cor.* Two nets of circles have in common a system of coaxal circles, the radical axis being the line joining the radical centres of the two nets. The conjugate coaxal system is determined by the orthotomic circles of the two nets.

## INVERSION.

**13.** Consider a fixed circle with centre $O$ and radius $k$, and any point $P$. The point $P'$ on $OP$, such that

$$OP \,.\, OP' = k^2,$$

is called the *inverse* of $P$ with respect to the fixed circle. The fixed circle is called the *circle of inversion*, its centre is the *centre of inversion*, $k^2$ is called the *constant of inversion*, and $k$ is the *radius of inversion.*

The relation between $P$ and $P'$ is symmetrical, so that $P$ is the inverse of $P'$.

The constant $k^2$ being positive, the points $P$, $P'$ lie on the same side of $O$. If $k^2$ were negative, $P$ and $P'$ would be on opposite sides of $O$. The circle of inversion in this case has an imaginary radius, though its centre is still real, and it is a virtual circle. We can avoid having to deal with a virtual circle of inversion by the following process. Let $P'$ and $P''$ be the inverse points of $P$ with respect to the positive and negative constants $k^2$ and $-k^2$ respectively, and the same centre of inversion $O$; so that

$$OP \,.\, OP' = k^2, \quad OP \,.\, OP'' = -k^2.$$

Then $OP'' = -OP'$. Hence we can obtain the inverse point $P''$ for the negative constant by finding first the inverse point $P'$ for the positive constant, and then rotating through two right angles about $O$.

**14. Formulae for inversion.** To find the coordinates of the inverse of $P \equiv (x, y)$ with regard to the circle

$$x^2 + y^2 = k^2.$$

Let the coordinates of $P'$ be $(x', y')$.

Then
$$\frac{x'}{x} = \frac{y'}{y} = \frac{OP'}{OP} = \frac{OP' \,.\, OP}{OP^2} = \frac{OP'^2}{OP \,.\, OP'}$$

$$= \frac{k^2}{x^2 + y^2} = \frac{x'^2 + y'^2}{k^2},$$

*i.e.*
$$x' = \frac{k^2 x}{x^2 + y^2}, \quad y' = \frac{k^2 y}{x^2 + y^2}.$$

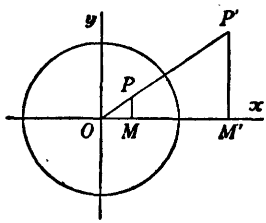



FIG. 52.

Using polar coordinates, with $O$ as pole, the equations of transformation are

$$\theta' = \theta, \quad r' = \frac{k^2}{r}.$$

The points $P$ and $P'$ will coincide if $x'=x$ and $y'=y$; then

$$x^2+y^2=k^2.$$

Hence the circle of inversion is the *locus of self-corresponding points.*

**15. Inverse of a curve.** If the coordinates $x$, $y$ are connected by any equation so that $P$ lies on a certain curve, the coordinates $x'$, $y'$ of $P'$ will be connected by a corresponding equation found by substituting

$$x=\frac{k^2x'}{x'^2+y'^2},\quad y=\frac{k^2y'}{x'^2+y'^2}.$$

The locus of the resulting equation is the inverse of the locus of $P$.

*The inverse of a straight line is a circle.*

The straight line $$lx+my+n=0$$

gives $$l\frac{k^2x'}{x'^2+y'^2}+m\frac{k^2y'}{x'^2+y'^2}+n=0,$$

or, multiplying up and dropping the dashes,

$$n(x^2+y^2)+k^2lx+k^2my=0,$$

which represents a circle passing through $O$, and having the tangent at $O$ parallel to the given straight line.

If the straight line passes through $O$, so that $n=0$, its inverse becomes $lx+my=0$, *i.e.* it is the same line. The points of the line are then interchanged in pairs, while the line as a whole is unaltered.

*The inverse of a circle is a circle.*

The circle $$x^2+y^2-2\alpha x-2\beta y+c=0$$

becomes $$\frac{k^2(x^2+y^2)}{(x^2+y^2)^2}-2\alpha\frac{k^2x}{x^2+y^2}-2\beta\frac{k^2y}{x^2+y^2}+c=0,$$

*i.e.* $$c(x^2+y^2)-2\alpha k^2x-2\beta k^2y+k^2=0.$$

This degenerates to a straight line when $c=0$, *i.e.* when the circle passes through the centre of inversion; and the straight line is parallel to the tangent at $O$.

**16.** *If two curves intersect in a point $P$, their inverses intersect in the inverse point $P'$.*

Let the polar coordinates of the point of intersection be $(r, \theta)$. The corresponding point on *either* of the inverse curves is $(k^2/r, \theta)$; hence the inverse curves cut in the point $(k^2/r, \theta)$, which is the inverse of $(r, \theta)$.

*If two curves touch at a point $P$, their inverses touch at $P'$, the inverse of $P$.*

Consider first two curves intersecting in two near points $P$, $Q$. Then, since to each point there corresponds a unique point, the two inverse curves will intersect in the distinct inverse points $P'$, $Q'$. Now, if the curves are varied slightly so that $Q$ comes to coincide with $P$, then $Q'$ will coincide with $P'$, *i.e.* when the original curves touch the inverse curves also touch.

**17.** *Two curves cut at the same angle as their inverses.* Let two curves cut at $P$, and let $PA$, $PB$ be the tangents at $P$; then the inverses of these lines are circles $P'A'O$, $P'B'O$ passing through $O$, and they touch the inverse curves at $P'$. Now the angle of intersection of the inverse curves is equal to the angle between the tangents to the circles at $P'$ or $O$; but the latter are parallel respectively to the lines $PA$ and $PB$. Hence the angles of intersection are equal.

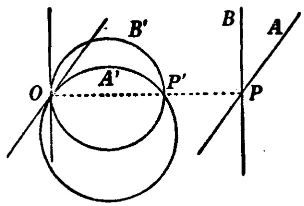


FIG 53.

From this property it follows that a small rectilinear triangle is transformed by inversion into a small curvilinear triangle whose angles are equal to those of the original triangle. The smaller the triangle the more nearly can its sides be regarded as being rectilinear, *i.e.* if the triangle is diminished and its inverse is magnified, as by a magnifying glass, so as to look always of the same size, it will become more and more nearly rectilinear. At the same time its angles remain constant, and we can say that it always remains similar to itself. We therefore say that the inverse figures are similar in their smallest parts, or in the infinitesimal. A transformation which keeps angles unaltered is said to be *conformal.*

**18.** The method of inversion or reciprocal radii is of use in simplifying certain classes of theorems, especially those which involve circles or systems of circles, or in deriving new theorems from simpler ones which are already known.

Exactly the same method is used in stereographic projection. This is a process by which a sphere is projected on a plane; the centre of projection $O$ is a point on the surface, say the north pole, and the plane of projection is the plane of the equator or any parallel plane. Actually the plane is the inverse of the sphere with $O$ as centre of inversion, and the two fundamental properties of inversion are true for stereographic projection, viz. circles are transformed into circles, and angles are unaltered in magnitude. Stereographic projection was known to the ancient geometers, and a treatise on the subject by Ptolemy (about 150 A.D.) entitled "The Planisphere" is in existence. The theory of plane inversion was not developed, however, until the nineteenth century; this curious fact, which is on a par with the fact that spherical trigonometry was developed before plane trigonometry, is without doubt due to the all-important influence of astronomy. One of the earliest applications of inversion, and it seems to amount to an independent discovery of the method, was made by W. Thomson (afterwards Lord Kelvin) in 1845. He called the inverse of a point its image in the circle or sphere, and he bases on this his theory of "electric images." Liouville, in his comments on this theory, gave to the geometrical process the name "Transformation by reciprocal radius-vectors."

**Examples.**

1. Three circles which pass through one point form a curvilinear triangle whose angle-sum is equal to two right angles.

Take the common point $O$ as centre of inversion. The three circles then become straight lines forming a rectilinear triangle. Since angles are unaltered by inversion, and since the angle-sum of a rectilinear triangle is equal to two right angles, the theorem follows.

2. Prove that three circles which cut a real fourth circle $O$ orthogonally and intersect in pairs form a curvilinear triangle within $O$ whose angle-sum is less than two right angles.

Take one of the vertices of the curvilinear triangle as centre of inversion.

3. If a system of circles touch two fixed circles, they also cut orthogonally one or other of two fixed circles.

This follows by inversion from the theorem that if a system of circles touch two fixed straight lines, they cut orthogonally one of the bisectors of the angles between the two straight lines.

4. Prove that a circle which cuts the circle of inversion orthogonally is unaltered by inversion.

5. Prove that a system of coaxal circles with real or imaginary common points is inverted into a system of coaxal circles with real or imaginary common points respectively.

6. Prove that a system of coaxal circles is unaltered individually by inversion in a circle of the conjugate coaxal system, the limiting points being unaltered.

7. Prove that a system of coaxal circles is unaltered as a whole by inversion in a circle of the same system, the limiting points being interchanged.

8. Prove that a system of coaxal circles and the conjugate system can be transformed by inversion into a system of concentric circles and their diameters.

9. Prove that a system of tangent coaxal circles and the conjugate system can be transformed by inversion into two sets of parallel straight lines intersecting at right angles.

10. Prove that three circles which cut each other mutually at right angles can be inverted into a circle and two perpendicular diameters.

11. Prove that a net of circles is inverted into a net of circles.

12. Prove that a net of circles which has a real orthotomic circle can be inverted into a system of circles with collinear centres.

**19. Invariant of two circles.** If two circles intersect at an angle $\theta$, and $t$ is the length of the common tangent, $r$ and $r'$ the radii, and $d$ the distance between the centres,

$$\begin{aligned} t^2 &= d^2 - (r - r')^2 \\ &= r^2 + r'^2 + 2rr' \cos\theta - (r - r')^2 \\ &= 2rr'(1 + \cos\theta). \end{aligned}$$

If the circles are inverted $\theta$ remains constant; therefore $\frac{t^2}{rr'}$ is also invariant.

If the circles do not intersect, and $t$ is a transverse common tangent,

$$t^2 = d^2 - (r + r')^2 = 2rr'(\cos\theta - 1).$$

Hence again, $\frac{t^2}{rr'}$ remains invariant. The angle $\theta$ is in this case of course imaginary, but since $\cos\theta$ is always determined by the formula

$$(r^2 + r'^2 - d^2)/(2rr'),$$

$\cos\theta$ is always real.

**20. Casey's Theorem.** If four circles touch a straight line at points $A$, $B$, $C$, $D$, their common tangents are connected by one of the relations

$$AB \,.\, CD \pm BC \,.\, AD \pm CA \,.\, BD = 0,$$

or, denoting the length of a common tangent of the circles with radii $r_1$ and $r_2$ by (12), etc., we have

$$(12)(34) \pm (13)(24) \pm (14)(23) = 0.$$

We may write this

$$\frac{(12)}{\sqrt{r_1 r_2}} \cdot \frac{(34)}{\sqrt{r_3 r_4}} \pm \frac{(13)}{\sqrt{r_1 r_3}} \cdot \frac{(24)}{\sqrt{r_2 r_4}} \pm \frac{(14)}{\sqrt{r_1 r_4}} \cdot \frac{(23)}{\sqrt{r_2 r_3}} = 0.$$

Now each of these terms is unaltered by inversion. Hence, we have the result: *If four circles all touch a fifth circle their* (*direct or transverse*) *common tangents* (*suitably chosen*) *are connected by one of the relations*

$$(12)(34) \pm (13)(24) \pm (14)(23) = 0.$$

Suppose the circle 4 reduces to a point-circle, then it lies on the fifth circle. The squares of the lengths of its common tangents with the other circles are just the powers of the point with regard to these circles. Hence, if we denote the circles 1, 2, 3 by $S_1$, $S_2$, $S_3$, where

$$S_1 \equiv x^2 + y^2 - 2\alpha_1 x - 2\beta_1 x + c_1, \text{ etc.,}$$

we obtain the equation of the fifth circle, *i.e.* the equation of a circle tc ching the three given circles, viz.:

$$(23)\sqrt{S_1} \pm (31)\sqrt{S_2} \pm (12)\sqrt{S_3} = 0.$$

When this is rationalized it is of the second degree in $S_1$, $S_2$, $S_3$, and therefore represents two circles. Three other pairs of circles are found by changing the signs of $S_1$, $S_2$, $S_3$.

The above theorem is analogous to Ptolemy's Theorem for a cyclic quadrilateral. It reduces, in fact, to Ptolemy's Theorem when the four circles become point-circles.

**21. Application of complex numbers.** There is a useful application of complex numbers, by which the formulae of inversion can be greatly simplified.

The complex number $x + iy$ may be denoted by the single letter $z$; then we shall denote the conjugate complex number $x - iy$ by $\bar{z}$. We have now two new variables or coordinates $z$, $\bar{z}$ instead of the cartesian coordinates $x$, $y$, and the equation of any locus can be expressed in terms of $z$, $\bar{z}$. It is just a sort of transformation of coordinates, but the new axes are imaginary. The equations connecting $x$, $y$ and $z$, $\bar{z}$ are

$$\begin{aligned} x + iy &= z, & 2x &= z + \bar{z}, \\ x - iy &= \bar{z}, & 2iy &= z - \bar{z}; \end{aligned} \quad \text{also } x^2 + y^2 = z\bar{z}.$$

The equation of a circle whose centre is the origin is expressed simply by

$$z\bar{z} = a^2.$$

Expressing in terms of $z$, $\bar{z}$ the general equation of a circle

$$x^2+y^2-2\alpha x-2\beta y+c=0,$$

we have
$$z\bar{z}-\alpha(z+\bar{z})+i\beta(z-\bar{z})+c=0,$$

or
$$z\bar{z}-(\alpha-i\beta)z-(\alpha+i\beta)\bar{z}+c=0.$$

Let $\alpha+i\beta=p$, so that $\alpha-i\beta=\bar{p}$; then we have

$$z\bar{z}-\bar{p}z-p\bar{z}+c=0.$$

This equation, therefore, represents a circle whose radius, $r$, is given by

$$r^2=\alpha^2+\beta^2-c=p\bar{p}-c,$$

and whose centre ($x=\alpha$, $y=\beta$) is determined by

$$x+iy=\alpha+i\beta, \quad \text{or} \quad z=p.$$

Ex. Show that the equation of any straight line can be reduced to the form

$$\bar{p}z+p\bar{z}=c.$$

22. The formulae for inversion in the circle $x^2+y^2=k^2$ or $z\bar{z}=k^2$ are

$$x'=\frac{k^2x}{x^2+y^2}=\frac{k^2x}{z\bar{z}}, \quad y'=\frac{k^2y}{x^2+y^2}=\frac{k^2y}{z\bar{z}},$$

whence
$$z'=x'+iy'=\frac{k^2(x+iy)}{z\bar{z}}=\frac{k^2}{\bar{z}},$$

$$\bar{z}'=x'-iy'=\frac{k^2(x-iy)}{z\bar{z}}=\frac{k^2}{z},$$

*i.e.*
$$z'\bar{z}=k^2=\bar{z}'z.$$

This may be represented by the single equation

$$z'\bar{z}=k^2,$$

since this implies also the conjugate equation $\bar{z}'z=k^2$, so that the equation for inversion is obtained from the equation of the circle by accenting one of the $z$'s.

We shall apply this now to find the formulae of inversion in the general circle
$$z\bar{z}-\bar{p}z-p\bar{z}+c=0.$$

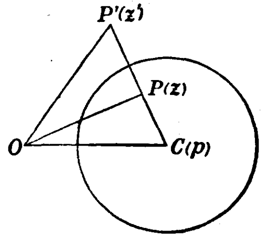



FIG. 54.

Let $P$, $P'$ be a pair of inverse points, and let the complex numbers which correspond to the vectors $OP$, $OP'$ be $z$, $z'$, and those which correspond to the vectors $CP$, $CP'$ be $\rho$, $\rho'$. The vector $OC$ is represented by $p=\alpha+i\beta$.

Then, by the composition of vectors,

$$z=p+\rho, \quad z'=p+\rho'.$$

But by the previous result

$$\bar{\rho}\rho'=r^2=p\bar{p}-c.$$

Therefore
$$(\bar{z}-\bar{p})(z'-p)=p\bar{p}-c,$$

*i.e.*
$$\bar{z}z'-\bar{p}z'-p\bar{z}+c=0,$$

or
$$z'=\frac{p\bar{z}-c}{\bar{z}-\bar{p}}.$$

**Examples.**

1. Prove that the last equation is equivalent to

$$\frac{x'-\alpha}{x-\alpha}=\frac{y'-\beta}{y-\beta}=\frac{\alpha^2+\beta^2-c}{(x-\alpha)^2+(y-\beta)^2}=\frac{(x'-\alpha)^2+(y'-\beta)^2}{\alpha^2+\beta^2-c}.$$

2. Show that when the circle of inversion becomes a straight line $\bar{p}z+p\bar{z}=c$, the formula for inversion becomes

$$\bar{p}z'+p\bar{z}=c.$$

3. Show that inversion in the straight line $y=x\tan\theta$ is represented by

$$z'=\bar{z}e^{2i\theta},$$

and deduce that inversion in a straight line is the same as reflexion in the straight line.

## EXAMPLES VII. A.

1. Find the coordinates of the limiting points of the pairs of circles:

$$\text{(i)}\ x^2+y^2-3x+1=0,\qquad 2x^2+2y^2-7x+2=0;$$
$$\text{(ii)}\ x^2+y^2-x-3y+3=0,\quad x^2+y^2+8x-6y-3=0.$$

2. Show that $(x-\alpha)^2+(y-\beta)^2+\lambda\{(x-\alpha')^2+(y-\beta')^2\}=0$ is the general equation of a system of coaxal circles having $(\alpha, \beta)$ and $(\alpha', \beta')$ as limiting points.

3. Show that (1, 2) is one of the limiting points of the system

$$(x-1)^2+(y-2)^2+\lambda(x^2+y^2+2x+5)=0,$$

and find the other one.

4. Find the equation of the orthotomic circle of the circles

$$x^2+y^2-2x-4y+6=0,\quad x^2+y^2+4y-6=0,\quad x^2+y^2-10x+18=0.$$

5. Show that the circle $x^2+y^2=4$ is cut by each of the circles $(x-2)^2+(y+1)^2=9$, $(x+4)^2+(y+4)^2=36$, $(x+3)^2+(y-6)^2=49$ at the ends of diameters.

6. Show that $x^2+y^2-10x+9+k(x^2+y^2+8x+9)=0$ represents a non-intersecting system of circles, and find the coordinates of the limiting points.

7. Find the equation of the circle which has for its diameter the chord cut off on the straight line $ax+by+c=0$ by the circle $(a^2+b^2)(x^2+y^2)=2c^2$. (Peterhouse, etc., 1899.)

8. The polars of a point $P$, with respect to two circles, meet in $Q$. Show that the radical axis of the circles bisects $PQ$.

9. If the circle $x^2+y^2-2\alpha x-2\beta y+c=0$ is a circle of a coaxal system having the origin as a limiting point, prove that the equation of the system is

$$\lambda(x^2+y^2)-2\alpha x-2\beta y+c=0,$$

and that the equation of the conjugate system is

$$(\alpha+\mu\beta)(x^2+y^2)-c(x+\mu y)=0.$$

10. Prove that the equations of the common tangents of the circle $x^2+y^2=289$ and the circle whose diameter is the chord of the first circle $x\cos\alpha+y\sin\alpha=15$ are

$$3(x\cos\alpha+y\sin\alpha)\pm4(y\cos\alpha-x\sin\alpha)=85.$$

(Peterhouse, etc., 1901.)

11. Prove that the locus of the pole of the line $y=k$ with respect to the family of coaxal circles $x^2+y^2+2\lambda x+c=0$, in which $c$ is constant but $\lambda$ varies, is the parabola $x^2=ky+c$. (Downing, 1914.)

12. Find the locus of the pole of the line $lx+my+n=0$ with respect to the family of coaxal circles $x^2+y^2+2\lambda x+c=0$, and interpret the result when $m=0$.

13. Find the condition that the circle $x^2+y^2-2\alpha x-2\beta y+c=0$ should cut the circle $x^2+y^2-2\alpha' x-2\beta' y+c'=0$ at ends of a diameter.

14. If $S_1=0$, $S_2=0$, $S_3=0$ are the equations of three fixed circles, prove that $lS_1+mS_2+nS_3=0$, where $l$, $m$, $n$ are variable, represents a variable circle cutting a fixed circle at the ends of a diameter, and that if this fixed circle is virtual the variable circle will cut a real fixed circle orthogonally.

15. If $S_1$ and $S_2$ are two circles which cut two given circles each at ends of diameters, prove that every circle of the system $S_1-\lambda S_2=0$ cuts the given circles in the same manner.

16. Prove that there are in general two circles in any coaxal system which are cut diametrically by any given circle.

17. Prove that there is in general just one circle of a given coaxal system which cuts a given circle diametrically.

18. Prove that there is a whole system of real circles each of which is cut diametrically by every circle of a coaxal system with real common points, and that all the circles of this system have double contact with a fixed ellipse of eccentricity $\frac{1}{2}\sqrt{2}$.

19. Sketch the figures obtained by inverting the following diagrams: (i) paper ruled in squares with a circle inscribed in each square, (ii) a tesselated pavement made of equilateral triangles with a circle inscribed in each triangle. State some of the more immediate properties which should be apparent in the new figures.

(Pembroke, etc., 1911.)

20. Two equal circles $A$, $B$ and a third circle $C$, inside $A$ and of half the radius, all touch a line at the same point. Show that inversion in $A$ followed by inversion in $B$ is equivalent to inversion in $C$ followed by reflexion in the common tangent.

(Pembroke, etc., 1912.)

## EXAMPLES VII. B.

1. Find the equations of the radical axes of the circles $(x-a)^2+(y-b)^2=b^2$, $(x-b)^2+(y-a)^2=a^2$, $(x-a-b-c)^2+y^2=ab+c^2$, and prove that they are concurrent. Find also the equation of the circle which cuts the three circles orthogonally. (Peterhouse, etc., 1900.)

2. Find the limiting points of the system of circles

$$x^2+y^2+2gx+c+\lambda(x^2+y^2+2fy+c')=0,$$

and show that the square of the distance between them is

$$\{(c-c')^2-4f^2g^2+4f^2c+4g^2c'\}/(f^2+g^2). \qquad \text{(Corpus, 1907.)}$$

3. If $r$, $r'$ are the radii of the circles $S$, $S'$, and $\theta$ their angle of intersection, prove that the equations $S/r-S'/r'.e^{\pm i\theta}=0$ represent the point-circles of the coaxal system determined by $S$ and $S'$.

4. Prove that the radius of the minimum circle of the coaxal system determined by two circles is $(rr'/d)\sin\theta$, where $r$, $r'$ are the radii of the circles, $d$ the distance between the centres, and $\theta$ the angle of intersection.

5. Show that the value of $\lambda$ for the minimum circle of the system $S+\lambda S'=0$ is $(d^2+r^2-r'^2)/(d^2-r^2+r'^2)$, where $r$, $r'$ are the radii, and $d$ the distance between the centres, of the circles $S$, $S'$.

6. A variable circle is one of a definite coaxal system, and a perpendicular is drawn from a fixed point to its polar with respect to the variable circle. Show that the locus of the foot of the perpendicular is a circle whose centre is on the common radical axis of the system of circles. (Math. Tripos I., 1913.)

7. Three circles $S_1$, $S_2$, $S_3$ are given, and the radical axis of each with respect to two given circles $O, O'$ form triangles $ABC$ and $A'B'C'$ respectively. Prove that these two triangles are in perspective. (Corpus, etc., 1912.)

8. If a conic through the common points of a coaxal system of circles is cut by any circle of the system in $P, Q$, prove that $PQ$ is parallel to a fixed direction.

9. Prove the following geometrical construction for the limiting points of the coaxal system of circles whose common points are the intersections of a given line $l$ with a given conic $S$: draw $d$, the diameter conjugate to $l$, and let $T$ and $T'$ be the ends of the equal diameter $d'$; let the tangents at $T$ and $T'$ cut $l$ in $C$ and $C'$; then the circles with centres $C, C'$ and radii $CT, C'T'$ intersect in the points required.

10. If $A, B, C, D$ are four concyclic points and $O$ any other point, prove that

$$OA^2 . (BCD)+OB^2 . (CAD)+OC^2 . (ABD)+OD^2 . (CBA)=0,$$

where $(BCD)$ represents the area, with the proper sign, of the triangle $BCD$, etc.

11. If $A, B, C, D$ are the centres of four circles which have a common orthotomic circle, and $P, Q, R, S$ are the powers of a point $O$ with regard to the four circles, prove that

$$P . (BCD)+Q . (CAD)+R . (ABD)+S . (CBA)=0.$$

12. If $A, B, C$ are the centres of three coaxal circles, and $t_1, t_2, t_3$ the lengths of the tangents drawn to them from any point, prove that

$$BC . t_1^2+CA . t_2^2+AB . t_3^2=0.$$

13. Show that the lengths of the tangents $t_1, t_2, t_3, t_4, t_5$ from any point to five fixed circles are connected by a fixed relation of the form

$$at_1^2+bt_2^2+ct_3^2+dt_4^2+et_5^2=0.$$

14. Show that the lengths of the tangents from any point to four fixed circles are connected by a fixed relation of the form

$$at_1^2+bt_2^2+ct_3^2+dt_4^2=e.$$

15. Show that the locus of a point such that the tangents from it to four fixed circles are connected by a fixed relation of the form

$$at_1^2+bt_2^2+ct_3^2+dt_4^2=0$$

is in general a circle. What is the locus if $a+b+c+d=0$?

16. Show that the equation

$$at_1^2+bt_2^2+ct_3^2=d,$$

where $t_1, t_2, t_3$ are the lengths of the tangents from a variable point $P$ to three fixed circles whose centres are not collinear, can represent any circle. What is the relation between $a, b, c$ if the locus is a straight line?

17. Prove that all the circles of a net which degenerate to straight lines form a pencil of lines.

18. If $\theta$ is the angle of intersection of the two circles $S, S'$, prove that the radii of the circles $S/r \pm S'/r'$ are $2rr'/(r+r') . \cos \frac{1}{2}\theta$ and $2rr'/(r+r') . \sin \frac{1}{2}\theta$. Deduce that if the circles $S$ and $S'$ cut in imaginary points, one of the circles $S/r \pm S'/r'$ is real and the other is virtual; and that if $S$ and $S'$ are orthogonal the two circles $S/r \pm S'/r'$ coincide.

19. Prove that if the Jacobian of three circles $S_1$, $S_2$, $S_3$ vanishes identically, the equations of the three circles are connected by a linear relation

$$lS_1+mS_2+nS_3=0.$$

20. If $x+iy=a\tan(\varphi+i\psi)$, where $i=\sqrt{-1}$, prove that $\varphi=\text{const.}$ and $\psi=\text{const.}$ represent conjugate systems of coaxal circles.

21. If $x+iy=(\varphi+i\psi)^{-1}$, prove that $\varphi=\text{const.}$ and $\psi=\text{const.}$ represent conjugate systems of coaxal circles having a common tangent.

22. Prove that the circles $S=0$ and $S'=0$ are inverse with regard to either of the circles $S/r\pm S'/r'=0$, where $r$, $r'$ are the radii of the circles $S$, $S'$.

23. A circle $S=0$ and a point $P$ are transformed by inversion in a certain circle with centre $O$ and become the circle $S'=0$ and the point $P'$. If $r$, $r'$ are the radii of these circles, $S$ the power of $P$ with regard to $S=0$, $S'$ that of $P'$ with regard to $S'=0$, and $\rho$, $\rho'$ the distances $OP$, $OP'$, prove that $S/(\rho r)=\pm S'/(\rho' r')$. Hence show that for a given circle of inversion and a given point $P$ the expression $S/r$ is transformed into $\lambda S'/r'$, where $\lambda$ is the same for all circles $S$.

24. If $S_1$, $S_2$, $S_3$, $S_4$ are the powers of a point with regard to four circles which cut in pairs orthogonally, and whose radii are $r_1$, $r_2$, $r_3$, $r_4$, prove that

$$S_1^2/r_1^2+S_2^2/r_2^2+S_3^2/r_3^2+S_4^2/r_4^2=0.$$

25. If $S_1$, $S_2$, $S_3$, $S_4$ are four mutually orthogonal circles, prove that the condition that the two circles

$$\lambda_1S_1+\lambda_2S_2+\lambda_3S_3+\lambda_4S_4=0,$$
$$\mu_1S_1+\mu_2S_2+\mu_3S_3+\mu_4S_4=0$$

should be orthogonal is

$$\lambda_1\mu_1r_1^2+\lambda_2\mu_2r_2^2+\lambda_3\mu_3r_3^2+\lambda_4\mu_4r_4^2=0.$$

26. Show that two inverse points with regard to a circle $S$ can be regarded as point-circles cutting $S$ in the same two (imaginary) points.

27. Prove that the equation of a circle which cuts each of the three circles $x^2+y^2-2\alpha_ix-2\beta_iy+c_i=0$ $(i=1, 2, 3)$ at the same angle $\varphi$ can be written

$$\begin{vmatrix} x^2+y^2 & x & y & 1 \\ c_1 & \alpha_1 & \beta_1 & 1 \\ c_2 & \alpha_2 & \beta_2 & 1 \\ c_3 & \alpha_3 & \beta_3 & 1 \end{vmatrix} + 2r\cos\varphi \begin{vmatrix} 0 & x & y & 1 \\ r_1 & \alpha_1 & \beta_1 & 1 \\ r_2 & \alpha_2 & \beta_2 & 1 \\ r_3 & \alpha_3 & \beta_3 & 1 \end{vmatrix} = 0,$$

where $r_1$, $r_2$, $r_3$ are the radii of the given circles and $r$ the radius of the required circle. Hence prove that all the circles which cut three given circles at equal (not specified) angles form four coaxal systems, the pairs of base-points being the points of intersection of the orthotomic circle with the homothetic axes, *i.e.* the (four) straight lines which cut the three circles at equal angles. (Each pair of circles has two homothetic centres, or centres of similitude, and the six homothetic centres of the three circles lie in sets of three on the four homothetic axes.)

28. Prove that the operations of inversion with respect to two coplanar circles in succession are commutative if the circles cut one another orthogonally.

(Pembroke, 1913.)

29. Prove that

$$|(z-z_1)/(z-z_2)|=\text{const.} \quad \text{and} \quad \text{am}\{(z-z_1)/(z-z_2)\}=\text{const.}$$

represent conjugate systems of coaxal circles.

# CHAPTER VIII.

## OBLIQUE AXES AND TRANSFORMATION OF COORDINATES.

**1.** In applying analysis to any geometrical problem the coordinate axes should be chosen so as to simplify the analysis as far as possible. If there are two lines which are distinctly indicated in the problem, it will generally be an advantage to take these as axes. But it will often happen that these two lines are not at right angles. In such cases we can use a convenient system of *oblique* coordinates defined as follows.

Let $x'Ox$ and $y'Oy$ be the oblique axes, inclined at an angle $\omega$. Let $P$ be any point, and draw $MP \parallel Oy$ and $NP \parallel Ox$. Then the coordinates

$$OM = NP = x, \quad ON = MP = y$$

determine the position of $P$.

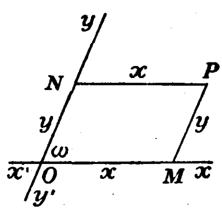



FIG. 55.

The convention of signs is the same as with rectangular coordinates, and the two axes divide the plane into four regions: in the first, $xOy$, the signs of $x$ and $y$ are $+\,+$; in the second, $yOx'$, they are $-\,+$; in the third, $x'Oy'$, $-\,-$; and in the fourth, $y'Ox$, $+\,-$.

This system is more general than the rectangular system, and includes it as a particular case when $\omega = 90°$. It may be called the *general cartesian system*.

**2. Elementary formulae in oblique coordinates.** Oblique axes are not very convenient for numerical work, or for problems in which the actual magnitudes of segments or angles are concerned. It is important, however, to note in what respects the elementary formulae differ when the axes are oblique.

The student may verify the following results:

*Distance between two points,*

$$PQ^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 + 2(x_2 - x_1)(y_2 - y_1)\cos\omega.$$

*Joachimsthal's formulae,*

$$x = \frac{lx_2 + mx_1}{l+m}, \qquad y = \frac{ly_2 + my_1}{l+m},$$

for the coordinates of a point dividing the join of two points in a given ratio, are unchanged.

*Area of a triangle OPQ,*

$$\Delta = \tfrac{1}{2}(x_1y_2 - x_2y_1)\sin\omega.$$

**3. Equation of a straight line; intercept form.** If the line is parallel to $Oy$, we have $x=\text{const.}$; if it is parallel to $Ox$, $y=\text{const.}$

If it cuts the axes at $A$ and $B$, let $OA=a$, $OB=b$. Let $P\equiv(x, y)$ be any point on the line, and draw the ordinate $MP$.

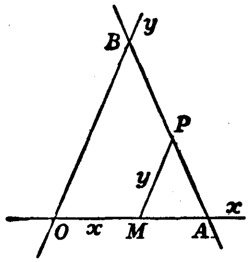


FIG. 56.

Then $$\frac{MP}{OB}=\frac{MA}{OA};$$

therefore $$\frac{y}{b}=\frac{a-x}{a}.$$

Hence the equation of the line is

$$\frac{x}{a}+\frac{y}{b}=1.$$

*The equation of a straight line is therefore always of the first degree*, and its intercept form is the same in rectangular and oblique coordinates.

**4.** Oblique axes are used in certain cases to simplify the equations of lines or curves when two of the principal lines which appear in the statement of the problem are not at right angles.

**Ex.** Perpendiculars $PM, PN$ are dropt from a variable point $P$ upon two fixed lines $OM, ON$, and $OM+ON$ is constant; find the locus of $P$.

Take $OM, ON$ as axes. Then

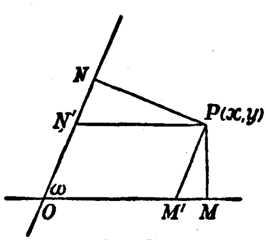


FIG. 57.

$$OM=x+y\cos\omega,$$
$$ON=y+x\cos\omega.$$

Therefore $(x+y\cos\omega)+(y+x\cos\omega)=k$,

*i.e.* $$x+y=k/(1+\cos\omega).$$

This represents a straight line parallel to one of the bisectors of the angles between $OM, ON$.

**5. The conic referred to oblique axes.** Since the equation of a straight line is of the first degree in $x, y$, an equation of the second degree represents a curve which has the property that it is cut by any straight line in two points, real, coincident, or imaginary; it therefore represents a curve of the second degree or a conic. We shall see presently what conics are represented by the simple equations $x^2/a^2+y^2/b^2=1$, $y^2=4ax$, etc. Many of the equations and formulae for conics are exactly the same in oblique coordinates. The student may verify the following results:

The equation of the tangent at $(x', y')$ to the curve

$$ax^2+by^2=1$$

is $$axx'+byy'=1.$$

This is also the equation of the polar of $(x', y')$.

The origin is the centre of the conic

$$ax^2+2hxy+by^2=1.$$

The lines $y=\mu x$, $y=\mu' x$ are conjugate diameters of the conic

$$ax^2+by^2=1,$$

if

$$a+b\mu\mu'=0.$$

The equation of the tangents from $(x', y')$ to the conic $ax^2+by^2=1$ is

$$(ax^2+by^2-1)(ax'^2+by'^2-1)=(axx'+byy'-1)^2.$$

**6. Equation of a central conic referred to a pair of conjugate diameters.** Since each diameter bisects all chords parallel to the conjugate diameter, the equation of the conic must be such that for every value of $x$ we have two values of $y$ equal but opposite in sign, and similarly for $y$; *i.e.* the equation contains only squared terms and a constant, and is therefore of the form

$$lx^2+my^2=1.$$

Let the conic be an ellipse with conjugate semi-diameters of lengths $a$ and $b$. When $y=0$, $x=\pm a$, and when $x=0$, $y=\pm b$; hence the equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

Similarly the equation of a hyperbola referred to conjugate diameters is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

**7. Equation of a hyperbola referred to its asymptotes.** Let the equation referred to the principal axes be

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

The lengths of the perpendiculars from the point $P\equiv(x, y)$ upon the asymptotes are $\frac{x}{a}\pm\frac{y}{b}$, each divided by $\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{\frac{1}{2}}$. Hence, if the lengths of these perpendiculars are $x'$ and $y'$, we have

$$x'y'=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)^{-1}=\frac{a^2b^2}{a^2+b^2}.$$

Also $\tan\frac{1}{2}\omega=b/a$; therefore $\sin\omega=\frac{2ab}{a^2+b^2}$.

Put $x'=x\sin\omega$, $y'=y\sin\omega$, and we get the equation in oblique coordinates,

$$xy=\tfrac{1}{4}(a^2+b^2).$$

**Examples.**

1. Show that the tangent at any point of a hyperbola makes with the asymptotes a triangle of constant area.

Taking the asymptotes as axes, the equation of the hyperbola is $xy=k$. The equation of the tangent at $(x', y')$ is

$$xy'+x'y=2k,$$

and its intercepts are $OA=2k/y'$ and $OB=2k/x'$. Hence the area of the triangle $OAB$ is $\frac{1}{2}OA\,.\,OB\sin\omega=2k^2\sin\omega/x'y'=2k\sin\omega$, which is constant.

2. **Prove that the segments of any chord of a hyperbola included between the curve and its asymptotes are equal.**

Let the equation of the chord be $lx+my+n=0$. This cuts the hyperbola $xy=k$ in points $P, Q$ whose abscissae are given by the equation

$$lx^2+nx+mk=0,$$

and the asymptotes in points $A, B$ whose abscissae are $-n/l$, 0. If $x_1, x_2$ are the roots of the last equation, the $x$ of the mid-point of $PQ=\frac{1}{2}(x_1+x_2)=-\frac{1}{2}n/l$. But this is equal to the $x$ of the mid-point of $AB$; therefore $AP=QB$.

3. **Prove that the segment of any tangent of a hyperbola included between the asymptotes is bisected at the point of contact.**

**8. Equation of a parabola referred to a diameter and the tangent at its extremity.** Since every chord parallel to the tangent at $O$ is bisected by the diameter, the equation must be of the form

$$y^2=ax^2+2gx+c;$$

and since every line parallel to the diameter meets the curve in one point at infinity, $a=0$; and finally, since the curve passes through the origin, $c=0$. Hence the equation reduces to $y^2=2gx$, or $\quad y^2=4px.$

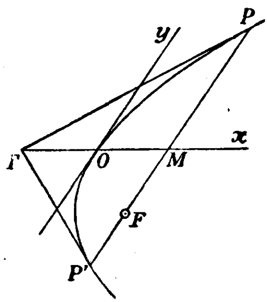


FIG. 58.

To find $p$, draw the focal chord $P'FP \parallel Oy$, and draw the tangents at $P$ and $P'$ intersecting in $T$, which lies on $Ox$. Then the coordinates of $P$ are $x=OM=OF$, and

$$y=MP=TM=2OM=2OF.$$

Therefore, substituting in the equation,

$$4OF^2=4p\,.\,OF;$$

hence $p=OF$.

**9. Equation of a conic referred to a tangent and the normal at the point of contact.** Let the equation of the conic be

$$ax^2+2hxy+by^2+2gx+2fy+c=0.$$

Since the curve passes through the origin, $c=0$. Put $x=0$, and we have

$$by^2+2fy=0.$$

This must give equal roots; one root is $y=0$; therefore the equation must reduce to $y^2=0$, and we have $f=0$. The equation of the conic then reduces to

$$ax^2+by^2+2hxy+2gx=0.$$

Ex. *All chords of a conic which subtend a right angle at a fixed point O on the conic cut the normal at O in a fixed point.*

Take the tangent and normal at $O$ as axes. The equation of the conic is then

$$ax^2+2hxy+by^2+2gx=0.$$

Let the equation of the chord $PQ$ be $lx+my+n=0$. The equation of the pair of lines $OP, OQ$ is then

$$n(ax^2+2hxy+by^2)-2gx(lx+my)=0.$$

The two lines will be at right angles if

$$an+bn-2gl=0.$$

Now the normal at $O$, *i.e.* the axis of $x$, cuts the line $PQ$ at $(-n/l, 0)$, *i.e.* $(-2g/(a+b), 0)$. Hence this is a fixed point. This point is called the *Frégier Point* corresponding to $O$.

## TRANSFORMATION OF COORDINATES.

**10.** We have seen that a problem is often simplified by a suitable choice of axes. Frequently, however, the axes are already determined, and in order to effect a simplification it is necessary to pass from one set of axes to another. By this transformation the coordinates of a given point will be changed, and the equation of a given locus will be transformed into another equation. We shall proceed to work out the relations which connect the two sets of coordinates of a given point in various transformations.

**11. To transform to new axes parallel to the old, but with a different origin. (Translation.)** Let $P$ be any point, whose coordinates referred to the old axes are $x$, $y$, and referred to the new axes $x'$, $y'$. Let the coordinates of the new origin referred to the old axes be $\alpha$, $\beta$. Then

$$\left.\begin{array}{l} x=x'+\alpha, \\ y=y'+\beta. \end{array}\right\} \;\ldots\ldots\ldots(A) \qquad \left.\begin{array}{l} x'=x-\alpha, \\ y'=y-\beta. \end{array}\right\} \;\ldots\ldots\ldots(A')$$

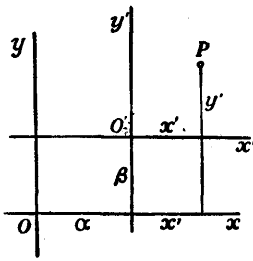


FIG. 59.

$(A)$ and $(A')$ are the equations of transformation from the old to the new axes, and the new to the old, respectively. $(A')$ is said to be the *inverse transformation* to $(A)$, for if it follows after $(A)$, it annuls its effect. Similarly $(A)$ is inverse to $(A')$. These equations also hold in the same form when the axes are oblique.

**12. To transform from one rectangular system to another with the same origin. (Rotation.)** Let the axes $Ox$, $Oy$ be rotated through the angle $\theta$ into the new positions $Ox'$, $Oy'$. Let $P$ be any point whose coordinates referred to the old and the new axes are $(x, y)$ and $(x', y')$. Draw $PM \perp Ox$, $PM' \perp Ox'$. Then the projection of $OP$ on any line being equal to the sum of the projections of $OM'$ and $M'P$, taken with the proper signs, we have, projecting first on $Ox$ and then on $Oy$,

$$\left.\begin{array}{l} x=x'\cos\theta-y'\sin\theta, \\ y=x'\sin\theta+y'\cos\theta. \end{array}\right\} \;\ldots\ldots\ldots\ldots(B)$$

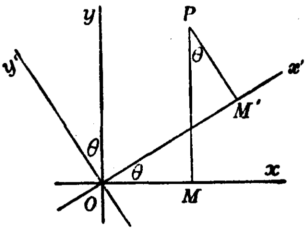


FIG. 60.

Similarly, by projecting $OP$ and also $OM$ and $MP$ on $Ox'$ and $Oy'$,

$$\left.\begin{array}{l} x'=\phantom{-}x\cos\theta+y\sin\theta \\ y'=-x\sin\theta+y\cos\theta. \end{array}\right\} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(B')$$

The second set of equations may also be found from the first by solving for $x'$ and $y'$. ($B$) and ($B'$) are, as before, inverse transformations.

The two sets of equations may be conveniently represented by a single scheme:

| | $x'$ | $y'$ |
|---|---|---|
| $x$ | $\cos\theta$ | $-\sin\theta$ |
| $y$ | $\sin\theta$ | $\cos\theta$ |

which may be read either horizontally or vertically.

**13. Examples.**

1. Transform the equation $2x^2+3y^2-12x+12y+24=0$ to parallel axes through the point $(3, -2)$.

Here $x=x'+3$, $y=y'-2$; hence the equation becomes

$$2(x'+3)^2+3(y'-2)^2-12(x'+3)+12(y'-2)+24=0,$$

*i.e.*

$$2x'^2+3y'^2=6.$$

When the transformation has been effected the accents may be dropt, so that the transformed equation is written

$$2x^2+3y^2=6.$$

2. Transform the equation $11x^2+24xy+4y^2=5$ in rectangular coordinates by rotating the axes through the angle $\tan^{-1}(-\frac{4}{3})$.

Taking the angle $\theta$ as a negative angle between 0 and $-90°$, we have

$$\sin\theta=-\tfrac{4}{5},\quad \cos\theta=\tfrac{3}{5},$$

and the equations of transformation are

$$x=(3x'+4y')/5, \qquad y=(-4x'+3y')/5.$$

The equation then becomes

$$11(3x+4y)^2+24(3x+4y)(-4x+3y)+4(-4x+3y)^2=125,$$

and this reduces to

$$-x^2+4y^2=1.$$

3. Transform the equation $14x^2-4xy+11y^2-36x+48y+41=0$ to rectangular axes through the point $(1, -2)$ inclined at the angle $-\tan^{-1}\frac{1}{2}$ to the original axes.

In this case it is best to effect the transformation in two stages. First change the origin to the point $(1, -2)$ by putting $x=x'+1$, $y=y'-2$. The equation then becomes

$$14(x'+1)^2-4(x'+1)(y'-2)+11(y'-2)^2-36(x'+1)+48(y'-2)+41=0,$$

which reduces to $14x^2-4xy+11y^2=25$.

Then rotate the axes through $-\tan^{-1}\frac{1}{2}$; $\sin\theta=-\dfrac{1}{\sqrt{5}}$, $\cos\theta=\dfrac{2}{\sqrt{5}}$, and the equations of transformation are $x=(2x'+y')/\sqrt{5}$, $y=(-x'+2y')/\sqrt{5}$. The equation then becomes

$$14(2x'+y')^2-4(2x'+y')(-x'+2y')+11(-x'+2y')^2=125,$$

which reduces to $3x^2+2y^2=5$.

*Note.* In carrying out the two successive transformations it is essential that the origin should be changed first, for if the axes were first rotated, the coordinates of the new origin would also be altered.

**14.** The general transformation from one set of rectangular axes to another is best carried out in two stages, first changing the origin and then rotating the axes. The whole transformation is said to be *compounded* of the two separate transformations, and it is very important to notice that the final result depends upon the order in which the two transformations are carried out.

First changing the origin to $(\alpha, \beta)$, the coordinates $(x, y)$ become $(x_1, y_1)$, where

$$\left.\begin{aligned} x_1 &= x-\alpha, \\ y_1 &= y-\beta; \end{aligned}\right\} \qquad \left.\begin{aligned} x &= x_1+\alpha, \\ y &= y_1+\beta. \end{aligned}\right\}.$$

Then, by rotating the axes through $\theta$, the coordinates $(x_1, y_1)$ become $(x', y')$, where

$$\left.\begin{aligned} x' &= x_1\cos\theta+y_1\sin\theta, \\ y' &= -x_1\sin\theta+y_1\cos\theta; \end{aligned}\right\} \qquad \left.\begin{aligned} x_1 &= x'\cos\theta-y'\sin\theta, \\ y_1 &= x'\sin\theta+y'\cos\theta. \end{aligned}\right\}.$$

Hence $(x, y)$ become $(x', y')$, where

$$\left.\begin{aligned} x &= x'\cos\theta-y'\sin\theta+\alpha, \\ y &= x'\sin\theta+y'\cos\theta+\beta. \end{aligned}\right\} \ldots(C) \qquad \left.\begin{aligned} x' &= (x-\alpha)\cos\theta+(y-\beta)\sin\theta, \\ y' &= -(x-\alpha)\sin\theta+(y-\beta)\cos\theta. \end{aligned}\right\} \ldots(C')$$

The transformations $(C)$ and $(C')$ are inverse, but it is not possible in this case to represent them by a single scheme read horizontally and vertically.

**Examples.**

Transform the following equations, both sets of axes being rectangular:

1. $12x^2+7xy-12y^2-17x-31y-7=0$ to axes through $(1, -1)$ turned through an angle $\tan^{-1}\frac{4}{3}$.

2. $x^2+3xy+2y^2-x-3y=2$ to axes through $(5, -3)$ rotated through $45°$.

3. $3x^2+2xy+3y^2-18x-22y+50=0$ to axes through $(2, 3)$ rotated through $45°$.

4. $6x^2+24xy-y^2=1$ so as to remove the term in $xy$.

5. $11x^2+4xy+14y^2=5$ so as to remove the term in $xy$.

6. $9x^2+24xy+2y^2-6x+20y+41=0$ so as to remove the terms in $x$, $y$ and $xy$.

7. $8x^2+12xy-8y^2+12x+4y+3=0$ so as to remove the terms in $x$, $y$, and $xy$.

**15.** Sometimes the new axes are given by their equations. In this case it is easy to write down the equations of transformation expressing $(x', y')$ in terms of $(x, y)$, for in the case of rectangular axes $x'$ and $y'$ are the perpendicular distances of the point $(x, y)$ from the new axes. If the new axes are oblique, we have to multiply these perpendicular distances by $\operatorname{cosec}\omega'$.

If the equations of the rectangular axes of $y'$ and $x'$ in the normal form are

$$l_1x+m_1y+n_1=0,$$
$$l_2x+m_2y+n_2=0,$$

the equations of transformation are

$$x'=l_1x+m_1y+n_1,$$
$$y'=l_2x+m_2y+n_2.$$

Ex. Transform the equation $x^2+4xy+4y^2-14x-8y+5=0$ to new axes of $x$ and $y$ whose equations are $x+2y-3=0$ and $2x-y+1=0$ respectively.

We have $x'=\pm(2x-y+1)/\sqrt{5}$, $y'=\pm(x+2y-3)/\sqrt{5}$.

In order to fix the signs the positive directions on the new axes must be fixed. Taking these as in the figure, we see that when referred to the new axes the coordinates of the old origin are $+,-$ respectively for $x$ and $y$. Hence the $+$ sign must be taken in both cases. We have then

$$5x=(2x'+y')\sqrt{5}+1,$$
$$5y=(-x'+2y')\sqrt{5}+7,$$

and the equation becomes

$$\sqrt{5}y'^2=4x'.$$

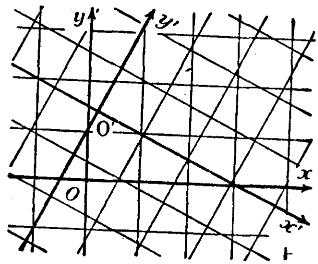


FIG. 61.

**16. Effect of a transformation of coordinates upon the general equation of the second degree.** We shall investigate the effect of the two simple transformations of rectangular coordinates upon the general equation

$$ax^2+by^2+2hxy+2gx+2fy+c=0.$$

**(1)** The transformation $\left.\begin{aligned}x&=x'+\alpha,\\ y&=y'+\beta.\end{aligned}\right\}$

The equation becomes, omitting accents,

$$a(x+\alpha)^2+b(y+\beta)^2+2h(x+\alpha)(y+\beta)+2g(x+\alpha)+2f(y+\beta)+c=0,$$

or, collecting terms,

$$ax^2+by^2+2hxy+2x(a\alpha+h\beta+g)+2y(h\alpha+b\beta+f)$$
$$+a\alpha^2+b\beta^2+2h\alpha\beta+2g\alpha+2f\beta+c=0,$$

*i.e.* the terms of the highest degree are unchanged, and the constant term is the result of substituting $\alpha$ for $x$ and $\beta$ for $y$ in the original expression.

Denoting the original equation by $F(x, y)=0$, the new constant term is $F(\alpha, \beta)$. The coefficients of $x$ and $y$ are the differential coefficients of $F(\alpha, \beta)$ with respect to $\alpha$ and $\beta$, viz. $\dfrac{\partial F}{\partial\alpha}$ and $\dfrac{\partial F}{\partial\beta}$. If $F(x, y)$ is made homogeneous by introducing $z$, we may write

$$F(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy.$$

Then we have the identity

$$2F(x, y, z)\equiv x\frac{\partial F}{\partial x}+y\frac{\partial F}{\partial y}+z\frac{\partial F}{\partial z}.$$

If $\gamma$ is put afterwards equal to 1, the new constant term

$$c'=F(\alpha, \beta, \gamma)=\frac{1}{2}\left(\alpha\frac{\partial F}{\partial\alpha}+\beta\frac{\partial F}{\partial\beta}+\gamma\frac{\partial F}{\partial\gamma}\right).$$

This last result often affords an easier way of obtaining the new constant term.

(2) The transformation $\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta, \\ y &= x'\sin\theta + y'\cos\theta. \end{aligned}\right\}$

Since these equations are homogeneous in $x'$, $y'$, each term will give rise only to terms of the same degree. We shall only investigate the transformation of the terms of the second degree

$$ax^2 + 2hxy + by^2,$$

which become, writing $c$ for $\cos\theta$ and $s$ for $\sin\theta$,

$$a(xc - ys)^2 + b(xs + yc)^2 + 2h(xc - ys)(xs - yc),$$

or $\quad x^2(ac^2 + bs^2 + 2hsc) + y^2(as^2 + bc^2 - 2hsc) + 2xy(-asc + bsc + h\,.\,\overline{c^2 - s^2}).$

The new coefficients may be written

$$\begin{aligned} a' &= \tfrac{1}{2}a(1 + \cos 2\theta) + \tfrac{1}{2}b(1 - \cos 2\theta) + h\sin 2\theta \\ &= \tfrac{1}{2}(a + b) + \tfrac{1}{2}(a - b)\cos 2\theta + h\sin 2\theta, \\ b' &= \tfrac{1}{2}(a + b) - \tfrac{1}{2}(a - b)\cos 2\theta - h\sin 2\theta, \\ h' &= -\tfrac{1}{2}(a - b)\sin 2\theta + h\cos 2\theta. \end{aligned}$$

**17. Invariants.** In the translation transformation, $a$, $b$ and $h$, which are unaltered, are called *invariants*. There are invariants also for the rotation transformation, for

$$a' + b' = a + b,$$

and $$a'b' - h'^2 = ab - h^2.$$

The first is obvious by adding the expressions for $a'$ and $b'$. To prove the second, we have

$$\begin{aligned} a'b' &= \tfrac{1}{4}(a + b)^2 - \{\tfrac{1}{2}(a - b)\cos 2\theta + h\sin 2\theta\}^2, \\ h'^2 &= \qquad\qquad\quad \{\tfrac{1}{2}(a - b)\sin 2\theta - h\cos 2\theta\}^2; \end{aligned}$$

therefore $$a'b' - h'^2 = \tfrac{1}{4}(a + b)^2 - \tfrac{1}{4}(a - b)^2 - h^2 = ab - h^2.$$

## EXAMPLES VIII.

1. Tangents are drawn from points $R$, $R'$, one on each asymptote, to touch a hyperbola in $P$, $P'$, and $T$ is the pole of $RR'$. Prove that $TP$, $TP'$ are parallel to the asymptotes.

2. Prove that if from two fixed points $O$ and $O'$ on a hyperbola there be drawn two rays to cut one another on the curve, the segment $PP'$ which these rays intercept on either of the asymptotes is of constant length.

(Trinity, 1911.)

3. $MM'$ is any chord of a hyperbola and $P$ is an extremity of the conjugate diameter. Lines $MK$, $PQ$, $M'K'$ are drawn parallel to one asymptote to meet the other in $K$, $Q$, $K'$. Show that $CK\,.\,CK' = CQ^2$, where $C$ is the centre.

(Trinity, etc., 1900.)

4. If the tangent at any point of a hyperbola with centre $C$ cuts the asymptotes in $L$, $L'$, prove that $CL\,.\,CL' = a^2 + b^2$. Hence show that if $F$, $F'$ are the foci, $L$, $L'$, $F$, $F'$ are concyclic.

5. Any two points $P, Q$ are taken on a hyperbola. Lines are drawn through $P, Q$ parallel to the asymptotes and intersecting in $K, K'$. Prove that $KK'$ passes through the centre of the hyperbola. (Queens', 1901.)

6. The tangent to an ellipse at a fixed point $D$ meets two parallel tangents in $S$ and $T$. Prove that $SD \,.\, DT$ is constant and equal to the square of the semi-diameter parallel to the fixed tangent.

7. Show that if the axes are inclined at an angle $\omega$ the directrix of the parabola $y^2=4cx$ is

$$x+y\cos\omega+c=0. \qquad \text{(St. Catharine's, 1900.)}$$

8. If $y=\mu x$, $y=\mu' x$ are conjugate diameters of the ellipse whose equation referred to another pair of conjugate diameters is $x^2/a^2+y^2/b^2=1$, show that

$$\mu\mu'=-b^2/a^2.$$

9. Prove that $(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}=1$ is the equation of a parabola, whether the axes are rectangular or oblique, and that it touches both of the coordinate axes.

10. Find the equation of the chord joining the points $(x_1, y_1)$ and $(x_2, y_2)$ on the parabola $(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}=1$, and deduce that the equation of the tangent at $(x', y')$ is

$$x/(ax')^{\frac{1}{2}}+y/(by')^{\frac{1}{2}}=1.$$

11. Prove that the line $lx+my+n=0$ will be a tangent to the parabola $(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}=1$ if $(al)^{-1}+(bm)^{-1}+n^{-1}=0$.

12. Find the equation of the directrix of the parabola $(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}=1$.

13. Show that the coordinates $(x, y)$ of the focus of the parabola

$$(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}=1$$

are given by the equations $x^2+2xy\cos\omega+y^2=ax=by$.

14. Prove that if $ab=k(a+b)$, where $k$ is constant, the locus of the foci of all the parabolas $(x/a)^{\frac{1}{2}}+(y/b)^{\frac{1}{2}}=1$, as $a$ and $b$ vary, is a circle.

15. A variable chord of a circle is divided harmonically by two fixed lines through the centre; prove that the envelope of the chord is a hyperbola with the lines as asymptotes. (Pembroke, etc., 1909.)

16. Through a given point a straight line is drawn to intersect two given straight lines. Show that the locus of the circumcentre of the triangle so formed is a hyperbola, and find the positions of its asymptotes. (Queens', 1910.)

17. Prove that the condition that the two circles

$$x^2+y^2+2xy\cos\omega+2gx+2fy+c=0, \quad x^2+y^2+2xy\cos\omega+2g'x+2f'y+c'=0$$

should cut orthogonally is

$$\begin{vmatrix} 1 & \cos\omega & g \\ \cos\omega & 1 & f \\ g' & f' & \frac{1}{2}(c+c') \end{vmatrix}=0.$$

18. If $\alpha, \beta$ are the angles which a line makes with the coordinate axes, and $\omega$ is the angle between the axes, prove that

$$\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\omega=\sin^2\omega.$$

19. If $\alpha$, $\beta$ and $\alpha'$, $\beta'$ are the angles which two lines make with the coordinate axes, prove that the two lines will be parallel if

$$\begin{vmatrix} \cos\alpha & \cos\beta \\ \cos\alpha' & \cos\beta' \end{vmatrix} = 0,$$

and perpendicular if

$$\begin{vmatrix} 1 & \cos\omega & \cos\alpha \\ \cos\omega & 1 & \cos\beta \\ \cos\alpha' & \cos\beta' & 0 \end{vmatrix} = 0.$$

20. Prove that the angle $\varphi$ between two lines whose direction-angles are $\alpha$, $\beta$ and $\alpha'$, $\beta'$ is given by either of the equations

$$\cos\varphi\sin^2\omega = -\begin{vmatrix} 1 & \cos\omega & \cos\alpha \\ \cos\omega & 1 & \cos\beta \\ \cos\alpha' & \cos\beta' & 0 \end{vmatrix}, \quad \sin\varphi\sin\omega = \begin{vmatrix} \cos\alpha & \cos\beta \\ \cos\alpha' & \cos\beta' \end{vmatrix}.$$

21. The points in a plane are displaced so that the point $(x, y)$ referred to rectangular coordinates takes the position $(X, Y)$, where $X = px + qy$, $Y = rx + sy$. Show that a unit square in any position becomes a parallelogram of area $ps \sim qr$, and that the parallelogram has the sum of the squares of the lengths of its sides constant. What is the least possible angle between the sides of the parallelogram ?

(Pembroke, etc., 1913.)

# CHAPTER IX.

## TRACING OF CONICS.

**1. Relation of the Ellipse, Hyperbola, Parabola, and Circle to the line at infinity.** (1) Consider first the Hyperbola, whose cartesian equation is

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1.$$

Making this homogeneous, we have the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=z^2.$$

To find where the curve cuts the line at infinity put $z=0$. Then the equation

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=0$$

represents the two (real) straight lines joining the centre $O$ to the points at infinity $H$, $K$ on the hyperbola. Hence *the hyperbola cuts the line at infinity in a pair of real points* $H$, $K$. The homogeneous coordinates of $H$, $K$ are $(a, b, 0)$ and $(a, -b, 0)$.

(2) For the Ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1,$$

the investigation is similar, but the two lines $OH$, $OK$, whose joint equation is

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=0,$$

are imaginary. Hence *the ellipse cuts the line at infinity in a pair of conjugate imaginary points* $H$, $K$; their coordinates are $(a, ib, 0)$ and $(a, -ib, 0)$.

(3) The equation of the Parabola, written homogeneously, is

$$y^2=4axz.$$

The two lines $OH$, $OK$ are then given by the equation $y^2=0$, and therefore coincide. Hence *the parabola cuts the line at infinity in two coincident points*, or *the line at infinity is a tangent to the parabola.*

(4) The general equation of a Circle, written homogeneously in $x$, $y$, $z$, is

$$x^2+y^2+2gxz+2fyz+cz^2=0.$$

Putting $z=0$, we get as the equation of the two lines $OH$, $OK$ in this case

$$x^2+y^2=0.$$

This represents a pair of conjugate imaginary straight lines, and, as the equation does not contain the particular constants $g, f, c$, it follows that *all circles cut the line at infinity in the same two points.* These points are called the *circular points at infinity*, and will usually be denoted by $I$ and $J$. Their homogeneous coordinates are $(1, \pm i, 0)$.

*Cor.* Two circles have in common four points, two of which, $I$ and $J$, are absolutely given. The existence of four points of intersection is in accordance with analysis, since two simultaneous equations, each of the second degree, have four sets of solutions.

**2. Asymptotes, centre, diameters, axes.** In the case of the hyperbola we saw (Chap. V. § 3), that the asymptotes are the tangents at the points where the curve cuts the line at infinity, and they pass through the centre. Then the asymptotes of the ellipse are imaginary. In both cases the line at infinity is the chord of contact of tangents drawn from the centre, and is therefore the polar of the centre. Further, the axes are the bisectors of the angles between the asymptotes. These relations are represented diagrammatically in Fig. 62.

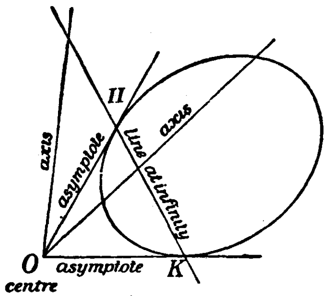



FIG. 62.

In the case of the parabola the asymptotes coincide with the line at infinity, and the pole of the line at infinity is the point of contact. Hence the centre of the parabola is the point at infinity on the curve, which is also the point at infinity on the axis. Every line through this point is therefore to be considered as a diameter, so that all diameters are parallel to the axis.

**3.** We shall postpone until next chapter the complete investigation of the converse theorems to those of § 1. Their object is to determine the character of a conic from its intersections with the line at infinity. But first we must introduce an analytical definition of a conic which will be more general than the geometrical definition given in Chap. IV. § 1.

*Def. A conic is the locus of the general equation of the second degree*

$$F(x, y) \equiv ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0.$$

The characteristic geometrical property of this locus is that it is cut by any straight line in two points, real, coincident, or imaginary, and it is the most general locus possessing this property. This property is, in fact, possessed by the different types of conics as already defined, including two straight lines. In this chapter, however, we shall assume that the equation represents a proper conic which is a hyperbola, an ellipse, or a parabola according as the locus cuts the line at infinity in real, imaginary, or coincident points.

Making the equation homogeneous in $x, y, z$, we have

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

Then, putting $z=0$, we find as the equation of the two lines $OH$, $OK$, joining the origin $O$ to the points at infinity $H$, $K$,

$$ax^2+by^2+2hxy=0.$$

*Case I.* If the two lines $OH$, $OK$ are real, so that $ab-h^2$ is negative, the curve is a hyperbola.

*Case II.* If the two lines $OH$, $OK$ are imaginary, so that $ab-h^2$ is positive, the curve is an ellipse.

*Case III.* If the two lines $OH$, $OK$ are coincident, so that $ab-h^2=0$, the curve is a parabola.

*Case IV.* If the points $H$, $K$ are the circular points, the equation of the two lines $OH$, $OK$ becomes $x^2+y^2=0$. Hence $a=b$ and $h=0$, and the curve is a circle.

We may prove the last result also as follows. The conditions that the locus should pass through the points $I$ and $J$, whose coordinates are $(1, \pm i, 0)$, are

$$a-b+2ih=0,$$
$$a-b-2ih=0.$$

Hence $a=b$ and $h=0$, and the locus is therefore a circle. Hence *every conic which passes through the two circular points is a circle.*

If the conic passes through just one of the circular points, its coefficients cannot be all real.

**4. Directions of the asymptotes.** Since the asymptotes are the tangents to the curve at $H$, $K$, they are parallel respectively to $OH$, $OK$. Hence *the asymptotes are parallel to the two straight lines represented by the equation* $ax^2+2hxy+by^2=0$.

**5. Coordinates of the centre.** The centre of a conic, if it exists, is a point $C$, such that every chord through $C$, terminated at each end by the curve, is bisected at $C$. If the origin is a centre, it follows that when $(x, y)$ is a point on the curve, so also is the point $(-x, -y)$. Now the only terms which are unaltered by changing the sign of both $x$ and $y$ are the terms of the second degree and the constant term. Hence *if the origin is a centre, there must be no terms of the first degree.*

Let the coordinates of the centre be $(\alpha, \beta)$. Then, transforming to $(\alpha, \beta)$ as origin without changing the direction of the axes, the terms of the second degree are unchanged, while the new coefficients of $2x$ and $2y$ are

$$\frac{1}{2}\frac{\partial F}{\partial \alpha}=a\alpha+h\beta+g,$$

and

$$\frac{1}{2}\frac{\partial F}{\partial \beta}=h\alpha+b\beta+f.$$

Equating these to zero, we have two equations of the first degree which determine $\alpha$, $\beta$ uniquely. Solving, we get

$$\alpha=\frac{hf-bg}{ab-h^2},\quad \beta=\frac{gh-af}{ab-h^2}.$$

These equations for the centre give a unique and definite solution except when $ab-h^2=0$, *i.e.* when the curve is a parabola or two parallel straight lines. The two equations $ax+hy+g=0$ and $hx+by+f=0$ represent straight lines through the centre, and therefore diameters. It will be seen later (see Chap. X. § 7) that they are the polars of the points at infinity on the axes of $x$ and $y$ respectively, and they therefore cut the curve at points at which the tangent is parallel to the axis of $x$ or $y$.

**6. Equation of the principal axes.** Having transformed the equation of the conic to parallel axes with the centre as origin, it reduces to the form

$$a\xi^2+2h\xi\eta+b\eta^2+c'=0,$$

where

$$\begin{aligned} c' &= a\alpha^2+2h\alpha\beta+b\beta^2+2g\alpha+2f\beta+c \\ &= \alpha(a\alpha+h\beta+g)+\beta(h\alpha+b\beta+f)+(g\alpha+f\beta+c) \\ &= g\alpha+f\beta+c. \end{aligned}$$

The equation of the asymptotes is now

$$a\xi^2+2h\xi\eta+b\eta^2=0.$$

The principal axes are the bisectors of the angles between the asymptotes, and are therefore at right angles, and also harmonic conjugates with regard to the asymptotes. As they are at right angles their equation is of the form

$$\xi^2+2\lambda\xi\eta-\eta^2=0,$$

and since they harmonically separate the asymptotes

$$-a+b=2h\lambda.$$

Hence the equation of the principal axes is

$$h(\xi^2-\eta^2)-(a-b)\xi\eta=0.$$

**7. Alternative method of finding the axes.** A convenient method of finding the positions and lengths of the axes, which has the advantage of definitely discriminating between the major and the minor axis, is as follows:

Consider the central conic

$$ax^2+2hxy+by^2=1,$$

and construct a concentric circle

$$x^2+y^2=r^2.$$

Starting with a small value of $r$, the circle will lie entirely inside the ellipse (or outside if it is a hyperbola). As $r$ increases the circle will come to cut the conic in four points, but just before this happens it will touch at the ends of the minor axis (or transverse axis, for the hyperbola). As $r$ increases the circle will come to surround the curve, if it is an ellipse, but just before this happens it will touch at the ends of the major axis; in the case of a hyperbola this of course will not happen, and there is only one case of real contact. When the circle cuts the conic the lines joining the origin to the points of intersection are given by the homogeneous equation

$$r^2(ax^2+2hxy+by^2)=(x^2+y^2),$$

*i.e.*

$$(ar^2-1)x^2+2hr^2xy+(br^2-1)y^2=0.$$

In order to get the principal axes, we have therefore to choose $r$ so that this equation may represent two coincident straight lines.

The condition is that

$$h^2r^4=(ar^2-1)(br^2-1),$$

*i.e.*

$$(ab-h^2)r^4-(a+b)r^2+1=0.$$

The positive roots of this equation are the lengths of the semi-axes. If $r_1$ is one of the roots, the equation of the corresponding axis is then

$$(ar_1^2-1)x+hr_1^2y=0,$$

or

$$hr_1^2x+(br_1^2-1)y=0.$$

If $ab-h^2$ is negative, so that the conic is a hyperbola, the roots of the quadratic in $r^2$ are of opposite sign. One of the values of $r$ is therefore imaginary, corresponding to imaginary contact of the circle with the conic, but the corresponding axis is always real, since its equation involves only $r^2$.

**8. Examples.**

1. Trace the curve $17x^2-12xy+8y^2+46x-28y+17=0$.

Making the equation homogeneous by means of $z$, we obtain

$$17x^2-12xy+8y^2+46xz-28yz+17z^2=0.$$

Putting $z=0$, we find the equation of the lines joining the origin to the points $H$, $K$, in which the curve cuts the line at infinity,

$$17x^2-12xy+8y^2=0.$$

These lines are imaginary; hence the curve is an ellipse.

The coordinates of the centre are given by

$$\frac{1}{2}\frac{\partial F}{\partial x}=17x-6y+23=0,$$

$$\frac{1}{2}\frac{\partial F}{\partial y}=-6x+8y-14=0;$$

hence the centre is $(-1, 1)$.

Transforming to parallel axes through the centre, the equation of the curve becomes

$$17\xi^2-12\xi\eta+8\eta^2=20.$$

The axes being at right angles, their equation is of the form

$$\xi^2+2\lambda\xi\eta-\eta^2=0,$$

and they harmonically separate the asymptotes $17\xi^2-12\xi\eta+8\eta^2=0$ (which in this case are imaginary). Hence $\lambda=\frac{3}{4}$.

The equation of the axes is therefore

$$2\xi^2+3\xi\eta-2\eta^2=0,$$

*i.e.*

$$(2\xi-\eta)(\xi+2\eta)=0.$$

The principal axis $2\xi-\eta=0$ cuts the curve in the two points $\left(\frac{2}{\sqrt{5}}, \frac{4}{\sqrt{5}}\right)$ and $\left(-\frac{2}{\sqrt{5}}, -\frac{4}{\sqrt{5}}\right)$, *i.e.* (0·89, 1·79) and (−0·89, −1·79). The principal axis $\xi+2\eta=0$ cuts the curve in the two points $\left(-\frac{2}{\sqrt{5}}, \frac{1}{\sqrt{5}}\right)$ and $\left(\frac{2}{\sqrt{5}}, -\frac{1}{\sqrt{5}}\right)$, *i.e.* (−0·89, 0·45) and (0·89, −0·45).

To determine the curve more accurately, we note that the new coordinate axes cut the curve in the points $(\pm\sqrt{\frac{20}{17}}, 0)$ and $(0, \pm\sqrt{\frac{5}{2}})$, *i.e.* (±1·08, 0) and (0, ±1·58). The curve is shown in Fig. 63.

The axes may be found alternatively by the concentric circle method. The equation of the lines joining the centre to the points of intersection with a concentric circle of radius $r$ is

$$r^2(17\xi^2 - 12\xi\eta + 8\eta^2) = 20(\xi^2 + \eta^2).$$

The condition that these should coincide is

$$36r^4 = (17r^2 - 20)(8r^2 - 20),$$

which reduces to
$$(r^2 - 4)(r^2 - 1) = 0.$$

Corresponding to $r = 2$, the major axis, we have

$$48\xi^2 - 48\xi\eta + 12\eta^2 = 0,$$

*i.e.*
$$(2\xi - \eta)^2 = 0,$$

and corresponding to $r = 1$, the minor axis, we have

$$(\xi + 2\eta)^2 = 0.$$

The equation of the ellipse referred to its principal axes is then

$$x^2/4 + y^2 = 1.$$

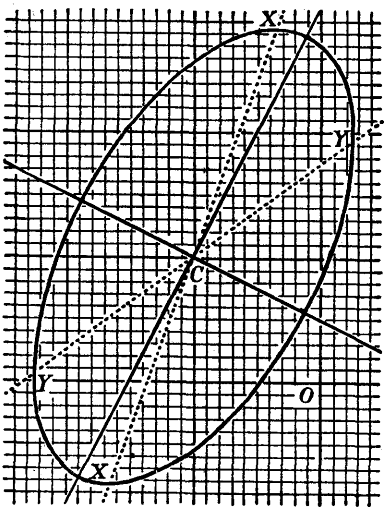



FIG. 63.

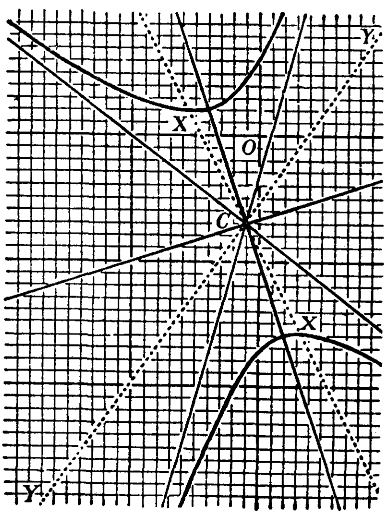



FIG. 64.

2. Sketch the curve

$$77x^2 + 78xy - 27y^2 + 70x - 30y + 29 = 0.$$

Making the equation homogeneous by means of $z$, we obtain

$$77x^2 + 78xy - 27y^2 + 70xz - 30yz + 29z^2 = 0.$$

Putting $z = 0$, we find the equation of the lines joining the origin to the points $H$, $K$, in which the curve cuts the line at infinity,

$$77x^2 + 78xy - 27y^2 = 0,$$

*i.e.*
$$(7x + 9y)(11x - 3y) = 0.$$

Hence the points $H$ and $K$ are real, and the curve is therefore a hyperbola.

The coordinates of the centre are given by

$$\frac{1}{2}\frac{\partial F}{\partial x}=77x+39y+35=0,$$

$$\frac{1}{2}\frac{\partial F}{\partial y}=39x-27y-15=0;$$

hence the coordinates of the centre are $(-\frac{1}{10}, -\frac{7}{10})$.

Transforming to parallel axes through the centre, the equation of the curve becomes

$$77\xi^2+78\xi\eta-27\eta^2+36=0,$$

and the asymptotes are

$$(7\xi+9\eta)(11\xi-3\eta)=0.$$

The axes being at right angles, their equation is of the form

$$\xi^2+2\lambda\xi\eta-\eta^2=0,$$

and they harmonically separate the asymptotes; therefore

$$-27-77-78\lambda=0,$$

whence

$$\lambda=-\tfrac{4}{3}.$$

Hence the equation of the axes is

$$3\xi^2-8\xi\eta-3\eta^2=0,$$

*i.e.*

$$(3\xi+\eta)(\xi-3\eta)=0.$$

The principal axis $3\xi+\eta=0$ cuts the curve in the two points $(\frac{3}{10}, -\frac{9}{10})$ and $(-\frac{3}{10}, \frac{9}{10})$, which are real. The principal axis $\xi-3\eta=0$ cuts the curve in the two points $(\frac{3}{5}i, \frac{1}{5}i)$ and $(-\frac{3}{5}i, -\frac{1}{5}i)$, which are imaginary.

To determine the curve more accurately, we note that the new coordinate axes cut the curve in the points $(\pm\sqrt{-\frac{36}{77}}, 0)$, which are imaginary, and in $\left(0, \pm\frac{2}{\sqrt{3}}\right)$, *i.e.* $(0, \pm 1{\cdot}16)$, which are real. The curve is shown in Fig. 64.

**9.** If the curve cuts the line at infinity in coincident points it is a parabola, and we may proceed as follows.

Since $ab-h^2=0$, the terms of the second degree form a perfect square, and hence we may write the equation

$$(ax+hy)^2+a(2gx+2fy+c)=0.$$

If now the two lines $ax+hy=0$ and $2gx+2fy+c=0$ are at right angles, this equation is of the form $\eta^2=4p\xi$, the standard form of a parabola. But this is not in general the case. We may, however, write the equation

$$(ax+hy+k)^2=2a(k-g)x+2(hk-af)y+k^2-ac,$$

and then we can determine $k$ so that the two lines

$$ax+hy+k=0$$

and

$$2a(k-g)x+2(hk-af)y+k^2-ac=0$$

may be at right angles, *i.e.* so that

$$a^2(k-g)+h(hk-af)=0.$$

$k$ is thus determined, and the equation is then reduced to the standard form

$$\eta^2=4p\xi,$$

where $\eta=0$ or $ax+hy+k=0$ is the principal axis, and $\xi=0$ is the tangent at the vertex.

**Example.** Trace the parabola

$$4x^2+y^2-4xy-10y-19=0.$$

The terms of the second degree form a perfect square, and the equation can be written

$$(2x-y)^2=10y+19.$$

Introduce the term $k$, and write the equation

$$(2x-y+k)^2=4kx+(10-2k)y+k^2+19.$$

In order that the straight lines on the left and right-hand sides, viz.

$$2x-y+k=0$$

and $\quad 4kx+(10-2k)y+k^2+19=0,$

may be at right angles, we must have

$$8k-(10-2k)=0\,;$$

hence $k=1$, and the equation becomes

$$(2x-y+1)^2=4(x+2y+5).$$

Putting $\eta=-(2x-y+1)/\sqrt{5}$

and $\qquad \xi=(x+2y+5)/\sqrt{5},$

the perpendicular distances from the new axes, and fixing the signs so that the positive directions of the new axes may be as in the figure (Fig. 65), we obtain

$$\eta^2=\frac{4}{\sqrt{5}}\,\xi\,;$$

hence the latus rectum is $\dfrac{4}{\sqrt{5}}$.

The coordinates of the vertex are

$$(-\tfrac{7}{5},\ -\tfrac{9}{5}).$$

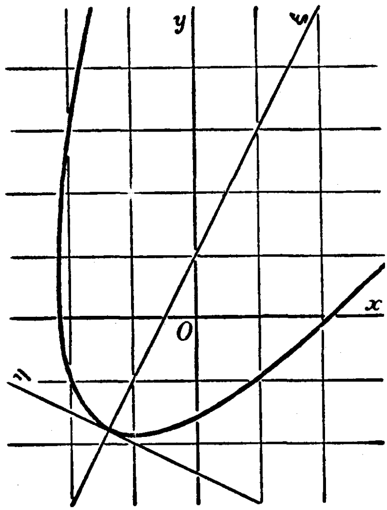



FIG. 65.

The coordinates of the focus referred to the $\xi$, $\eta$ axes are $\left(\dfrac{1}{\sqrt{5}},\ 0\right)$. Hence substituting in the above formulae, giving $\xi$ and $\eta$ in terms of $x$ and $y$, we obtain

$$0=\frac{2x-y+1}{\sqrt{5}},\quad \frac{1}{\sqrt{5}}=\frac{x+2y+5}{\sqrt{5}}\,;$$

hence the coordinates of the focus, referred to the original axes, are $(-\tfrac{6}{5},\ -\tfrac{7}{5})$.

## 10. Miscellaneous examples.

1. Trace the curve $4x^2-9y^2-24x-36y-36=0$.

We may write the equation thus:

$$4(x-3)^2-9(y+2)^2=36.$$

Transferring to the point $(3,\ -2)$ as origin, the equation reduces to

$$\frac{x^2}{9}-\frac{y^2}{4}=1\,;$$

hence the curve is a hyperbola with centre $(3,\ -2)$ and axes parallel to the axes of $x$ and $y$.

2. $$6x^2-10y^2+11xy-4x+9y=0.$$

The terms of the second degree factorize, and we have

$$(3x-2y)(2x+5y)-4x+9y=0.$$

Write this $$(3x-2y+\alpha)(2x+5y+\beta)=\alpha\beta,$$

and determine $\alpha$ and $\beta$.

The coefficient of $x$ is $2\alpha+3\beta=-4$,

and the coefficient of $y$ is $5\alpha-2\beta=\ \ 9$.

Hence $\alpha=1$, $\beta=-2$, and the equation becomes

$$(3x-2y+1)(2x+5y-2)=-2.$$

This is the equation of a hyperbola referred to its asymptotes. The centre is the point of intersection of the asymptotes, viz. $(-\frac{1}{19}, \frac{8}{19})$.

## EXAMPLES IX.

1. Find the coordinates of the centres of the conics:
(i) $2x^2+3y^2+4xy-8x-14y+9=0$, (ii) $7x^2-2y^2-12xy+2x+4y+2=0$,
(iii) $x^2+4y^2-6xy-5x+10y+3=0$.

2. Find the equations and the lengths of the semi-axes of the conics:
(i) $5x^2+5y^2-6xy+18x-14y+9=0$,
(ii) $13x^2+37y^2-32xy-14x+38y-35=0$,
(iii) $5x^2-5y^2-24xy+14x+8y-16=0$.

3. Find the equations of the asymptotes of the hyperbolas:
(i) $6xy+9x+4y=0$, (ii) $2x^2-3y^2-xy+4x-1=0$,
(iii) $3x^2+2y^2+6xy+6x+10y+1=0$.

4. Sketch the conics:
(i) $x^2-4xy-2y^2+10x+4y=0$, (ii) $x^2+4y^2-2x-16y+1=0$,
(iii) $41x^2+24xy+9y^2-130x-60y+116=0$,
(iv) $2x^2-4xy+5y^2-4x-2y-31=0$,
(v) $(2x+y+1)^2=x-2y$, (vi) $5x^2-4xy+8y^2-6x-12y-36=0$,
(vii) $7x^2-48xy-7y^2+110x-20y+100=0$.

5. Find the equation of the ellipse $x^2-xy+2y^2=6$ referred to its principal axes.

6. Find the magnitudes and directions of the axes of the conic
$$x^2+xy+y^2-2x+2y-6=0.$$ (King's, etc., 1913.)

7. Reduce to its principal axes the conic $9x^2-4xy+6y^2-10x-7=0$, and prove that its area is $\pi\sqrt{2}$. (Trinity, etc., 1911.)

8. Find the equation of the conic through the points $(0, \frac{1}{3})$, $(\frac{1}{2}, 0)$, $(0, -2)$, $(2, 0)$, $(2, 1)$, and trace it roughly.

9. Prove that the six points $(2, 3)$, $(3, 2)$, $(3, 1)$, $(1, 3)$, $(1, 2)$, $(2, 1)$ are on a conic whose equation referred to its axes is $3x^2+y^2-2=0$. (Trinity, 1909.)

10. Trace the curve $y^2-4xy-5x^2+6y+42x-63=0$. (Peterhouse, 1900.)

11. Draw the curve $9x^2+6xy+y^2+2x+3y+4=0$, and find its latus rectum. (Queens', 1901.)

12. Find the species, the eccentricity, and the position of the axes of the conic $x^2-11y^2-16xy+10x+10y-7=0$, and sketch the curve. (Math. Tripos II., 1911.)

13. Trace the conic $34x^2+24xy+41y^2+48x+14y-108=0$, and find its eccentricity. (Corpus, etc., 1913.)

14. Find the coordinates of the focus and the vertex of the parabola

$$x^2-4xy+4y^2+10x-8y+13=0.$$ (King's, 1912.)

15. Find the equation of the directrix and the coordinates of the focus of the parabola $x^2+2xy+y^2-3x+6y-4=0$. (Magdalene, 1910.)

16. Find the equation of an ellipse with focus $(-1, 1)$, directrix $x-y+3=0$, and eccentricity $\frac{1}{2}$.

17. A conic is given by the equation

$$x^2+2(2\lambda-1)xy+\lambda^2y^2+2\lambda x+2\lambda^2y+\lambda^2+\tfrac{1}{4}\lambda-\tfrac{1}{4}=0,$$

where $\lambda$ is a parameter which takes all real values. Show that the conic is always real, and find the values of $\lambda$ for which the conic degenerates. (King's, 1912.)

# CHAPTER X.

## THE GENERAL CONIC.

**1.** The general equation of the second degree in $x$, $y$ may be written

$$F(x, y) \equiv ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0.$$

The first thing we notice about this equation is that it has six terms, each with a coefficient to which we can give an arbitrary value. If, however, every term is multiplied by the same factor, the equation is unaltered. Hence only the *ratios* of the six coefficients are significant. It follows that a conic can be made to satisfy *five* conditions, *e.g.* to pass through five given points. In order to fix a conic it is sufficient to determine five of its elements. For example, an ellipse will be determined if we know the lengths of its axes (2 data), their orientation (1), and the position of the centre (2). The problem of determining these elements depends upon the reduction of the general equation to one of the simple forms which have been considered in the previous chapters.

**2. Joachimsthal's ratio equation.** To find the ratios in which the line joining the points $P \equiv (x', y')$ and $Q \equiv (x, y)$ is cut by the conic: let the line $PQ$ cut the conic in a point $X$, which divides $PQ$ in the ratio $k : 1$. The coordinates of $X$ are then

$$\frac{kx + x'}{k+1}, \quad \frac{ky + y'}{k+1}.$$

Since $X$ lies on the curve, we have

$$a(kx + x')^2 + b(ky + y')^2 + 2h(kx + x')(ky + y') + 2(k+1)\{g(kx + x') + f(ky + y')\} + c(k+1)^2 = 0.$$

Writing this in descending powers of $k$, we have

$$k^2(ax^2 + by^2 + 2hxy + 2gx + 2fy + c) + 2k\{axx' + byy' + h(xy' + x'y) + g(x + x') + f(y + y') + c\} + (ax'^2 + by'^2 + 2hx'y' + 2gx' + 2fy' + c) = 0.$$

The two roots of this equation correspond to the two points $X$, $Y$ in which $PQ$ cuts the conic.

**3. Condition that P and Q may be conjugate points with regard to the conic.** If $(PQ, XY)$ is harmonic, $X$ and $Y$ divide $PQ$ internally and externally in the same ratio, and therefore the values of $k$ are equal but of opposite sign, *i.e.* their sum is zero. Hence

$$axx' + byy' + h(xy' + x'y) + g(x + x') + f(y + y') + c = 0,$$

which may also be written

$$x(ax'+hy'+g)+y(hx'+by'+f)+(gx'+fy'+c)=0.$$

**4. Pole and polar.** The polar of a point $P\equiv(x', y')$ is defined as the locus of points $Q\equiv(x, y)$, which are conjugate to $P$; hence, from § 3, the equation of the polar of $P\equiv(x', y')$ is

$$x(ax'+hy'+g)+y(hx'+by'+f)+(gx'+fy'+c)=0,$$

which may also be written

$$x\frac{\partial F}{\partial x'}+y\frac{\partial F}{\partial y'}+z\frac{\partial F}{\partial z'}=0.$$

Since the tangent at $(x', y', z')$ is the polar of the point of contact, this equation is also the equation of the tangent at $(x', y', z')$, provided this point lies on the curve.

**5. The centre.** The centre is the pole of the line at infinity. Let its homogeneous coordinates be $(x_1, y_1, z_1)$. Then the equation of the polar is

$$x\frac{\partial F}{\partial x_1}+y\frac{\partial F}{\partial y_1}+z\frac{\partial F}{\partial z_1}=0.$$

But this is to reduce to $z=0$. Therefore the coordinates of the centre satisfy the two equations

$$\frac{\partial F}{\partial x}=0,\qquad \frac{\partial F}{\partial y}=0,$$

*i.e.*

$$ax+hy+gz=0,$$

and

$$hx+by+fz=0,$$

whence

$$\frac{x}{hf-bg}=\frac{y}{gh-af}=\frac{z}{ab-h^2}.$$

6. The expressions in the denominators of these fractions are most conveniently represented by determinants. Consider the determinant

$$\Delta\equiv\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix},$$

whose vanishing expresses the condition that the equation

$$F(x, y, z)\equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$$

should represent two straight lines (Chap. II. § 14).

Now the denominator under $x$ is the minor of $g$ in this determinant, obtained by striking out the row and column containing $g$; similarly the denominator under $y$ is the minor of $f$ with the sign reversed. We shall use the term *cofactors* for the minors with signs prefixed as follows: a positive sign if the element is at any of the four corners or the centre, and a negative sign for any other element. Then the cofactor of each element will be represented by the corresponding capital letter. Thus

$$\left.\begin{array}{ll} A=bc-f^2, & F=gh-af, \\ B=ca-g^2, & G=hf-bg, \\ C=ab-h^2, & H=fg-ch. \end{array}\right\} \quad \text{.............................(1}$$

It will then be seen that the determinant can be expanded in different ways, thus

$$\Delta \equiv aA + hH + gG, \qquad \text{(2)}$$

while we have also relations of the form

$$aH + hB + gF \equiv 0, \qquad \text{(3)}$$

which represent the determinant with two rows made the same.

The determinant

$$\nabla \equiv \begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix}$$

is also important. By actual multiplication it may be verified that

$$\left.\begin{aligned} BC - F^2 &= \Delta a, & GH - AF &= \Delta f, \\ CA - G^2 &= \Delta b, & HF - BG &= \Delta g, \\ AB - H^2 &= \Delta c, & FG - CH &= \Delta h, \end{aligned}\right\} \qquad \text{(4)}.$$

and then

$$\nabla = A(BC - F^2) + H(FG - CH) + G(HF - BG) = \Delta(aA + hH + gG) = \Delta^2.$$

With this notation the homogeneous coordinates of the centre of the general conic are $(G, F, C)$.

**7. Diameters.** A diameter is a line through the centre, and the pole of a diameter is a point at infinity. The polar of $(x', y', z')$ is

$$x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} + z'\frac{\partial F}{\partial z} = 0;$$

hence

$$\frac{\partial F}{\partial x} = 0 \quad \text{and} \quad \frac{\partial F}{\partial y} = 0$$

are the polars of (1, 0, 0) and (0, 1, 0) which are the points at infinity on the coordinate axes. These are therefore diameters, and any diameter can be represented by

$$\lambda\frac{\partial F}{\partial x} + \mu\frac{\partial F}{\partial y} = 0.$$

Conjugate diameters are diameters which are conjugate lines with regard to the conic, and each passes through the pole of the other. The pole of the diameter $x'\frac{\partial F}{\partial x} + y'\frac{\partial F}{\partial y} = 0$ is $(x', y', 0)$, and the gradient of the conjugate diameter is therefore $\mu' = y'/x'$, while the gradient of the given diameter is $\mu = -(ax' + hy')/(hx' + by') = -(a + h\mu')/(h + b\mu')$. Hence the gradients of two conjugate diameters are connected by the symmetrical equation

$$a + h(\mu + \mu') + b\mu\mu' = 0.$$

This expresses that the two pairs of lines

$$\left.\begin{aligned} y &= \mu x \\ y &= \mu' x \end{aligned}\right\} \quad \text{and} \quad ax^2 + 2hxy + by^2 = 0$$

should be apolar. Hence *a pair of conjugate diameters are harmonic conjugates with regard to the asymptotes.*

The same relation also expresses that *the points at infinity on the two conjugate diameters are conjugate with regard to the conic, or harmonic conjugates with regard to the points at infinity, $H$, $K$, on the conic,* for the homogeneous coordinates of the points at infinity on the diameters are $(1, \mu, 0)$ and $(1, \mu', 0)$.

**8. The tangents from a given point.** Let $P \equiv (x', y')$ be the given point and $Q \equiv (x, y)$ any point on a tangent through $P$. Then the line $PQ$ meets the conic in two coincident points. Hence, in Joachimsthal's equation (§ 2), the values of $k$ are equal, and therefore

$$\{axx' + byy' + h(xy' + x'y) + g(x + x') + f(y + y') + c\}^2$$
$$= (ax^2 + by^2 + 2hxy + 2gx + 2fy + c)(ax'^2 + by'^2 + 2hx'y' + 2gx' + 2fy' + c).$$

This equation is the relation which connects the coordinates $(x, y)$ of any point on a tangent through $(x', y')$, and is therefore the joint-equation of the two tangents. The equation may be written in the form

$$F(x, y, z)\,F(x', y', z') = \frac{1}{4}\left(x\frac{\partial F}{\partial x'} + y\frac{\partial F}{\partial y'} + z\frac{\partial F}{\partial z'}\right)^2.$$

**9. The orthoptic circle.** The orthoptic locus is the locus of points at which the conic subtends a right angle. Hence, if $(x', y')$ is any point on the orthoptic locus, the tangents from $(x', y')$ to the conic are at right angles. The condition for this is that the sum of the coefficients of $x^2$ and $y^2$ in the equation of § 8 should vanish. Hence the equation of the locus is

$$(a + b)(ax^2 + by^2 + 2hxy + 2gx + 2fy + c) - (ax + hy + g)^2 - (hx + by + f)^2 = 0.$$

With the notation of § 6, this can be written

$$C(x^2 + y^2) - 2Gx - 2Fy + A + B = 0,$$

which represents a circle, concentric with the conic.

When the conic is a parabola, $C = 0$, and the locus degenerates to a straight line

$$2Gx + 2Fy = A + B,$$

the directrix of the parabola.

**10. The asymptotes.** The asymptotes are defined as the tangents to the curve at infinity. The points at infinity, $H$, $K$, on the curve are determined by the equations

$$ax^2 + 2hxy + by^2 \equiv b(y - \lambda x)(y - \mu x) = 0, \quad z = 0,$$

so that the homogeneous coordinates of $H$, $K$ are $(1, \lambda, 0)$ and $(1, \mu, 0)$ The equation of the tangent at $(1, \lambda, 0)$ is

$$\frac{\partial F}{\partial x} + \lambda\frac{\partial F}{\partial y} = 0,$$

and the joint equation of the two asymptotes is

$$\left(\frac{\partial F}{\partial x} + \lambda\frac{\partial F}{\partial y}\right)\left(\frac{\partial F}{\partial x} + \mu\frac{\partial F}{\partial y}\right) = 0,$$

*i.e.*

$$\left(\frac{\partial F}{\partial x}\right)^2 + (\lambda + \mu)\frac{\partial F}{\partial x}\frac{\partial F}{\partial y} + \lambda\mu\left(\frac{\partial F}{\partial y}\right)^2 = 0.$$

But $\lambda+\mu=-2h/b$, $\lambda\mu=a/b$; therefore the equation of the asymptotes becomes

$$b\left(\frac{\partial F}{\partial x}\right)^2-2h\frac{\partial F}{\partial x}\frac{\partial F}{\partial y}+a\left(\frac{\partial F}{\partial y}\right)^2=0.$$

**11. Reduction of equation to axes through the centre.** If $(\alpha, \beta)$ are the coordinates of the centre, the equations of transformation are

$$x=\xi+\alpha,$$
$$y=\eta+\beta.$$

By this transformation the terms in $\xi$ and $\eta$ vanish, and the terms of the second degree are unchanged. To find the constant term $c'$, we have

$$c'=a\alpha^2+b\beta^2+2h\alpha\beta+2g\alpha+2f\beta+c$$
$$=\alpha(a\alpha+h\beta+g)+\beta(h\alpha+b\beta+f)+(g\alpha+f\beta+c).$$

Now, since $(\alpha, \beta)$ is the centre,

$$a\alpha+h\beta+g=0,$$
$$h\alpha+b\beta+f=0;$$

therefore $$c'=g\alpha+f\beta+c=(gG+fF+cC)/C=\Delta/C.$$

The equation therefore reduces to

$$a\xi^2+2h\xi\eta+b\eta^2+\frac{\Delta}{C}=0.$$

This is the equation of a *central conic* referred to axes through its centre. If the conic is a parabola, $C=0$, and the equation cannot be reduced to this form. If $\Delta=0$, the equation becomes homogeneous and represents two straight lines.

**12. Reduction of the equation to the principal axes.** By transforming to parallel axes through the centre, the equation has been reduced to the form

$$a\xi^2+2h\xi\eta+b\eta^2+\Delta/C=0.$$

We have next to rotate the axes through an angle $\theta$, so as to remove the term in $\xi\eta$. The equations of transformation are

$$\xi=\xi'\cos\theta-\eta'\sin\theta,$$
$$\eta=\xi'\sin\theta+\eta'\cos\theta.$$

The coefficient of $2\xi'\eta'$ is

$$-a\cos\theta\sin\theta+b\cos\theta\sin\theta+h(\cos^2\theta-\sin^2\theta).$$

Equating this to zero, we have

$$\tan 2\theta=\frac{2h}{a-b},$$

which determines the angle $\theta$ which one of the principal axes makes with the axis of $x$.

The equation then reduces to the form

$$a'\xi'^2+b'\eta'^2+\Delta/C=0,$$

where $$a'=a\cos^2\theta+b\sin^2\theta+2h\cos\theta\sin\theta$$
$$=\tfrac{1}{2}(a+b)+\tfrac{1}{2}(a-b)\cos 2\theta+h\sin 2\theta,$$
$$b'=\tfrac{1}{2}(a+b)-\tfrac{1}{2}(a-b)\cos 2\theta-h\sin 2\theta.$$

Putting $R^2=(a-b)^2+4h^2=(a+b)^2-4C,$

we have $\cos 2\theta=(a-b)/R, \quad \sin 2\theta=2h/R;$

hence $a'=\frac{1}{2}(a+b)+\frac{1}{2}(a-b)^2/R+2h^2/R=\frac{1}{2}(a+b+R)$

and $b'=\frac{1}{2}(a+b-R).$

The equation then reduces to

$$(a+b+R)\xi'^2+(a+b-R)\eta'^2+2\Delta/C=0.$$

From this we find at once the squares of the semi-axes $r_1^2$, $r_2^2$

$$=-2\Delta/\{C(a+b\pm R)\}.$$

**13. The eccentricity.** If $a$, $b$ are the semi-axes, the eccentricity is given by

$$e^2=(a^2-b^2)/a^2.$$

If no distinction is made between the axes, there are therefore two values for the eccentricity,

$$e^2=1-r_2^2/r_1^2 \quad \text{and} \quad e'^2=1-r_1^2/r_2^2,$$

*i.e.*

$$e^2=\frac{2R}{a+b+R}=\frac{R(a+b-R)}{2C}, \quad e'^2=\frac{-2R}{a+b-R}.$$

From these we get the relations

$$(1-e^2)(1-e'^2)=1, \qquad (1)$$

$$e^2e'^2=-R^2/C, \qquad (2)$$

and

$$\frac{1}{e^2}+\frac{1}{e'^2}=1. \qquad (3)$$

From (1) it follows that the two values of $e^2$ are either both $>1$ or both $<1$, and from (2) $e^2$ and $e'^2$ are either both positive (when $C<0$), or one positive and one negative (when $C>0$), but never both negative, since the sum of their reciprocals $=+1$.

We may therefore classify as follows:

(1) $C<0$. *Hyperbola.* $e^2$ and $e'^2$ are both positive, and therefore, by equation (3), both $>1$.

(2) $C>0$. *Ellipse.* $e^2>0$ and $e'^2<0$, and therefore, by (1), $e^2<1$. $e'^2$ may be numerically $>$ or $<1$, according as $e^2<$ or $>\frac{1}{2}$.

The two values of the eccentricity refer to the real and the imaginary foci, but they are also the eccentricities of the two conjugate hyperbolas $x^2/a^2-y^2/b^2=\pm 1$, or the real and the virtual conjugate ellipses,

$$x^2/a^2+y^2/b^2=\pm 1,$$

with reference in each case to the real foci.

If $e^2=e'^2$, then, by (3), each $=2$, and the curve is a *rectangular hyperbola* with eccentricity $\sqrt{2}$.

If $e^2=-e'^2$, then, by (1), $e^4=0$; hence, by (2), $R=0$. This requires, for real values, that $a=b$ and $h=0$, so that the curve is a *circle*.

If $e^2=1$, then, by (3), $e'^2$ is infinite, and hence, by (2), $C=0$. The curve is thus a *parabola*. This is the case also for two coincident or parallel straight lines. For two distinct straight lines, considered as a conic, the eccentricity is the same as that of a conic of which the two lines are asymptotes.

If $e'^2 = -1$, then, by (3), $e^2 = \frac{1}{2}$. This is a particular case of an ellipse which has some special properties. It is called *Fagnano's Ellipse.* In it the minor auxiliary circle passes through the foci.

## EXAMPLES X.

1. A hyperbola touches the axis of $y$ at the origin, and the line $y = 7x - 5$ at the point (1, 2). One of its asymptotes is parallel to the axis of $x$. Find the equation of the curve. (Trinity, 1909.)

2. Four points $A, B, C, D$ lie on a circle. From $C$ and $D$ lines $CE, DF$ are drawn parallel to $AB$ and equal to $AB$. Prove that $BCDEF$ lie on a rectangular hyperbola. (King's, 1913.)

3. For what value of $\lambda$ will the equation $ax^2 + by^2 + 2gx + 2fy + 2\lambda xy = 0$ represent two straight lines? Prove that one of the lines is the tangent at the origin to the curve $ax^2 + by^2 + 2gx + 2fy + 2\lambda xy = 0$ for any value of $\lambda$.

4. Show that the lines $y = mx$, $y = m'x$ are equal diameters of the conic $ax^2 + 2hxy + by^2 = 1$ if $(a - b)(m + m') - 2h(1 - mm') = 0$. Hence find the equation of the pair of equi-conjugate diameters. (Pembroke, 1911.)

5. Prove that the area of the ellipse $ax^2 + by^2 + 2hxy + 2gx + 2fy + c = 0$ is equal to $\pi \Delta C^{-\frac{3}{2}}$.

6. Find the equation of the orthoptic circle of the conic $(ax + by - 1)^2 = 2\lambda xy$, and prove that for different values of $\lambda$ the orthoptic circles are coaxal. (Magdalene, 1907.)

7. The tangents at the ends of the axes of an ellipse form a rectangle $ABCD$, and a conic $S$ is drawn through these four points. A pair of conjugate semi-diameters of the ellipse cut the conic $S$ in points $P, Q$. Prove that $PQ$ is a tangent to the ellipse.

8. A circle is inscribed and a rectangular hyperbola is circumscribed to an equilateral triangle. Show that each curve passes through the centre of the other. (Peterhouse, 1913.)

9. A conic $ax^2 + by^2 = 1$ and a point $P(h, k)$ being given, prove that the locus of a point $Q$ whose polar makes a constant angle with $QP$ is a conic passing through $P$ and the origin. What is the nature of the locus when the constant angle is (i) zero, (ii) a right angle? (Math. Tripos I., 1915.)

10. Prove that if $A, A'$ are two fixed points, the locus of $P$, such that

$$\angle PAA' - \angle PA'A$$

is constant, is a rectangular hyperbola with $AA'$ as a diameter. Is the constant angle the same for both branches?

11. Any conic is inscribed in a quadrilateral $ABCD$ having a right angle at $B$. If $BA, BC$ touch the conic in $E, F$ respectively, prove that the foot of the perpendicular from $B$ on $EF$ lies on a fixed circle. (St. Catharine's, 1912.)

12. Prove that the four normals drawn from the origin to the conic

$$ax^2 + 2hxy + by^2 + 2fy = 0$$

form a harmonic pencil if $54ah^2 + (2a - b)^3 = 0$. (Corpus, 1910.)

13. Any two diameters of a conic are at right angles to each other and meet the tangent at a fixed point $P$ in $Q$ and $R$. The other two tangents to the conic through $Q$ and $R$ intersect in $T$. Show that $T$ lies on a fixed straight line.

(Trinity, 1901.)

14. $P$ is a variable point on a fixed line, and the polar of $P$ with regard to a given conic meets the conic in $Q$ and $R$. Show that the locus of the centroid of the triangle $PQR$ is a cubic curve having a double-point at the centre of the conic.

Explain the result when the fixed line is a diameter.

15. With the point $(x', y')$ as centre a family of circles is drawn to cut the conic $ax^2+2hxy+by^2+2gx+2fy+c=0$. Prove that the locus of the middle points of the chords of intersection is the rectangular hyperbola,

$$(x-x')(hx+by+f)-(y-y')(ax+hy+g)=0.$$

(Math. Tripos I., 1910.)

16. Show that a pair of conjugate diameters of the ellipse $ax^2+by^2=1$ lie along the lines $ax^2+2hxy-by^2=0$. Prove also that there is one conic with these lines as asymptotes which cuts $ax^2+by^2=1$ orthogonally at the common points.

(Pembroke, 1912.)

17. The two diagonals $AOB$ and $COD$ of the quadrilateral $ACBD$ intersect at right angles in $O$, and $OA\,.\,OB>OC\,.\,OD$. Prove that the eccentricity of the ellipse of smallest eccentricity which can be described through the four points $ABCD$ is $\sqrt{(1-OC\,.\,OD/OA\,.\,OB)}$. (Trinity, 1900.)

18. The parabola which has four-point contact with the conic

$$ax^2+2hxy+by^2+2gx+2fy=0$$

at the origin is drawn; prove (i) that the axis of the parabola makes with the tangent at the origin the angle $\phi$, where $\tan\phi=(fF+gG)/(fg\sim gF)$; (ii) that the equation of the directrix of the parabola is $2Gx+2Fy+g^2+f^2=0$.

(Queens', 1901.)

19. $A$, $B$ are two fixed points, and $P$ a variable point. The angle $PAB=\theta$ and $PBA=\phi$.

(i) Prove that if $a\tan\theta+b\tan\phi=c$, the locus of $P$ is in general a hyperbola passing through $A$ and $B$; but if $c=0$ it is a straight line perpendicular to $AB$; and if $a=b$ it is a parabola whose axis is perpendicular to $AB$.

(ii) If $a\cot\theta+b\cot\phi=c$, show that the locus of $P$ is a straight line, perpendicular to $AB$ if $c=0$ and parallel to $AB$ if $a=b$.

(iii) If $\sin\theta=\mu\sin\phi$, show that the locus of $P$ is a circle.

(iv) If $a\cot(\theta-\alpha)+b\cot(\phi-\beta)=c$, where $\alpha$ and $\beta$ are given angles, prove that the locus of $P$ is a conic circumscribing the triangle $ABC$, where $C$ is determined by making $\angle CAB=\alpha$, $\angle CBA=\beta$.

# CHAPTER XI.

## LINE-COORDINATES AND ENVELOPES.

**1. Line-coordinates.** We have considered a straight line as the locus of a point which moves according to a certain geometrical law which is expressed by an equation connecting the coordinates of the point. But a line may be considered as a geometrical element, to be fixed, like a point, by certain data or coordinates. Thus a straight line $AB$ will be fixed if we know the lengths of the intercepts $OA$, $OB$ on the axes; or the perpendicular $ON$ from the origin and the angle $XON$; or the gradient and the intercept $OB$; and so on. Just as in the case of a point, two data are always required, but there is a wide choice of suitable date. Any pair of data which suffice to fix the line can be called the coordinates of the line, but in choosing data we shall make it a condition that the line is to be uniquely determined by the coordinates, and conversely that only one pair of coordinates belong to any line. We shall find it convenient, for reasons which will appear later, to take as our coordinates the *negative reciprocals of the intercepts* on the axes, and we shall denote these by $l$ and $m$, so that

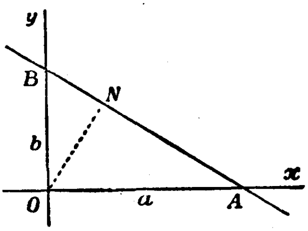



FIG. 66.

$$l = -\frac{1}{a}, \quad m = -\frac{1}{b}.$$

We can then speak of the line $(l, m)$, which is the same as the line whose equation is

$$lx + my + 1 = 0.$$

The signs as well as the magnitudes of $l$, $m$ being thus determined, to every pair of values of $l$ and $m$ there corresponds one line, and conversely. Fig. 67 represents the four lines $(\pm 2, \pm 3)$.

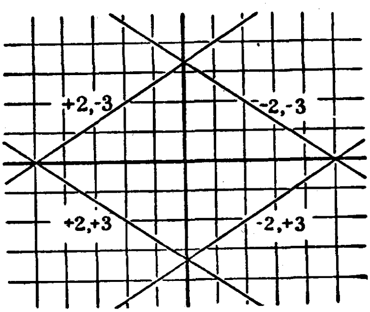



FIG. 67.

As $l$ and $m$ vary, we can obtain—with certain possible exceptions—all the lines of the plane. When $m=0$, we get lines parallel to the axis of $y$; when $l=0$, lines parallel to the axis of $x$. If $l=0$ and $m=0$, however, we do not get any line, and we cannot represent a straight line through the origin with any finite values of $l$ and $m$. These apparent exceptional cases will be considered later.

**2.** If $l$ and $m$ are not free to take all values independently, but are connected by an equation, we shall get, not all lines in the plane, but only a class of lines.

Take the equation $$2l+3m+6=0.$$
We can draw up a table of corresponding values of $l$ and $m$, thus

| $l$ | $-6$ | $-3$ | $0$ | $3$ | $6$ |
|---|---|---|---|---|---|
| $m$ | $2$ | $0$ | $-2$ | $-4$ | $-6$ |

The corresponding values of the intercepts are

| $a$ | 0·17 | 0·33 | $\infty$ | $-0·33$ | $-0·17$ |
|---|---|---|---|---|---|
| $b$ | $-0·5$ | $\infty$ | 0·5 | 0·25 | 0·17 |

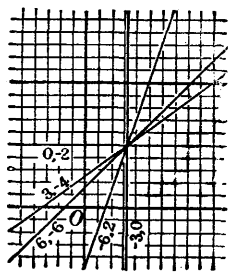


FIG. 68.

When the lines are drawn they all appear to pass through the same point $(\frac{1}{3}, \frac{1}{2})$ (Fig. 68).

Consider any equation of the first degree in $l$, $m$,
$$Al+Bm+C=0.$$
The equation of the line with coordinates $l$, $m$ is
$$lx+my+1=0.$$
The given equation of the first degree then expresses that the line always passes through the fixed point $(A/C, B/C)$. Hence *an equation of the first degree in $l$, $m$ represents a system of straight lines through a fixed point.*

**3.** We shall next consider some equations of the second degree in $l$, $m$.

Ex. 1. $2l^2+m^2=1$.

| $l$ | 0 | ·2 | ·4 | ·5 | ·6 | ·65 | ·7 |
|---|---|---|---|---|---|---|---|
| $m$ | 1 | ·96 | ·83 | ·71 | ·53 | ·39 | ·14 |

The corresponding intercepts are, in magnitude,

| $a$ | $\infty$ | 5 | 2·5 | 2 | 1·7 | 1·5 | 1·4 |
|---|---|---|---|---|---|---|---|
| $b$ | 1 | 1·04 | 1·2 | 1·4 | 1·9 | 2·5 | 7·1 |

and each pair may be taken $\pm$. The lines in this case envelop a curve (ellipse, Fig. 69).

Thus an equation in $l$, $m$ represents a class of lines which envelop a curve, just as an equation in $x$, $y$ represents a class of points whose locus is a curve

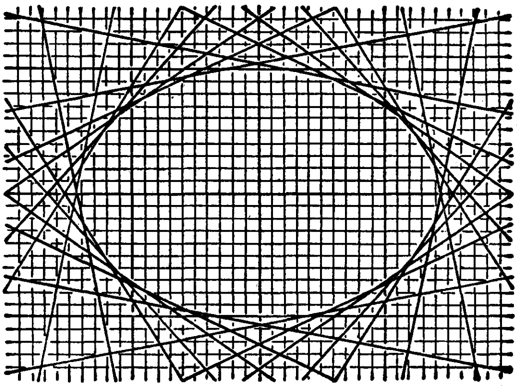


FIG. 69.

Ex. 2. $lm+l+m=0$. (Parabola touching the coordinate axes. Fig. 70.)

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $l$ | −4 | −2 | −1·5 | −1 | −0·8 | −0·7 | 2 | 4 | ∞ |
| $m$ | −1·3 | −2 | −3 | ∞ | 4 | 2·3 | −0·7 | −0·8 | −1 |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | 0·25 | 0·5 | 0·67 | 1 | 1·25 | 1·43 | −0·5 | −0·25 | 0 |
| $b$ | 0·75 | 0·5 | 0·33 | 0 | −0·25 | −0·43 | 1·5 | 1·25 | 1 |

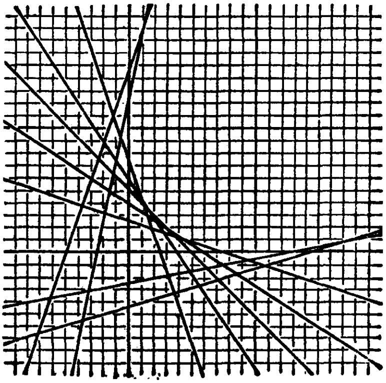


FIG. 70.

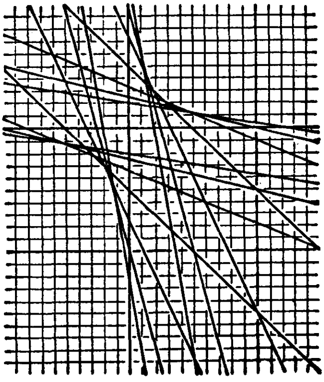


FIG. 71.

Ex. 3. $lm=4$. (Hyperbola. Fig. 71.)

| $l$ | $-3$ | $-2$ | $-1$ | $0$ | $1$ | $2$ | $3$ | $4$ | $5$ |
|---|---|---|---|---|---|---|---|---|---|
| $m$ | $-1{\cdot}3$ | $-2$ | $-4$ | $\infty$ | $4$ | $2$ | $1{\cdot}3$ | $1$ | $0{\cdot}8$ |

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| $a$ | | | | $\infty$ | $-1$ | $-0{\cdot}5$ | $-0{\cdot}33$ | $-0{\cdot}25$ | $-0{\cdot}2$ |
| $b$ | | | | $0$ | $-0{\cdot}25$ | $-0{\cdot}5$ | $-0{\cdot}75$ | $-1$ | $-1{\cdot}25$ |

Examples.

Draw the envelopes of the following equations:

1. $2l-m+1=0$. 2. $l+2m=3$. 3. $3l-2m=5$.
4. $l^2=m$. 5. $l^2+m^2=1$. 6. $l^2=2m+1$.

**4.** We shall consider now a few examples of the converse problem, to find the equation of the envelope of a line which moves according to some assigned law.

Ex. Find the equation of the envelope of a line which is at a fixed distance 2 from the point (3, 1).

Let the coordinates of the line be $(l, m)$. Then we have

$$2=\frac{3l+m+1}{\sqrt{l^2+m^2}}.$$

Rationalizing, $$4(l^2+m^2)=(3l+m+1)^2,$$

or $$5l^2-3m^2+6lm+6l+2m+1=0.$$

The envelope is a circle with centre (3, 1) and radius $=2$.

**Examples.**

Find the equation of the envelope in the following cases, and interpret the result geometrically:

1. A line moves so that the algebraic sum of its distances from the points $(2, 1)$, $(-1, 3)$, $(-1, -4)$ is equal to 3.

2. $A, B, C, \ldots$ are any number of points such that the origin $O$ is their centroid, and a line moves so that the algebraic sum of its distances from these points is constant.

3. A line moves so that the product of its distances from two points $(\pm 3, 0)$ on the same side of the line is constant, $=16$.

4. A line moves so that the product of its distances from the points $(\pm 13, 0)$ on opposite sides of the line is constant, $=25$.

5. A line moves so that 5 times the product of its distances from the points $(1, 2)$, $(1, -2)$ is equal to the square of its distance from the point $(5, 0)$.

6. A line moves so that it makes with the coordinate axes a triangle whose perimeter is constant, $=1$.

7. A line moves so that the algebraic sum of its intercepts on the axes is constant, $=1$.

8. The vertex of a right angle moves along a fixed line $(x=0)$ while one arm passes through a fixed point $(1, 0)$. Find the envelope of the other arm.

**5. Tangential equations.** We have seen that an equation in $l$, $m$ represents an envelope, or curve generated by a moving line. It may be viewed also from another point of view. The equation in $l$, $m$ is the condition that the straight line $lx+my+1=0$ should be a tangent to the curve. In exactly the same way an equation in point-coordinates $x$, $y$ represents a locus, or curve generated by a moving point; it also represents the condition that the point $(x, y)$ should lie on the curve.

From the second point of view the equation in $l$, $m$ is called the *tangential equation* of the curve, and we have already met examples of tangential equations. In future, however, we shall generally speak of the equation in $l$, $m$ as the *line-equation*, the equation in $x$, $y$ being the *point-equation* of the curve.

We shall illustrate three methods, each of general application, for finding the tangential or line-equation of a curve.

*To find the line-equation of the circle whose centre is at the origin and radius equal to $r$.*

METHOD I., using a geometrical property of the curve.

If the line $(l, m)$ or $lx+my+1=0$ is a tangent, the length of the perpendicular upon it from the origin is equal to $r$. Hence

$$\frac{1}{\sqrt{l^2+m^2}}=r,$$

whence, rationalizing, $$r^2(l^2+m^2)=1.$$

METHOD II., from the point-equation by expressing that the line

$$lx+my+1=0$$

cuts the curve in two coincident points.

The equation of the lines joining the origin to the points of intersection of the line $lx+my+1=0$ with the circle $x^2+y^2=r^2$ is

$$x^2+y^2=r^2(lx+my)^2,$$

*i.e.* $$(r^2l^2-1)x^2+2r^2lmxy+(r^2m^2-1)y^2=0.$$

These lines will coincide if

$$r^4l^2m^2=(r^2l^2-1)(r^2m^2-1),$$

*i.e.* if $$r^2(l^2+m^2)=1.$$

METHOD III., from the point-equation by identifying the equation $lx+my+1=0$ with the equation of the tangent at a given point.

The equation of the tangent at $(x', y')$ to the circle $x^2+y^2=r^2$ is

$$x'x+y'y-r^2=0.$$

If this is the same line as $lx+my+1=0$, we have

$$l=-x'/r^2, \quad m=-y'/r^2.$$

But $(x', y')$ lies on the line; therefore $lx'+my'+1=0$.
Hence, substituting for $x'$ and $y'$,

$$r^2(l^2+m^2)=1.$$

Ex. Deduce the following line-equations from the point-equations (cf. Chaps IV. §7; V. §4; VI. §3):

| | | |
|---|---|---|
| Circle, | $x^2+y^2-2\alpha x-2\beta y+c=0,$ | $(l\alpha+m\beta+1)^2=(l^2+m^2)(\alpha^2+\beta^2-c).$ |
| Ellipse, | $x^2/a^2+y^2/b^2=1,$ | $a^2l^2+b^2m^2=1.$ |
| Hyperbola, | $x^2/a^2-y^2/b^2=1,$ | $a^2l^2-b^2m^2=1.$ |
| Parabola, | $y^2=4ax,$ | $am^2=l.$ |

**6. Homogeneous line-coordinates.** In the former examples of tangential equations we expressed always the condition that the line $lx+my+n=0$ should be a tangent. Thus we found that the tangential equation of the ellipse $x^2/a^2+y^2/b^2=1$ was $a^2l^2+b^2m^2=n^2$, and in each case we found as the tangential equation an equation *homogeneous in* $l$, $m$, $n$. The line-equations which were considered in the last section are derived from these by putting $n=1$.

For many reasons there is an advantage in leaving $n$ variable, and then we shall call $l$, $m$, $n$ the *homogeneous coordinates* of the line $lx+my+n=0$. Since the line is not altered if we multiply each term of the equation by the same factor, any multiple $(kl, km, kn)$ of the homogeneous coordinates $(l, m, n)$ will represent the same straight line. It is therefore only necessary to consider the ratios $l:m:n$, and the actual values have no significance.

One great advantage of employing the homogeneous coordinates $l$, $m$, $n$ is that it avoids the necessity of dealing with infinite values. There was a difficulty in representing a line through the origin by the coordinates $l$, $m$, which represent the negative reciprocals of the intercepts, since these intercepts vanish; but using homogeneous coordinates, the condition that the line should pass through the origin is simply $n=0$, so that, *e.g.*, (1, 2, 0) will represent a definite line through the origin.

**7. Simultaneous equations in l, m, n.** Two simultaneous homogeneous equations in $l$, $m$, $n$ determine one or more sets of values of the ratios $l:m:n$, *i.e.* one or more straight lines. These are the straight lines common to the two envelopes. For example, if the two equations are of the first degree,

$$a_1l+b_1m+c_1n=0,$$
$$a_2l+b_2m+c_2n=0,$$

the single set of values of the ratios

$$l:m:n=b_1c_2-b_2c_1:c_1a_2-c_2a_1:a_1b_2-a_2b_1$$

which satisfy these two equations are the line-coordinates of the line common to the two pencils, or the line joining the two points represented by the two equations.

**8. Degree of an equation in line-coordinates.** The degree of an equation in cartesian coordinates has a definite geometrical meaning, being equal to the number of points in which the curve is cut by any straight line, or the *degree* of the curve. Similarly, in line-coordinates the degree of the equation is equal to the number of lines which the envelope has in common with any pencil, or the number of tangents that can be drawn to the curve from any point; for if we associate with the given equation of degree $r$

the equation of a pencil, which is of the first degree, the two equations determine $r$ sets of values of the ratios $l:m:n$. This number is called the *class* of the curve. A curve of the first degree is a straight line, a curve of the first class is a point.

**9. The general equation of the second degree in l, m, n.** The general equation of the second degree will be written

$$\varphi(l, m, n) \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0.$$

This equation represents a curve of the second class, that is, having the property that two tangents can be drawn to it from any point. For if the equation is solved simultaneously with an equation of the first degree, we get two sets of values of $l:m:n$, and therefore the envelope has two lines in common with any pencil. It should be possible, therefore, to identify the envelope of this equation with the general conic. We shall do this by finding the tangential or line-equation of the general conic.

**10. To find the tangential or line-equation of the conic**

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

The line-coordinates of the tangent at $(x', y', z')$ are $(l, m, n)$, where

$$\lambda l = ax' + hy' + gz',$$
$$\lambda m = hx' + by' + fz',$$
$$\lambda n = gx' + fy' + cz'.$$

If $x', y', z'$ are eliminated between these three equations and the equation

$$0 = lx' + my' + nz',$$

we get an equation in $l, m, n$ which is the relation between the line-coordinates of any tangent, or the condition that the line $(l, m, n)$ or

$$lx + my + nz = 0$$

should be a tangent. This gives us, therefore, the tangential equation of the conic in the form

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix} = 0.$$

When this is expanded, we get

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0,$$

where, as usual, the capital letters stand for the cofactors of the corresponding small letters in the determinant $\Delta$.

**11. To find the pole of a given line**

$$lx + my + nz = 0.$$

Let the coordinates of the pole be $(x', y', z')$; then comparing this equation with that of the polar of $(x', y', z')$, we have

$$\lambda l = ax' + hy' + gz',$$
$$\lambda m = hx' + by' + fz',$$
$$\lambda n = gx' + fy' + cz'.$$

Multiply these equations respectively by $A$, $H$, $G$, and add, and we get

$$\Delta x' = \lambda(Al + Hm + Gn).$$

Similarly
$$\Delta y' = \lambda(Hl + Bm + Fn),$$
$$\Delta z' = \lambda(Gl + Fm + Cn).$$

We can therefore write the homogeneous coordinates of the pole of the line $(l, m, n)$ in the form $\left(\frac{\partial\phi}{\partial l}, \frac{\partial\phi}{\partial m}, \frac{\partial\phi}{\partial n}\right)$. We can also write the line-equation of the pole of the line $(l_1, m_1, n_1)$ in the form

$$l\frac{\partial\phi}{\partial l_1} + m\frac{\partial\phi}{\partial m_1} + n\frac{\partial\phi}{\partial n_1} = 0.$$

When $(l, m, n)$ is a tangent to the conic, $\frac{\partial\phi}{\partial l}, \frac{\partial\phi}{\partial m}, \frac{\partial\phi}{\partial n}$ are the coordinates of the point of contact.

These expressions for the point-coordinates of the point of contact of a tangent are related to the line-equation of the conic in exactly the same way as the line-coordinates of the tangent at a given point are related to the point-equation. We can therefore carry out the process of § 10 with line-coordinates and point-coordinates interchanged, and thus the point-equation is derived from the line-equation by exactly the same analytical process. This is in accordance with the identities (4) in Chap. X. § 6.

**12. Conjugate points and lines.** Two points $(x, y, z)$ and $(x', y', z')$ are conjugate when their join is cut harmonically by the conic, or when one lies on the polar of the other. Then their coordinates are connected by the symmetrical relation

$$axx' + byy' + czz' + f(yz' + y'z) + g(zx' + z'x) + h(xy' + x'y) = 0.$$

Similarly, two lines are conjugate if each passes through the pole of the other. Hence, if $(l, m, n)$ and $(l', m', n')$ are conjugate lines,

$$l'(Al + Hm + Gn) + m'(Hl + Bm + Fn) + n'(Gl + Fm + Cn) = 0,$$

or
$$All' + Bmm' + Cnn' + F(mn' + m'n) + G(nl' + n'l) + H(lm' + l'm) = 0.$$

**13.** If the equation

$$Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0$$

reduces, each factor when equated to zero represents a point, and the equation then represents two points. The condition for this is (cf. Chap. II. § 14)

$$\begin{vmatrix} A & H & G \\ H & B & F \\ G & F & C \end{vmatrix} = 0.$$

The equation of the line joining these two points is then (cf. Chap. X. § 5)

$$(HF - BG)x + (GH - AF)y + (AB - H^2) = 0.$$

**Examples.**

1. Find the line-equations corresponding to the following point-equations:

(i) $x^2 - y^2 + 2xy - 4x + 6y + 1 = 0$, (ii) $4x^2 + 4xy + y^2 - 6x + 2y + 1 = 0$,

(iii) $2x^2 - 3xy + 4y^2 + x - 2y + 3 = 0$.

2. Find the point-equations corresponding to the following line-equations: (i) $12l^2+7m^2-n^2+6mn-4nl+12lm=0$, (ii) $(l+m-2n)^2=0$, (iii) $l^2+m^2=0$.

3. Show that $x+y=0$ is a tangent to the conic $4x^2+3y^2+6xy-8x-14y+9=0$, and find its point of contact.

4. Show that the conic $3n^2+2mn-4nl+6lm=0$ touches both of the coordinate axes, and find the coordinates of the points of contact.

5. Find the coordinates of the point of intersection of the tangents to the conic $3x^2-2y^2+6xy-4x+6y+3=0$ at the points where it is cut by the line $x+11y-4=0$.

6. Find the equation of the polar of the point $(-2, 3)$ with respect to the conic $2l^2+3m^2-n^2+2mn-4nl+6lm=0$.

**14. Line-equation of the circular points.** The condition that the line $lx+my+nz=0$ should pass through the point $I\equiv(1, i, 0)$, is

$$l+im=0,$$

and through $J\equiv(1, -i, 0)$, $\quad l-im=0.$

Hence the assemblage of lines through $I$ or $J$ is represented by the quadratic line-equation

$$(l+im)(l-im)\equiv l^2+m^2=0.$$

**15. Foci of a conic, in relation to the circular points.** Consider the ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

From each of the circular points two tangents can be drawn to the curve. The equation of any line through one of the circular points is

$$y-y_1=i(x-x_1),$$

*i.e.* $\quad ix-y+(y_1-ix_1)=0.$

The condition that this should be a tangent to the ellipse is

$$-a^2+b^2=(y_1-ix_1)^2.$$

Hence the equations of the four tangents from $I$ and $J$ to the conic are

$$\pm ix-y\pm i\sqrt{a^2-b^2}=0.$$

Taking these four lines two at a time, we find the coordinates of their points of intersection, other than the points $I$ and $J$, to be

$$(\pm\sqrt{a^2-b^2}, 0)$$

and $\quad (0, \pm i\sqrt{a^2-b^2}).$

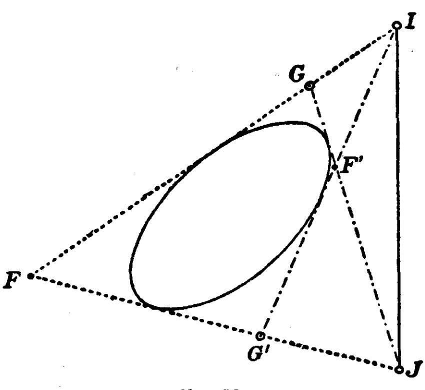


FIG. 72.

But these are the coordinates of the real and imaginary foci. Hence *the foci of a conic are the points of intersection of the tangents to the conic from the circular points.*

This is represented in Fig. 72, where corresponding dotted lines represent conjugate imaginary lines, which intersect in real points.

In the case of a parabola the curve touches the line at infinity $IJ$, and the tangents $IF'$ and $JF'$ coincide. $F'$ becomes the point at infinity on the curve, and $G$, $G'$ coincide with the circular points.

**16. Coordinates of the foci of the conic**

$$ax^2+by^2+2hxy+2gx+2fy+c=0.$$

Let $F\equiv(\alpha, \beta)$ be a focus; then the equation of $FI$ is

$$y-\beta=i(x-\alpha),$$

or, using homogeneous coordinates,

$$x+iy-(\alpha+i\beta)z=0.$$

This line will be a tangent to the conic if its homogeneous line-coordinates $\{1, i, -(\alpha+i\beta)\}$ satisfy the line-equation of the conic

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0.$$

Hence $\quad A-B+C(\alpha+i\beta)^2-2Fi(\alpha+i\beta)-2G(\alpha+i\beta)+2Hi=0,$

*i.e.* $\quad C(\alpha^2-\beta^2)-2G\alpha+2F\beta+A-B+2i(C\alpha\beta-F\alpha-G\beta+H)=0.$

Similarly, the condition that $FJ$ should be a tangent leads to an equation of the same form with the sign of $i$ changed. Adding and subtracting, we get

$$C(\alpha^2-\beta^2)-2G\alpha+2F\beta+A-B=0, \qquad (1)$$

$$C\alpha\beta-F\alpha-G\beta+H=0. \qquad (2)$$

These two equations, being each of the second degree, determine four sets of values of $\alpha$, $\beta$, and therefore four foci. Regarding $\alpha$, $\beta$ as current coordinates, each equation represents a rectangular hyperbola, and it is easily verified that the centre of each is at $(G, F, C)$, *i.e.* both hyperbolas are concentric with the given conic.

Transforming to the common centre as origin, the equations become

$$C^2(\xi^2-\eta^2)=(a-b)\Delta, \qquad (3)$$

$$C^2\xi\eta=h\Delta. \qquad (4)$$

From these we get

$$C^4(\xi^2+\eta^2)^2=C^4(\xi^2-\eta^2)^2+4C^4\xi^2\eta^2=\{(a-b)^2+4h^2\}\Delta^2;$$

hence $$C^2(\xi^2+\eta^2)=\pm R\Delta. \qquad (5)$$

By adding and subtracting (3) and (5), we get the values of $\xi^2$ and $\eta^2$, and hence we get finally the coordinates of the foci, $G/C+\xi$, $F/C+\eta$.

17. A more convenient method of finding the foci in a numerical case is the following. If $F\equiv(\alpha, \beta)$ is a focus, the lines joining $(\alpha, \beta)$ to the circular points, *i.e.* $(x-\alpha)^2+(y-\beta)^2=0$, are tangents to the conic. If the chord of contact of the tangents is $lx+my+n=0$, the equation of the conic, which touches the two lines $FI$, $FJ$ at the points of intersection with this line, is of the form

$$\lambda\{(x-\alpha)^2+(y-\beta)^2\}+(lx+my+n)^2=0;$$

hence, identifying this with the equation of the conic, we have

$$ax^2+by^2+2hxy+2gx+2fy+c-\lambda\{(x-\alpha)^2+(y-\beta)^2\}\equiv(lx+my+n)^2,$$

*i.e.* $\lambda$ must be chosen so that the expression on the left-hand side may be a perfect square. The conditions for this will be found to lead to the same equations (1) and (2) as in § 16.

Examples.

1. Find the foci of the conic $x^2-5y^2+8xy-2x+6y-6=0$.

If $(\alpha, \beta)$ is a focus, and $lx+my+n=0$ the corresponding directrix, we can find $\lambda$ so that

$$x^2-5y^2+8xy-2x+6y-6+\lambda\{(x-\alpha)^2+(y-\beta)^2\}\equiv(lx+my+n)^2,$$

*i.e.* $\lambda$ is to be chosen so that

$$(\lambda+1)x^2+(\lambda-5)y^2+8xy-2(\lambda\alpha+1)x-2(\lambda\beta-3)y+\lambda(\alpha^2+\beta^2)-6 \quad\text{..........(1)}$$

may be a perfect square.

One condition is

$$(\lambda+1)(\lambda-5)=16, \qquad i.e.\ \lambda^2-4\lambda-21=0;$$

whence $\lambda=-3$ or 7.

Taking $\lambda=7$, the expression (1) becomes

$$8x^2+2y^2+8xy-2(7\alpha+1)x-2(7\beta-3)y+7(\alpha^2+\beta^2)-6, \quad\text{...............(2)}$$

*i.e.* $$(2x+y)^2-(7\alpha+1)x-(7\beta-3)y+\tfrac{7}{2}(\alpha^2+\beta^2)-3,$$

and since it is a perfect square it can be written

$$\{2x+y-\tfrac{1}{2}(7\beta-3)\}^2. \quad\text{...............(3)}$$

Identifying this with (2), we have

$$14\beta-6=7\alpha+1, \quad i.e.\ \alpha=2\beta-1, \quad\text{...............(4)}$$

and $$(7\beta-3)^2=14(\alpha^2+\beta^2)-12=14(5\beta^2-4\beta+1)-12;$$

hence $$3\beta^2-2\beta-1=0,$$

giving $\beta=1$ or $-\frac{1}{3}$.

The coordinates of the real foci are therefore, from (4), (1, 1) and $(-\frac{5}{3}, -\frac{1}{3})$.

Similarly $\lambda=-3$ will give the imaginary foci.

Then the equations of the directrices are found by putting the values of $\beta$ in the expression (3), and equating to zero, viz.

$$2x+y-2=0 \quad\text{and}\quad 6x+3y+8=0.$$

2. Prove directly that the conditions that the origin should be a focus are $A=B$ and $H=0$.

3. Find the real foci of the conic $5x^2+10y^2-12xy+6x-10y+6=0$, and the corresponding directrices.

4. Find the real foci, directrices, and eccentricity of the conic

$$3x^2-4xy+2x+4y-9=0.$$

**18. Coordinates of the foci of a conic whose line-equation is given.** The coordinates of the foci of a conic are most easily found when the line-equation is given,

$$\phi\equiv Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0. \quad\text{...............(1)}$$

This represents the assemblage of all tangents to the conic. Now we have to consider specially the tangents which pass through one or other of the circular points. The equation which represents the assemblage of all lines through one or other of the circular points, *i.e.* the line-equation of the circular points, is $$l^2+m^2=0. \quad\text{...............(2)}$$

Solving (1) and (2) simultaneously, we would get the line-coordinates of the tangents $FI$, $FJ$, etc., from the circular points to the conic. We require, however, the coordinates of the points $F$, $F'$, etc., or their line-equation. Now the equation

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm+\lambda(l^2+m^2)=0 \quad \text{.........(3)}$$

is the line-equation of a conic, and it is satisfied by the values of $l$, $m$, $n$, which satisfy both (1) and (2), *i.e.* it represents a conic having the four lines $FI$, $FJ$, $F'I$, $F'J$ as tangents, and therefore $F$, $F'$ and $G$, $G'$ as foci. But if the equation (3) breaks up into factors it represents two pencils of lines, and these are the pencils through a pair of foci. The condition for this is

$$\begin{vmatrix} A+\lambda & H & G \\ H & B+\lambda & F \\ G & F & C \end{vmatrix}=0,$$

*i.e.*

$$C\lambda^2+(a+b)\Delta\lambda+\Delta^2=0.$$

Let $\lambda_1$ and $\lambda_2$ be the roots of this equation; then the line-equations of the two pairs of foci are

$$\phi+\lambda_1(l^2+m^2)=0 \quad \text{and} \quad \phi+\lambda_2(l^2+m^2)=0.$$

Each of these expressions will factorize, the one into real and the other into imaginary factors. If the real factors are

$$(x_1l+y_1m+n)(x_2l+y_2m+n),$$

the coordinates of the real foci are $(x_1, y_1)$ and $(x_2, y_2)$.

Ex. Find the foci of the conic $15x^2+12y^2-4xy-44x+8y-4=0$. The line-equation is

$$4l^2+34m^2-11n^2+2mn-32nl+12lm=0.$$

The equation of any conic with the same foci is

$$(4-\lambda)l^2+(34-\lambda)m^2-11n^2+2mn-32nl+12lm=0.$$

This will break up into factors if $\lambda=25$ or $\frac{400}{11}$.

Taking $\lambda=25$, we find the factors

$$(l-m+n)(21l+9m+11n)=0.$$

Hence the real foci are $(1, -1)$ and $(\frac{21}{11}, \frac{9}{11})$. The other value of $\lambda$ would give the imaginary foci.

**19. Focus and axis of a parabola.** When the conic is a parabola $C=0$, and the two hyperbolas in § 16 reduce to straight lines

$$2Gx-2Fy=A-B, \quad \text{.........(1)}$$

$$Fx+Gy=H. \quad \text{.........(2)}$$

In each case the remainder of the curve is the straight line at infinity. We get then only one finite focus, the other three being at infinity.

Solving these equations, we get the coordinates of the single focus

$$2(G^2+F^2)x=2HF+G(A-B),$$

$$2(G^2+F^2)y=2GH-F(A-B).$$

**20.** For the parabola we may also use the method of § 18. In this case, since $C=0$, only one value is obtained for $\lambda$, the second root being infinite. The line-equation of the real foci then becomes

$$(a+b)(Al^2+Bm^2+2Fmn+2Gnl+2Hlm)-\Delta(l^2+m^2)=0.$$

Since there is no term in $n^2$, this must break up into factors of the form

$$(x_1 l+y_1 m+n)(l+\mu m)=0.$$

The real and only finite focus is then $(x_1, y_1)$. The second factor represents a point at infinity in the direction of the line $y=\mu x$; this therefore gives the direction of the axis.

**Examples.**

1. Find the focus and axis of the parabola

$$4x^2+y^2-4xy+22x-6y+24=0.$$

The line-equation is $\qquad 3l^2-5m^2-4mn-2nl+6lm=0.$

The equation of any confocal conic is

$$(3+\lambda)l^2+(-5+\lambda)m^2-4mn-2nl+6lm=0.$$

This will break up into factors if $\lambda=1$, and we get the factors

$$(2l-m-n)(l+2m)=0.$$

Hence the focus is $(-2, 1)$, and the axis is parallel to $y=2x$. Since the axis passes through the focus, the equation of the axis is

$$y-2x=1+4=5.$$

2. Find the coordinates of the real foci of the conics whose line-equations are

(i) $3l^2+n^2+3nl-lm=0$, (ii) $4l^2-4m^2+n^2-mn+5nl-lm=0$,

(iii) $l^2-4m^2-3mn-nl+5lm=0$.

3. Find the coordinates of the real foci of the conics:

(i) $3x^2+4xy-2x-6y-4=0$,

(ii) $19x^2+11y^2-6xy+76x-132y+256=0$,

(iii) $4x^2+y^2-4xy+6x-18y+36=0$,

(iv) $3x^2+3y^2-2xy-8x+8y+15=0$.

## EXAMPLES XI.

1. Prove that the conic $30x^2+35y^2=12xy+24x+16y+16$ has one focus at the origin. Find the equation of the corresponding directrix, the eccentricity, and the coordinates of the second focus. (Pembroke, 1909.)

2. Find the equations of the two conics which touch the coordinate axes (supposed rectangular), have a focus at the point (1, 1), and pass through the point $(\frac{1}{2}, 1)$. (Pembroke, 1912.)

3. Trace the conic $14x^2-4xy+11y^2+20x-20y-4=0$, and find the coordinates of the foci. (Corpus, 1912.)

4. Prove that the conic $9x^2-24xy+41y^2=15x+5y$ has one extremity of its major axis at the origin, and one extremity of its minor axis on the axis of $x$. Find the coordinates of its centre and foci. (Pembroke, 1910.)

5. The conic $ax^2+2hxy+by^2+2x+2y=0$ is such that its real foci lie one on each of the coordinate axes; show that $ab-h^2=2(h-a)(h-b)$, and that the lengths of its semi-axes are $-\frac{1}{2h}$ and $\frac{1}{\sqrt{(ab-h^2)}}$. (Selwyn, 1907.)

6. Prove that if the normals at the points in which the conic $ax^2+by^2=1$ is cut by the lines $lx+my=1$ and $l'x+m'y=1$ meet in a point, then

$$ll'/a=mm'/b=-1.$$

Show that if one of these lines passes through a fixed point, the other touches a fixed parabola. (Corpus, 1914.)

7. From a given point on a conic two chords are drawn equally inclined to a fixed direction. Prove that the line joining their other extremities passes through a fixed point. (Pembroke, 1906.)

8. Show that the line-equation of the two points at infinity on the conic $ax^2+by^2+2hxy+2gx+2fy+c=0$ is $bl^2+am^2-2hlm=0$.

9. Show that the line-equation of the two points in which the line $(l', m', n')$, cuts the conic $\Sigma\equiv Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0$ is

$$\Sigma\Sigma'=\{l(Al'+Hm'+Gn')+m(Hl'+Bm'+Fn')+n(Gl'+Fm'+Cn')\}^2.$$

10. A chord $PQ$ of $x^2/a^2+y^2/b^2=1$ is drawn through the fixed point $(f, g)$. Prove that, if the circle through $P$, $Q$, and $C$, the centre of the ellipse, cuts the curve again in the points $R$, $S$, then $RS$ will touch a fixed parabola whose focus is $C$ and tangent at the vertex $(a^2-b^2)(gy-fx)+a^2b^2=0$. (Trinity, 1906.)

11. Show that the envelope of the polars, with respect to $a'x^2+b'y^2+c'=0$, of points on $ax^2+by^2+2gx+2fy=0$ is

$$(a'fx-b'gy)^2+2c'(a'bgx+ab'fy)-abc'^2=0.$$

12. The perpendiculars from fixed points $P_1$, $P_2$, ... on a variable line have lengths $p_1$, $p_2$, ..., and the line moves so that the sum $\Sigma k_1p_1{}^2=0$. Show that the envelope of the line is a conic; and that if $\Sigma k_1=0$ it is a parabola, and if $\Sigma k_1k_2(P_1P_2)^2=0$ it is a rectangular hyperbola. (Pembroke, 1910.)

13. $O$ and $A$ are fixed points on the circumference of a circle of radius $a$, $OA$ subtending at the centre an angle $2\alpha$. The tangent at a variable point $P$ on the circle intersects the tangent at $O$ in $T$, $TQ$ is drawn parallel to $AP$; show that $TQ$ envelops a parabola, and find its position and magnitude. (Queens', 1910.)

14. Show that if $x\cos\psi+y\sin\psi=p_1$, $x\cos\psi+y\sin\psi=p_2$ represent a variable pair of parallel tangents of a fixed conic, the lines $x\cos\psi+y\sin\psi=\lambda p_1+\mu p_2$, $x\cos\psi+y\sin\psi=\lambda p_2+\mu p_1$, where $\lambda$, $\mu$ are constants, envelop another fixed conic with parallel asymptotes. Explain the case when $\lambda+\mu=1$. (Pembroke, 1914.)

15. Find the conditions that the general equation of the second degree in $l$, $m$, $n$ should represent a circle, and obtain the coordinates of the centre.

16. Show that the coordinates of the foci of the general conic are given by the equations

$$Cx^2-2Gx+A=\lambda\Delta,$$
$$Cy^2-2Fy+B=\lambda\Delta,$$

where $\lambda$ is either root of the quadratic

$$C\lambda^2-(a+b)\lambda+1=0.$$ (King's, 1913.)

17. Show that the equation of the polars of the circular points with respect to the conic $S=0$ is

$$\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2=0.$$

18. Prove that

$$b\left(\frac{\partial S}{\partial x}\right)^2+a\left(\frac{\partial S}{\partial y}\right)^2-2h\frac{\partial S}{\partial x}\frac{\partial S}{\partial y}\equiv 4(CS-\Delta).$$

Interpret the equations obtained by equating each side to zero.

19. Show that the equation of the directrices of the conic $S=0$ can be written in the form $\left(\frac{\partial S}{\partial x}\right)^2+\left(\frac{\partial S}{\partial y}\right)^2-4\lambda S=0$, where $\lambda$ is a root of the quadratic equation

$$C\lambda^2-(a+b)\lambda+1=0.$$

20. Writing the line-equation of a conic in the form

$$Al^2+2Hlm+Bm^2+n(2Gl+2Fm+Cn)=0,$$

interpret the equations $Al^2+2Hlm+Bm^2=0$ and $2Gl+2Fm+Cn=0$.

21. Show that the line-equation of the conic $S=0$ can be written in the form

$$(Gl+Fm+Cn)^2+\Delta(bl^2-2hlm+am^2)=0$$

and interpret it.

# CHAPTER XII.

## PROJECTIVE GEOMETRY AND HOMOGENEOUS COORDINATES.

**1.** THE history of geometry reveals three successive stages in generalization. First there is the geometry of Euclid and its developments by Apollonius and others: Elementary Metrical Geometry; in this the theory of parallel straight lines always formed a sort of stumbling-block and necessitated the frequent statement of exceptional cases, the simplest of which is contained in the proposition: "two straight lines intersect in a point unless they are parallel." The next stage was inaugurated by Desargues' conception of parallel lines meeting at infinity, and is characterized by the introduction of points and lines at infinity. By this convenient notation it was possible to get rid of the exceptional cases and state general theorems which could be interpreted in the language of metrical geometry. It was shown by Poncelet how by the process of projection a figure which involves parallel lines could be transformed into one in which the corresponding lines intersect, and that thus there is no real distinction between the two cases. In the third stage, that of Pure Projective Geometry, the scaffolding of metrical geometry was taken away and a uniform system of geometry revealed in which points and lines at infinity are irrelevant and two coplanar lines intersect in a point without exception.

**2.** For metrical geometry the most convenient system of coordinates is the rectangular cartesian system, and in this the exceptional case of non-intersecting lines survives. The device of replacing the cartesian coordinates $x$ and $y$ by the ratios $X/Z$ and $Y/Z$, thus introducing homogeneous cartesian coordinates, enables points at infinity to be represented on the same footing as ordinary points. We proceed now to take the third step and introduce a system of coordinates which is entirely non-metrical, and forms a convenient medium for projective geometry.

**3. Projective coordinates.** A point is represented by or defined as a set of three numbers or coordinates $(x, y, z)$—an ordered triad—such that if $k$ is any number, not zero, the triad $(kx, ky, kz)$ represents always the same point as $(x, y, z)$. A straight line is defined as the class of points whose coordinates satisfy a homogeneous equation of the first degree in $x, y, z$,

$$lx + my + nz = 0.$$

With these definitions we verify the axioms that two distinct points uniquely determine a straight line, and two distinct lines uniquely determine a point.

If $u$ and $v$ are homogeneous linear expressions in $x$, $y$, $z$, the equation $u+\lambda v=0$ represents a straight line through the intersection of $u=0$ and $v=0$.

The equations $x=0$, $y=0$, $z=0$ represent three straight lines, called the lines of reference, and it is essential for this representation that they should form a triangle and not have a point in common. For if they all had a point $P$ in common, the equation $lx+my+nz=0$ would represent a line through $P$ and could not represent an arbitrary line. The triangle thus formed is called the *triangle of reference.*

**4. Unit-point.** In fixing a system of coordinates of this kind we may take for triangle of reference any three lines which form a triangle $ABC$. But obviously something more is required before we can assign the coordinates of a given arbitrary point. There is one unique point with the coordinates (1, 1, 1). We call this the *unit-point* $I$. If this point is known it can be shown that the coordinates of any other point can be determined if that point can be derived from $A$, $B$, $C$ and $I$ by a construction involving only intersections of lines and joining of points (projective construction).

**5. Harmonic ranges and pencils.** As an example, which is fundamental, consider the following construction. Let $AI$, $BI$, $CI$ cut the opposite sides of the triangle in $L$, $M$, $N$; let $MN$ cut $BC$ in $L'$.

The equation of $AI$ is $y-z=0$, and the point $L$ is therefore $(0,1,1)$. $BI$ is $x-z=0$, and therefore $NM$ is $x-z+\mu y=0$ where $\mu$ is to be determined. Also $CI$ is $x-y=0$ and therefore $NM$ is also $x-y+\nu z=0$. Identifying these two equations we find $\mu=-1$ and $\nu=-1$, so that $NM$ is $x-y-z=0$. Hence $L'$ is $(0, 1, -1)$, and $AL'$ is $y+z=0$.

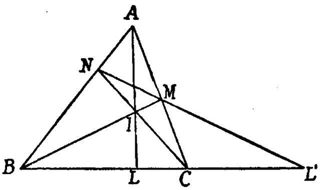


FIG. 73.

More generally, if $AL$ is $y-\lambda z=0$, then $AL'$ is $y+\lambda z=0$.

This is the construction for a harmonic range $(BC, LL')$ or a harmonic pencil $A(BC, LL')$. We find therefore that *the two pairs of lines*

$$y=0,\ z=0 \quad \text{and} \quad y=\pm\lambda z$$

*form a harmonic pencil.*

**6.** Still more generally we shall find the condition that the two pairs of lines $y=\lambda x$, $y=\mu x$ and $y=\lambda' x$, $y=\mu' x$ should form a harmonic pencil.

Let $y-\lambda x\equiv X$, $y-\mu x\equiv Y$,

then $$(\lambda-\mu)x\equiv Y-X, \quad (\lambda-\mu)y\equiv\lambda Y-\mu X.$$

Hence $$(\lambda-\mu)(y-\lambda' x)\equiv(\lambda Y-\mu X)-\lambda'(Y-X)\equiv(\lambda-\lambda')Y-(\mu-\lambda')X.$$

Similarly $(\lambda-\mu)(y-\mu' x)\equiv(\lambda-\mu')Y-(\mu-\mu')X.$

Applying the previous result we find then that the condition that the two pairs of lines

$$\left.\begin{matrix}X=0,\\ Y=0,\end{matrix}\right\} \text{ and } \left.\begin{matrix}(\lambda-\lambda')Y-(\mu-\lambda')X=0\\ (\lambda-\mu')Y-(\mu-\mu')X=0\end{matrix}\right\}$$

should form a harmonic pencil is

$$\frac{\lambda-\lambda'}{\mu-\lambda'}\bigg/\frac{\lambda-\mu'}{\mu-\mu'}=-1.$$

The construction for a harmonic range or pencil is a purely projective one, and we take this construction in fact as defining a harmonic range and no longer define it metrically in terms of ratios of segments. In the present discussion we wish to exclude segments and angles entirely from our consideration.

Ex. Show that $y=\lambda x$, $x=\lambda y$ are harmonic w.r.t. $y=\pm x$.

**7. Line-coordinates.** A system of coordinates for the line as element can be defined in a precisely similar way. A line is represented by a set of three numbers $(l, m, n)$—ordered triad—such that if $k$ is any number, not zero, the coordinates $(kl, km, kn)$ always represent the same line as $(l, m, n)$. A homogeneous equation of the first degree in $l, m, n$,

$$al+bm+cn=0,$$

represents the class of lines whose coordinates satisfy this equation. Two such equations determine the ratios $l:m:n$ uniquely; and two distinct lines, *i.e.* two distinct sets of values of $l, m, n$, determine uniquely a set of coefficients $a, b, c$. We can thus identify the class of lines represented by the homogeneous linear equation with a pencil of lines through a fixed point. A point in line-coordinates is thus represented by a homogeneous equation of the first degree. But the point is also determined by the ratios of the three coefficients $(a, b, c)$. We may thus connect the system of line-coordinates with the already defined system of point-coordinates. The equation

$$lx+my+nz=0$$

connects the point-coordinates $(x, y, z)$ of a point with the line-coordinates $(l, m, n)$ of a line; the point and line are said to be *incident* with one another, the point lies on the line and the line passes through the point. If $l, m, n$ are fixed, the equation represents the locus of points lying on the line $(l, m, n)$; if $x, y, z$ are fixed, it represents the assemblage of lines passing through the point $(x, y, z)$.

$l=0$ is the equation of the point $(1, 0, 0)$, *i.e.* the vertex $A$ of the triangle of reference. Thus $l=0$, $m=0$, $n=0$ represent the three vertices.

**8.** If $u$ and $v$ are homogeneous expressions of the first degree in $l, m, n$, the equation $u+\lambda v=0$ represents a point lying on the line joining the points $u=0$ and $v=0$. In particular if

$$u\equiv x_1 l+y_1 m+z_1 n \text{ and } v\equiv x_2 l+y_2 m+z_2 n,$$

$u+\lambda v=0$ represents the point whose coordinates are

$$(x_1+\lambda x_2,\ y_1+\lambda y_2,\ z_1+\lambda z_2).$$

Hence the equations

$$\left.\begin{aligned}\rho x &= x_1 + \lambda x_2,\\ \rho y &= y_1 + \lambda y_2,\\ \rho z &= z_1 + \lambda z_2,\end{aligned}\right\} \qquad \text{.........(1)}$$

where $\rho$ is a factor of proportionality, are freedom-equations of the line joining the points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$. Similarly

$$\left.\begin{aligned}\rho\, l &= l_1 + \lambda l_2,\\ \rho m &= m_1 + \lambda m_2,\\ \rho\, n &= n_1 + \lambda n_2,\end{aligned}\right\} \qquad \text{.........(2)}$$

are freedom-equations of the point of intersection of the lines $(l_1, m_1, n_1)$ and $(l_2, m_2, n_2)$.

**9. Conditions for collinearity and concurrency.** The equation of the line joining two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ is found by eliminating $\rho$ and $\lambda$ between equations (1).

The condition for collinearity of three points $(x_1, y_1, z_1)$, $(x_2, y_2, z_2)$, $(x_3, y_3, z_3)$ is found by substituting the coordinates of one of the points in the equation of the line joining the other two; hence it is

$$\begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix} = 0.$$

To find the point of intersection of the two lines

$$l_1 x + m_1 y + n_1 z = 0,$$
$$l_2 x + m_2 y + n_2 z = 0,$$

we have $\qquad x : y : z = m_1 n_2 - m_2 n_1 : n_1 l_2 - n_2 l_1 : l_1 m_2 - l_2 m_1.$

The condition that the three lines

$$l_1 x + m_1 y + n_1 z = 0,$$
$$l_2 x + m_2 y + n_2 z = 0,$$
$$l_3 x + m_3 y + n_3 z = 0$$

should be concurrent is found by substituting in one equation the coordinates of the point of intersection of the other two lines; hence it is

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$

**10. Principle of Duality.** The exact correspondence between point-coordinates and line-coordinates is expressed by the *Principle of Duality.* Any theorem involving only the incidence of points and lines can give rise to a dual or reciprocal theorem obtained by interchanging the terms "point" and "line," "point of intersection" and "line joining," "concurrent" and "collinear." Both theorems can be expressed by the same algebraic work, merely interpreting $x$, $y$, $z$ in the one case as point-coordinates and in the other as line-coordinates, and *vice versa* for $l$, $m$, $n$.

**11.** The figure formed of four arbitrary straight lines $a$, $b$, $c$, $d$, and their six points of intersection or vertices, $(bc)\equiv A$, $(ca)\equiv B$, $(ab)\equiv C$, $(ad)\equiv A'$, $(bd)\equiv B'$, $(cd)\equiv C'$, is called a *complete quadrilateral*; the reciprocal figure, formed of four arbitrary points and their six joins, is called a *complete quadrangle*. The joins of opposite vertices $AA'$, $BB'$, $CC'$ of the complete quadrilateral form a triangle called its *harmonic triangle*; and similarly the points of intersection of pairs of opposite sides of the complete quadrangle form its harmonic triangle.

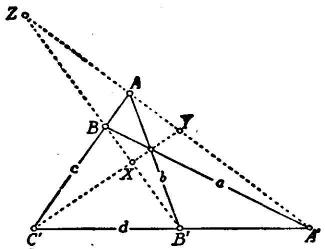


FIG. 74.

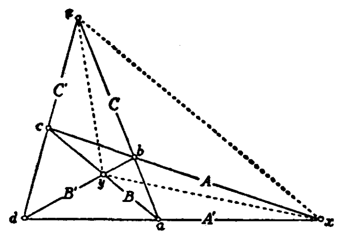


FIG. 75.

Taking the harmonic triangle of the complete quadrilateral (quadrangle) as triangle of reference, let the equation of one side (vertex) $d$ be

$$x+y+z=0.$$

The side (vertex) $a$ passes through (lies on) the intersection (join) of

$$x+y+z=0 \quad \text{and} \quad x=0,$$

hence its equation is of the form

$$\lambda x+y+z=0.$$

Similarly the equations of $b$ and $c$ are of the form

$$x+\mu y+\ z=0,$$

and

$$x+\ y+\nu z=0.$$

But $b$ and $c$ intersect in (are joined by) $A$, which lies on (passes through) $x$. Hence the last two equations are satisfied simultaneously by $x=0$ and $y/z=-1/\mu$ or $-\nu$, therefore

$$\mu\nu=1.$$

Similarly $\qquad \nu\lambda=1 \quad \text{and} \quad \lambda\mu=1.$

Therefore $\qquad \lambda^2\mu^2\nu^2=1.$

$\lambda\mu\nu=+1$ would give $\lambda=\mu=\nu=1$ and $a$, $b$, $c$ would coincide. Excluding this we have $\lambda\mu\nu=-1$, and then $\lambda=\mu=\nu=-1$. Hence the equations of the four lines (points) are

$$\pm x\pm y\pm z=0.$$

We deduce from this that it is always possible by proper choice of coordinates to represent four arbitrary lines, no three of which are concurrent, by the equations

$$\pm x \pm y \pm z = 0,$$

and four arbitrary points, no three of which are collinear, by the coordinates

$$(\pm 1, \pm 1, \pm 1).$$

**12. Examples.**

Ex. 1. $AA'BB'CC'$ is a complete quadrilateral and $LMN$ its harmonic triangle. $P$, $Q$, $R$ are three collinear points on the sides of the triangle $LMN$, and $P'$, $Q'$, $R'$ are their harmonic conjugates with regard to the pairs of opposite vertices $AA'$, $BB'$, $CC'$ which lie on the sides $MN$, $NL$, $LM$. Prove that $P'$, $Q'$, $R'$ are collinear.

| | |
|---|---|
| If the harmonic triangle $LMN$ of a complete quadrilateral be taken as the triangle of reference the point-coordinates of the pairs of opposite vertices $A, A'$; $B, B'$; $C, C'$ of the complete quadrilateral can be taken to be | If the harmonic triangle $lmn$ of a complete quadrangle be taken as the triangle of reference the line-coordinates of the pairs of opposite sides $a, a'$; $b, b'$; $c, c'$ of the complete quadrangle can be taken to be |

$$(0, 1, \pm 1), \quad (\pm 1, 0, 1), \quad (1, \pm 1, 0).$$

| | |
|---|---|
| Let the three collinear points $P, Q, R$, one on each of the sides of the harmonic triangle, lie on the line whose point-equation is | Let the three concurrent lines $p, q, r$, one through each of the vertices of the harmonic triangle, pass through the point whose line-equation is |

$$lx + my + nz = 0.$$

| | |
|---|---|
| $P$ is the intersection of this line with the side $MN$, and its point-coordinates are therefore, putting $x=0$, | $p$ is the join of this point to the vertex $mn$, and its line-coordinates are therefore, putting $x=0$, |

$$(0, -n, m).$$

| | |
|---|---|
| The harmonic conjugate $P'$, of $P$ with regard to $A, A'$, *i.e.* $(0, 1, \pm 1)$, is | The harmonic conjugate $p'$, of $p$ with regard to $a, a'$, *i.e.* $(0, 1, \pm 1)$, is |

$$(0, m, -n).$$

| | |
|---|---|
| Similarly the point-coordinates of $Q'$ and $R'$, the harmonic conjugates of $Q$ and $R$ with regard to $B, B'$ and $C, C'$ respectively, are | Similarly the line-coordinates of $q'$ and $r'$, the harmonic conjugates of $q$ and $r$ with regard to $b, b'$ and $c, c'$ respectively, are |

$$(-l, 0, n) \quad \text{and} \quad (l, -m, 0).$$

| | |
|---|---|
| But | But |

$$\begin{vmatrix} 0 & m & -n \\ -l & 0 & n \\ l & -m & 0 \end{vmatrix} = lmn \begin{vmatrix} 0 & 1 & -1 \\ -1 & 0 & 1 \\ 1 & -1 & 0 \end{vmatrix} = 0.$$

Hence $P', Q', R'$ are collinear. Therefore, if a straight line cuts the diagonals $AA', BB', CC'$ of a complete quadrilateral in the three points $P, Q, R$, and $P', Q', R'$ are the harmonic conjugates of $P, Q, R$ with regard to the corresponding pairs of vertices, then $P', Q', R'$ are collinear.

Hence $p', q', r'$ are concurrent. Therefore, if a point is joined to the diagonal points $aa', bb', cc'$ of a complete quadrangle by the three lines $p, q, r$, and $p', q', r'$ are the harmonic conjugates of $p, q, r$ with regard to the corresponding pairs of sides, then $p', q', r'$ are concurrent.

**2.** Let $P \equiv (f, g, h)$ be any point, and let $AP, BP, CP$ cut the sides $BC, CA, AB$ of the triangle $ABC$ in $D, E, F$. Let $EF, FD, DE$ cut $BC, CA, AB$ in $X, Y, Z$. Then $X, Y, Z$ are collinear.

The equation of $AD$ is $\frac{y}{g}=\frac{z}{h}$, and similarly for $BE$ and $CF$.

The equation of $EF$, which joins the points $(f, 0, h)$ and $(f, g, 0)$, is

$$\begin{vmatrix} x & y & z \\ f & 0 & h \\ f & g & 0 \end{vmatrix}=0,$$

*i.e.*

$$-\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0,$$

and this line cuts $x=0$ at its intersection with the line

$$\frac{x}{f}+\frac{y}{g}+\frac{z}{h}=0.$$

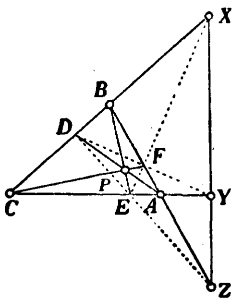


FIG. 76.

The symmetry of this equation shows that this line passes through $Y$ and $Z$ also.

The line $XYZ$ is called the *polar of the point P with respect to the triangle.* It is the polar line of the point $(f, g, h)$ with respect to the triangle regarded as a curve of the third degree. For the equation of the curve is

$$\phi(x, y, z) \equiv xyz=0,$$

and the polar line is defined by the equation

$$x\frac{\partial\phi}{\partial f}+y\frac{\partial\phi}{\partial g}+z\frac{\partial\phi}{\partial h}=0,$$

*i.e.*

$$xgh+yhf+zfg=0.$$

If we form the analogous equation

$$f\frac{\partial\phi}{\partial x}+g\frac{\partial\phi}{\partial y}+h\frac{\partial\phi}{\partial z}=0,$$

we obtain the equation of a conic

$$fyz+gzx+hxy=0,$$

which is called the *polar conic* of the point $(f, g, h)$ with regard to the triangle. Its equation is also given by

$$x^2\frac{\partial^2\phi}{\partial f^2}+\ldots+2yz\frac{\partial^2\phi}{\partial g\,\partial h}+\ldots=0.$$

**13. Cross-ratio.** The expression

$$\frac{\lambda_1-\lambda_3}{\lambda_1-\lambda_4}\bigg/\frac{\lambda_2-\lambda_3}{\lambda_2-\lambda_4},$$

which is a function of the parameters of four lines of a pencil,

$$y=\lambda_1 x, \quad y=\lambda_2 x, \quad y=\lambda_3 x, \quad y=\lambda_4 x,$$

referred to the base-rays $x=0$ and $y=0$, and which has the value $-1$ when the two pairs 1, 2 and 3, 4 are harmonic, is called the *cross-ratio* of the four parameters and is written $(\lambda_1\lambda_2, \lambda_3\lambda_4)$. Four collinear points also may be determined by four parameters, when referred to two base-points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$; viz. the coordinates are

$$(x_1+\lambda x_2, \quad y_1+\lambda y_2, \quad z_1+\lambda z_2),$$

where $\lambda$ is given the values $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

But the same set of points may be represented by different parameters, depending upon the choice of base-points and the "unit-point" or point corresponding to $\lambda=1$. The great importance of the cross-ratio of the parameters depends upon the fact that it is *independent of the choice of base-points and unit-point.* The proof of this depends upon a general theorem (for the proof of which see Chap. XVIII, § 4), that *when a one-to-one correspondence exists between two sets of numbers $t$, $t'$ the cross-ratio of any four values of $t$ is equal to the cross-ratio of the four corresponding values of $t'$.*

Between the points of a range and the values of the parameter there is a one-to-one correspondence; to every point corresponds one value of $\lambda$ and conversely. If $\lambda'$ is another parameter by which the points are determined, there is again a (1, 1) correspondence between the points and the values of $\lambda'$. Hence there is a (1, 1) correspondence between $\lambda$ and $\lambda'$. Similar relations hold for a pencil of lines. We therefore speak of the cross-ratio of a range or a pencil, as it is a characteristic of the figure itself and does not depend upon the frame of reference.

**Ex.** Show that the two points $(x_1+\lambda x_2, \ldots)$ and $(x_1-\lambda x_2, \ldots)$ are harmonic conjugates w.r.t. the base-points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$.

**14.** Two ranges of points which are transversals of the same pencil are said to be in *perspective*; the joins of corresponding points are concurrent. Similarly two pencils of lines such that the intersections of corresponding lines are collinear are said to be in perspective. Two ranges in perspective are in (1, 1) correspondence, and therefore the cross-ratio of any four points is equal to that of the four corresponding points; this is also equal to the cross-ratio of the pencil of lines joining the corresponding points.

The processes by which new ranges and pencils are formed in this way are called projection and section, or simply projection. Two ranges or two pencils which can be connected by a finite number of projections are said to be projective and have the same cross-ratio.

**15. The six different cross-ratios of four numbers.** The cross-ratio of four numbers $a$, $b$, $c$, $d$ depends upon the order in which they are taken. We can write the cross-ratio $(ab, cd)$ in the form

$$(ab, cd) = \frac{a-c}{a-d} \Big/ \frac{b-c}{b-d} = \frac{C-A}{C-B},$$

where $A \equiv bc + ad$, $B \equiv ca + bd$, $C \equiv ab + cd$. There are 24 orders of the four numbers $a$, $b$, $c$, $d$, but only 6 different orders of $A$, $B$, $C$. Hence there are only six different values of the cross-ratio.

$A$, $B$, $C$ are each unaltered if we interchange any two of the four numbers $a$, $b$, $c$, $d$ and at the same time interchange the other two. Therefore

$$(ab, cd) = (ba, dc) = (cd, ab) = (dc, ba).$$

Hence they fall into six sets of four.

Further, if $a$ and $b$ *or* $c$ and $d$ are interchanged, $A$ is interchanged with $B$, and $C$ is unaltered; hence the cross-ratio $(ab, cd)$ is changed into its reciprocal.

Again,
$$(ac, bd) = \frac{B-A}{B-C} = 1 - \frac{C-A}{C-B} = 1 - (ab, cd).$$

Hence, denoting $(ab, cd)$ by $\lambda$, and applying in alternate succession the two transformations, taking the reciprocal and subtracting from unity, we get the six values

$$\lambda, \quad \frac{1}{\lambda}, \quad 1 - \frac{1}{\lambda}, \quad \frac{\lambda}{\lambda - 1}, \quad \frac{1}{1-\lambda}, \quad 1 - \lambda.$$

**16. Special cases of equality among the cross-ratios.** If $\lambda = \frac{1}{\lambda}$, we have $\lambda = \pm 1$.

(1) Let $\lambda = 1$. Then the six values reduce to three, viz.

$$\lambda = \frac{1}{\lambda} = 1, \quad 1 - \lambda = 1 - \frac{1}{\lambda} = 0, \quad \frac{1}{1-\lambda} = \frac{\lambda}{\lambda-1} = \infty.$$

In this case, $(a-b)(c-d) = 0$, so that either $a = b$ or $c = d$.

(2) Let $\lambda = -1$. The six values again reduce to three, viz.

$$\lambda = \frac{1}{\lambda} = -1, \quad 1 - \lambda = 1 - \frac{1}{\lambda} = 2, \quad \frac{1}{1-\lambda} = \frac{\lambda}{\lambda - 1} = \frac{1}{2}.$$

The pairs $a$, $b$ and $c$, $d$ are harmonic.

(3) There is only one other distinct case of equalities, viz. when $\lambda = 1 - \frac{1}{\lambda}$. This gives $\lambda^2 - \lambda + 1 = 0$, *i.e* $\lambda = -\omega$ or $-\omega^2$, the imaginary cube roots of $-1$. The six values then reduce to two, viz.

$$\lambda = 1 - \frac{1}{\lambda} = \frac{1}{1-\lambda} = -\omega, \quad \frac{1}{\lambda} = \frac{\lambda}{\lambda - 1} = 1 - \lambda = -\omega^2.$$

In this case the four numbers, which cannot be all real, are said to be *equianharmonic.*

**Examples.**

1. Prove that the six cross-ratios of four real points in a straight line, taken in all orders, can be represented by $\sin^2\theta$, $\cos^2\theta$, $-\tan^2\theta$, $\operatorname{cosec}^2\theta$, $\sec^2\theta$, $-\cot^2\theta$.

2. Prove that $(XY, PQ).(XY, QR)=(XY, PR)$.

3. If $X, Y, A, B$ are fixed points on a straight line, show that there are two distinct points $P$ for which $(XY, AP)=(XY, PB)$, and that they are harmonic conjugates with regard to $X, Y$.

4. Two points $X, Y$ separate harmonically each of the three pairs of points $P, P'$; $Q, Q'$; $R, R'$. Prove that $(PP', QR)=(P'P, Q'R')$.

5. Prove that the three points whose cartesian coordinates are $(-1, -\omega)$, $(-\omega, -\omega^2)$, $(-\omega^2, -1)$ each form a harmonic range with the three points $(1, \omega)$, $(\omega, \omega^2)$, $(\omega^2, 1)$, $\omega$ denoting one of the complex cube roots of unity.

6. Show that the four points whose cartesian coordinates are $(0, 0)$, $(1, \omega)$, $(\omega, \omega^2)$, $(\omega^2, 1)$ form an equianharmonic range.

7. $OP, OQ$ are two lines through the origin containing an angle 30°; prove that they form with the imaginary lines $x^2+y^2=0$ an equianharmonic pencil.

**17. Expressions for the homogeneous coordinates by cross-ratios.**

Let $P\equiv(X, Y, Z)$ be any point, and $I\equiv(1, 1, 1)$ the unit-point, referred to the triangle $ABC$. Then we have a pencil $A(BC, IP)$, the equations of whose rays are $z=0$, $y=0$, $y-z=0$, $Zy-Yz=0$. These are all of the form $y-\lambda z=0$, where $\lambda$ has respectively the values $\infty$, 0, 1, $Y/Z$. Hence the cross-ratio

$$A(BC, IP)=(\infty, 0;\ 1, Y/Z)=Y/Z.$$

Similarly $\quad B(CA, IP)=Z/X \quad$ and $\quad C(AB, IP)=X/Y.$

In a similar way it may be proved that if $p\equiv(l, m, n)$ is any line, and $i$ is the unit-line $(1, 1, 1)$ or $x+y+z=0$, the cross-ratio of the range formed by the intersections of the side $a$ or $BC$ with the other two sides of the triangle of reference and the two lines $i$ and $p$ is

$$a(bc, ip)=m/n.$$

**18. Transformation to a new triangle of reference.** A system of homogeneous coordinates is determined by a triangle of reference $XYZ$ together with a unit-point $I\equiv(1, 1, 1)$. Let the figure be referred to another triangle of reference $X'Y'Z'$, the equations of whose sides, referred to $XYZ$, are

$$\left.\begin{array}{ll}(Y'Z') & u\equiv l_1x+m_1y+n_1z=0,\\ (Z'X') & v\equiv l_2x+m_2y+n_2z=0,\\ (X'Y') & w\equiv l_3x+m_3y+n_3z=0,\end{array}\right\} \quad\ldots\ldots(1)$$

and let the new unit-point $I'$ be $(x_0, y_0, z_0)$. Let the coordinates of an arbitrary point $P$ be $(X, Y, Z)$ referred to the original triangle and $(X', Y', Z')$ referred to the new. The equations of the four lines $Z'X'$, $Z'Y'$, $Z'I'$, $Z'P$ are

$$v=0, \quad u=0, \quad v_0u-u_0v=0, \quad Vu-Uv=0,$$

where $\quad u_0\equiv l_1x_0+m_1y_0+n_1z_0, \quad U\equiv l_1X+m_1Y+n_1Z$, etc.

Then we have by § 17

$$X'/Y' = Z'(X'Y', I'P) = (\infty, 0;\ u_0/v_0,\ U/V) = \frac{U}{V}\Big/\frac{u_0}{v_0}.$$

If we suppose that the expressions $u$, $v$, $w$ have first been prepared by multiplying by the appropriate factors so that $u_0 = v_0 = w_0$ we obtain the simple results:

$$X' : Y' : Z' = U : V : W,$$

*i.e.* for any point $(x, y, z)$ its coordinates referred to the new triangle are given by

$$\left.\begin{aligned} \rho x' &= l_1 x + m_1 y + n_1 z, \\ \rho y' &= l_2 x + m_2 y + n_2 z, \\ \rho z' &= l_3 x + m_3 y + n_3 z, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A)$$

where $\rho$ is a factor of proportionality.

It is essential that the three lines (1) be not concurrent; hence the determinant of the transformation

$$\Delta \equiv \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \neq 0.$$

The inverse transformation is found by solving for $x$, $y$, $z$ in terms of $x'$, $y'$, $z'$, and may be written

$$\left.\begin{aligned} \rho' x &= L_1 x' + L_2 y' + L_3 z', \\ \rho' y &= M_1 x' + M_2 y' + M_3 z', \\ \rho' z &= N_1 x' + N_2 y' + N_3 z', \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (A')$$

where the capital letters are the cofactors of the corresponding small letters in the determinant.

Further, the equation of a line $\xi x + \eta y + \zeta z = 0$ becomes

$$\xi(L_1 x' + L_2 y' + L_3 z') + \eta(M_1 x' + \ldots) + \zeta(N_1 x' + \ldots) = 0.$$

Identifying this with $\xi' x' + \eta' y' + \zeta' z' = 0$ we have

$$\left.\begin{aligned} \sigma\xi' &= L_1\xi + M_1\eta + N_1\zeta, \\ \sigma\eta' &= L_2\xi + M_2\eta + N_2\zeta, \\ \sigma\zeta' &= L_3\xi + M_3\eta + N_3\zeta, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B)$$

and inversely

$$\left.\begin{aligned} \sigma'\xi &= l_1\xi' + l_2\eta' + l_3\zeta', \\ \sigma'\eta &= m_1\xi' + m_2\eta' + m_3\zeta', \\ \sigma'\zeta &= n_1\xi' + n_2\eta' + n_3\zeta'. \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (B')$$

**19.** Although homogeneous coordinates are best suited for treating projective geometry, they may be usefully applied also to metrical geometry. We assume now the metrical definition of the cross-ratio of four points on a straight line:

$$(AB, CD) = \frac{AC}{BC}\Big/\frac{AD}{BD}.$$

This is consistent with the property of the projective cross-ratio as being equal to the cross-ratio of the parameters by which the points are determined. For the parameter of a point may be its distance from a fixed origin $O$ on the line. If the distances of $A$, $B$, $C$, $D$ from $O$ are $a$, $b$, $c$, $d$, $AC=c-a$, etc., and the cross-ratio

$$(AB, CD)=\frac{c-a}{c-b}\Big/\frac{d-a}{d-b}=(cd, ab)=(ab, cd).$$

Different special systems of coordinates are obtained according to their relation to the line at infinity.

**20. Cartesian coordinates.** The simplest special system is that in which the line at infinity is taken as one side $z=0$ of the triangle of reference. We have then a system of *cartesian coordinates*, the other two sides of the triangle, $OX$, $OY$, being the coordinate-axes, and the vertex $O$ the origin.

Let $I\equiv(1, 1, 1)$ be the unit-point, and $P\equiv(x, y, z)$ any point. Draw parallels through $I$ and $P$ to each coordinate-axis cutting the other axis in $L$, $N$ and $K$, $M$ respectively.

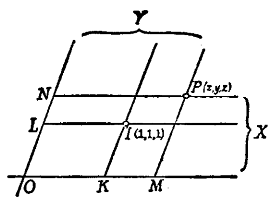



FIG. 77.

Then $x/z=$ the cross-ratio of the pencil $Y(OX, PI)$

$$=(OX, MK)=OM/OK=OM, \text{ if } OK=1.$$

Similarly, if $OL=1$,

$$y/z=ON.$$

$x$, $y$, $z$ are thus identified with the homogeneous cartesian coordinates as previously defined.

**21. Areal coordinates.** If the line at infinity is taken as unit-line $x+y+z=0$, the unit-point $(1, 1, 1)$ is its pole, so that if $XI$, $YI$, $ZI$ cut the opposite sides of the triangle in $D$, $E$, $F$, $FE$ cuts $YZ$ on the line at infinity, *i.e.* $FE\parallel YZ$. Similarly $DF\parallel ZX$ and $ED\parallel XY$. $I$ is therefore the centroid of the triangle.

If $P\equiv(x, y, z)$ is any point

$$\frac{y}{z}=X(YZ, IP)=(YZ, DL)$$

$$=\frac{LZ}{YL} \text{ (since } YD=DZ) \qquad =\frac{\Delta PZX}{\Delta PXY}.$$

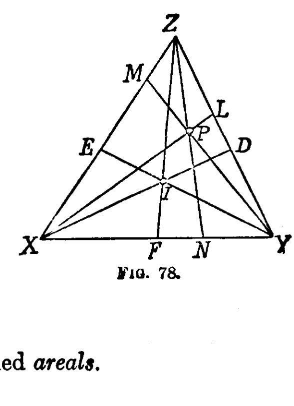



FIG. 78.

Hence $\quad x:y:z=\Delta PYZ:\Delta PZX:\Delta PXY.$

This system of coordinates is therefore called *areals*.

If three particles of masses $x$, $y$, $z$ respectively are placed at the vertices $X$, $Y$, $Z$, $L$ is the centre of mass of the two particles $y$ and $z$, and then since

$$\frac{XP}{PL}=\frac{\Delta PZX}{\Delta PLZ}=\frac{\Delta PXY}{\Delta PYL}=\frac{\Delta PZX+\Delta PXY}{\Delta PYZ}=\frac{y+z}{x},$$

$P$ is the centre of mass of the three particles. From this point of view areal coordinates are called also *barycentric coordinates.*

**22. Trilinear coordinates.** If $a$, $b$, $c$ are the lengths of the sides of the triangle of reference, and $\alpha$, $\beta$, $\gamma$ are the perpendicular distances of $P$ from the sides, the area of the triangle $PYZ=\frac{1}{2}a\alpha$. Hence *the areal coordinates are proportional to*

$$a\alpha,\quad b\beta,\quad c\gamma.$$

Numbers proportional to the distances $\alpha$, $\beta$, $\gamma$ themselves are called the *trilinear coordinates.* Since in areals the equation of the line at infinity is $x+y+z=0$, *in trilinears the equation of the line at infinity is*

$$ax+by+cz=0,$$

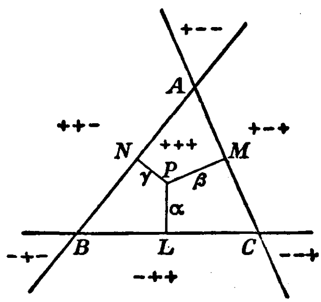


FIG. 79.

and the unit-point is the point equidistant from the three sides, *i.e.* the centre of the inscribed circle.

In both areals and trilinears the unit-point is in the interior of the triangle, and the ratios of the coordinates of any interior point are all positive (Fig. 79).

**23.** The actual perpendiculars $\alpha$, $\beta$, $\gamma$ from $P$ on the sides of the triangle of reference could be taken as a symmetrical system of metrical coordinates, but since only two coordinates are required to fix a point in a plane these are *superabundant.* They are, in fact, connected by an identical relation. We have

$$\Delta PYZ+\Delta PZX+\Delta PXY=\Delta XYZ=\Delta, \text{ say,}$$

therefore $$a\alpha+b\beta+c\gamma=2\Delta.$$

We shall call these the *metrical* or non-homogeneous trilinear coordinates.

Similarly the three quantities

$$X\equiv\frac{\Delta PYZ}{\Delta XYZ},\quad Y\equiv\frac{\Delta PZX}{\Delta XYZ},\quad Z\equiv\frac{\Delta PXY}{\Delta XYZ}$$

could be taken as superabundant areal coordinates, connected by the identical relation

$$X+Y+Z=1.$$

To pass from the metrical coordinates $\alpha$, $\beta$, $\gamma$ to the trilinear coordinates $x$, $y$, $z$ we have the equations

$$\frac{\alpha}{x}=\frac{\beta}{y}=\frac{\gamma}{z}=\frac{2\Delta}{ax+by+cz}.$$

Similarly if $x$, $y$, $z$ are the areal coordinates

$$\frac{\Delta PYZ}{x}=\frac{\Delta PZX}{y}=\frac{\Delta PXY}{z}=\frac{\Delta}{x+y+z}.$$

Thus, the equation $\beta^2=p\alpha+q$, which represents a parabola with axis $CA$, becomes in trilinears

$$4\Delta^2y^2=2p\Delta x(ax+by+cz)+q(ax+by+cz)^2$$

which is homogeneous in $x$, $y$, $z$.

Ex. Find the equation of the locus of a point such that the algebraic sum of its distances from the sides of the triangle is equal to the perimeter.

We have $$\alpha+\beta+\gamma=a+b+c,$$
hence the equation in trilinears is

$$2\Delta(x+y+z)=(a+b+c)(ax+by+cz).$$

**24. Condition for parallelism.** The condition for parallelism of two lines

$$l_1x+m_1y+n_1z=0,$$
$$l_2x+m_2y+n_2z=0,$$

is the same as the condition for concurrency of the two lines and the line at infinity. Hence if the equation of the line at infinity is

$$l_0x+m_0y+n_0z=0$$

the condition for parallelism is

$$\begin{vmatrix} l_0 & m_0 & n_0 \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix}=0.$$

The equation $(lx+my+nz)+\lambda(l_0x+m_0y+n_0z)=0$ represents any line parallel to the line $lx+my+nz=0$.

**25. Joachimsthal's section-formulae.** Since $\alpha$ is the same as the ordinate of the point $P$ referred to $BC$ as axis of $x$ in rectangular cartesian coordinates, the formulae for the coordinates of a point $(\alpha, \beta, \gamma)$ which divides the join of two points $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ in a given ratio $k:1$ are

$$\alpha=\frac{\alpha_1+k\alpha_2}{1+k},\quad \beta=\frac{\beta_1+k\beta_2}{1+k},\quad \gamma=\frac{\gamma_1+k\gamma_2}{1+k}.$$

These hold also for the metrical areal coordinates $X$, $Y$, $Z$ connected by the relations $X+Y+Z=1=X_1+Y_1+Z_1=X_2+Y_2+Z_2$.

**26. Some important points and lines connected with a triangle.** Areal or trilinear coordinates are very useful in treating of the geometry of the triangle. The following results are tabulated for convenience of reference. The verifications are left as exercises to the student.

| | Trilinears. | Areals. |
|---|---|---|
| $D$, the mid-point of the side $BC$ | $0 : 1/b : 1/c$ | $0:1:1$ |
| Median $AD$, - - - | $b\beta = c\gamma$ | $y=z$ |
| Line through $A \parallel BC$, - | $b\beta = -c\gamma$ | $y=-z$ |
| Centroid $G$, - - - | $1/a : 1/b : 1/c$ | $1:1:1$ |
| Symmedian or Lemoine point | $a : b : c$ | $a^2 : b^2 : c^2$ |
| Interior bisector of angle $A$, | $\beta = \gamma$ | $cy = bz$ |
| Exterior bisector of angle $A$, | $\beta = -\gamma$ | $cy = -bz$ |
| In-centre $I$, - - - | $1:1:1$ | $a:b:c$ |
| Ex-centre $I_1$, - - - | $-1:1:1$ | $-a:b:c$ |
| Circumcentre $S$, - - | $\cos A : \cos B : \cos C$ | $\sin 2A : \sin 2B : \sin 2C$ |
| Altitude $AL$, - - - | $\beta \cos B = \gamma \cos C$ | $y \cot B = z \cot C$ |
| Orthocentre $O$, - - | $\sec A : \sec B : \sec C$ | $\tan A : \tan B : \tan C$ |
| Centre of nine-point circle, | $\cos(B-C)$ : etc. | $\tan A(1+\tan B \tan C)$ : etc. |

**Isogonal and isotomic conjugates.**

*Def.* 1. If $AX$ is any line through $A$, the line $AX'$, such that

$$\angle X'AC = BAX,$$

is said to be *isogonally conjugate to $AX$ with respect to the angle $BAC$.*

Prove that, if $AX, BY, CZ$ are three concurrent lines through a point $P \equiv (\alpha_1, \beta_1, \gamma_1)$, the isogonal conjugates $AX', BY', CZ'$ are concurrent in a point $P' \equiv \left(\frac{1}{\alpha_1}, \frac{1}{\beta_1}, \frac{1}{\gamma_1}\right)$, the coordinates being trilinears.

*Def.* 2. The point $P'$ is called the *isogonal conjugate of $P$ with respect to the triangle $ABC$.*

*Def.* 3. If $P$ is any point on the segment $BC$, the point $P'$ which lies on $BC$ and is such that $BP = P'C$ is called the *isotomic conjugate of $P$ with respect to the segment $BC$.*

Prove that, if $AX, BY, CZ$ are three concurrent lines through a point $P$ whose areal coordinates are $(x_1, y_1, z_1)$, and $X', Y', Z'$ are the isotomic conjugates of $X, Y, Z$ with respect to the sides of the triangle, the three lines $AX', BY', CZ'$ are concurrent in a point $P' \equiv \left(\frac{1}{x_1}, \frac{1}{y_1}, \frac{1}{z_1}\right)$.

*Def.* 4. The point $P'$ is called the *isotomic conjugate of the point $P$ with respect to the triangle $ABC$.*

**Examples.**

1. The points $X$, $Y$ are taken on the sides $BC$, $CA$ respectively of the triangle $ABC$, so that $BX = \frac{1}{2}XC$ and $CY = 2YA$. Show that $AX$ passes through the mid-point of $BY$.

Since this is a metrical problem, we must take a definite system of metrical coordinates and take account of the identical relation. Taking areal coordinates, the equation of $AX$ is $y=3z$, and that of $BY$ is $x=2z$. Hence the coordinates of $P$ are $(2:3:1)$. $Y\equiv(2:0:1)$ and its metrical coordinates are $(\frac{2}{3}, 0, \frac{1}{3})$, while those of $B$ are $(0, 1, 0)$. Hence the coordinates of the mid-point of $BY$ are $(\frac{1}{3}, \frac{1}{2}, \frac{1}{6})$ or $(2:3:1)$, and therefore it coincides with $P$.

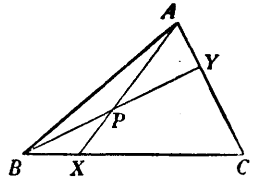


FIG. 80.

If it were required to find the ratio $AP:PX=\lambda$, we have $A\equiv(1,0,0)$, $X\equiv(0,\frac{3}{4},\frac{1}{4})$. Then

$$P\equiv(1:\tfrac{3}{4}\lambda:\tfrac{1}{4}\lambda)\equiv\left(\frac{4}{\lambda}:3:1\right)\equiv(2:3:1).$$

Hence $\lambda=2$.

2. $F$ is the mid-point of the side $AB$ of the triangle $ABC$. Any line through $F$ meets $BC$ in $P$ and $CA$ in $Q$. $AP$ and $BQ$ meet in $R$. Show that $CR$ is parallel to $AB$.

Taking areal coordinates, $F\equiv(1:1:0)$, and the equation of a line through this is

$$x-y+kz=0.$$

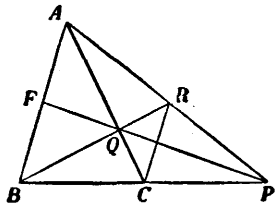


FIG. 81.

The equation of $AP$ is of the forms

$$\lambda x+(x-y+kz)=0$$

and $$y+\mu z=0.$$

Therefore $\lambda=-1$ and $AP$ is $y=kz$. Similarly $BQ$ is $x+kz=0$. Hence $CR$ is

$$(y-kz)+(x+kz)\equiv x+y=0 \quad\text{or}\quad x+y+z=z.$$

Therefore $CR \parallel AB$.

**27. Line-coordinates.** If $a$, $b$, $c$ denote the three lines of reference which form the triangle $ABC$, $i$ the unit-line, and $l$ any line $(\xi, \eta, \zeta)$, the line-coordinates are expressed in terms of cross-ratios as follows:

$$\eta/\zeta=a(bc, il), \quad \zeta/\xi=b(ca, il), \quad \xi/\eta=c(ab, il).$$

Let $i$ and $l$ cut the sides of the triangle in $U$, $V$, $W$ and $P$, $Q$, $R$, and let $p$, $q$, $r$ be the perpendicular distances of $A$, $B$, $C$ from the line $l$. Let the coordinates be areals, so that $i$ is the line at infinity. Then

$$\eta/\zeta=a(bc, il)$$
$$=(CB, UP)=BP/CP=q/r.$$

Hence $$\xi:\eta:\zeta=p:q:r,$$

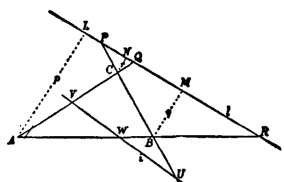


FIG. 82.

*i.e.* when the line at infinity is $x+y+z=0$, the line-coordinates $(\xi, \eta, \zeta)$ of the line $\xi x+\eta y+\zeta z=0$ are proportional to the distances of the line from the vertices of the triangle of reference.

In trilinears let the equation of the line be $l\alpha+m\beta+n\gamma=0$. Transforming this to areals by the transformation $x=a\alpha$, $y=b\beta$, $z=c\gamma$, it becomes

$$xl/a+ym/b+zn/c=0.$$

Hence $$l:m:n=ap:bq:cr.$$

Ex. Show that the point-equation $F(\alpha, \beta, \gamma)=0$ in trilinears is equivalent to the equation $F(x/a, y/b, z/c)=0$ in areals; and the line-equation $\phi(l, m, n)=0$ in trilinears is equivalent to the equation $\phi(al, bm, cn)=0$ in areals.

**28.** The three perpendiculars $p$, $q$, $r$ from $A$, $B$, $C$ to the straight line may be taken as superabundant coordinates of the line, and are connected by an identical relation. In Fig. 82 let $MN=u$, $NL=v$, $LM=w$; then we have

$$u^2=a^2-(q-r)^2,\quad v^2=b^2-(r-p)^2,\quad w^2=c^2-(p-q)^2.$$

But one of the four relations $u\pm v\pm w=0$ is true; therefore

$$\Sigma u^4-2\Sigma v^2w^2=0.$$

Hence $$\Sigma\{a^2-(q-r)^2\}^2-2\Sigma\{b^2-(r-p^2)\}\{c^2-(p-q)^2\}=0,$$

or, multiplying out and collecting terms,

$$\begin{aligned}(q-r)^4+\ldots-2(r-p)^2(p-q)^2-\ldots\\ +2(q-r)^2(-a^2+b^2+c^2)+\ldots\\ +\Sigma a^4-2\Sigma b^2c^2=0.\end{aligned}$$

The last line factorizes to

$$-(a+b+c)(-a+b+c)(a-b+c)(a+b-c)=-16\Delta^2.$$

The first line similarly breaks up into factors, one of which is

$$(q-r)+(r-p)+(p-q),$$

and it therefore vanishes.

The second line reduces to

$$\begin{aligned}2p^2(-c^2+a^2+b^2-b^2+c^2+a^2)+\ldots-4qr(-a^2+b^2+c^2)-\ldots\\ =4\Sigma a^2p^2-8\Sigma qrbc\cos A.\end{aligned}$$

Hence the identical relation reduces to

$$a^2p^2+b^2q^2+c^2r^2-2bqcr\cos A-2crap\cos B-2apbq\cos C=4\Delta^2.$$

Then we may pass from the metrical coordinates $p$, $q$, $r$ to the areal line-coordinates $\xi$, $\eta$, $\zeta$ by the equations

$$\frac{p}{\xi}=\frac{q}{\eta}=\frac{r}{\zeta}=\frac{2\Delta}{\{\Sigma a^2\xi^2-2\Sigma\eta\zeta bc\cos A\}^{\frac{1}{2}}}.$$

**29. Distance of a point from a straight line.** As a rule metrical formulae are not very simple in homogeneous coordinates, but there is one formula which is of importance from theoretical considerations. This is the formula for the distance of a point from a straight line.

Let $x_0$ $y_0$, $z_0$ be the metrical areal coordinates of the point $P$, $lx+my+nz=0$ the equation of the line. $l$, $m$, $n$ are proportional to the perpendiculars $AL$, $BM$, $CN$ from the vertices (§ 27). We shall take them as actually equal to these distances. Join $AP$ cutting $BC$ in $D$.

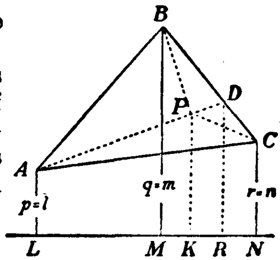



FIG. 88.

Then $$\frac{BD}{DC}=\frac{\triangle PAB}{\triangle PCA}=\frac{z_0}{y_0};$$

therefore $$DR=\frac{y_0 m+z_0 n}{y_0+z_0}.$$

Also $$\frac{AP}{PD}=\frac{\triangle PAB}{\triangle PBD}=\frac{\triangle PCA}{\triangle PDC}=\frac{\triangle PAB+\triangle PCA}{\triangle PBC}=\frac{y_0+z_0}{x_0};$$

therefore $$\delta=PK=\left\{(y_0+z_0)\frac{y_0 m+z_0 n}{y_0+z_0}+x_0 l\right\}/(x_0+y_0+z_0)$$
$$=lx_0+my_0+nz_0,$$
since $x_0+y_0+z_0=1$.

This expression involves the metrical point-coordinates and the metrical line-coordinates. In order to find an expression for the distance depending only on the ratios of the coordinates, we must, with the help of the identical relations, make the expression of zero dimensions in both $x$, $y$, $z$ and $l$, $m$, $n$. We shall find it most useful, however, if we write the expression in terms of the natural distances $\alpha$, $\beta$, $\gamma$ of the point from the sides, and $p$, $q$, $r$ of the line from the vertices. Then we have

$$\delta=\frac{2\Delta(ap\alpha+bq\beta+cr\gamma)}{(a\alpha+b\beta+c\gamma)(a^2p^2+\ldots-2bqr\cos A-\ldots)^{\frac{1}{2}}};$$

and from this, by putting $x:y:z=A\alpha:B\beta:C\gamma$ and $l:m:n=Pp:Qq:Rr$, where $AP:BQ:CR=a:b:c$, we can get the expression in any system of homogeneous coordinates.

**30. Statical proof of distance-formula.** The point whose barycentric or areal coordinates are $x$, $y$, $z$ is the centre of gravity of three particles of masses $x$, $y$, $z$ placed at the vertices of the triangle. Hence, since the sum of the moments of these three particles about any line is equal to the moment of a particle of mass $x+y+z$, placed at their C.G., we have

$$\delta(x+y+z)=px+qy+rz.$$

**31. Homogeneous cartesian coordinates.** We have shown that the ordinary system of cartesian coordinates, rendered homogeneous by the introduction of $z$, is a particular case of the general system of homogeneous coordinates, in which the equation of the line at infinity is $z=0$. It is instructive to demonstrate this by another method.

Take $CA$ and $CB$ as axes of $x$ and $y$; let $\angle ACB=\omega$, and $CN$, the perpendicular from $C$ on $AB$, $=k$, so that $ck=2\Delta$. Then, using actual values of the coordinates,

$$\alpha_0=x_0 \sin \omega, \quad \beta_0=y_0 \sin \omega,$$

but, instead of taking $\gamma_0=z_0 \sin \omega$, let $z_0=\gamma_0/k$.

This makes $x_0$ and $y_0$ the usual oblique coordinates, and they are definite multiples of the trilinear coordinates.

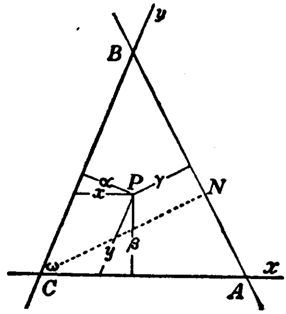




FIG. 83A.

The identical relation

$$a\alpha_0+b\beta_0+c\gamma_0=2\Delta$$

becomes $$(ax_0+by_0) \sin \omega+2\Delta z_0=2\Delta=ab \sin \omega.$$

Hence $$1-z_0=\frac{ax_0+by_0}{ab}=\frac{x_0}{b}+\frac{y_0}{a}.$$

Now let $a\to\infty$ and $b\to\infty$, so that $AB$ becomes the line at infinity. Then the identical relation becomes

$$z_0=1.$$

The equation of the line at infinity $a\alpha+b\beta+c\gamma=0$ becomes $ax+by+abz=0$; dividing by $ab$ and letting $a\to\infty$ and $b\to\infty$, this becomes $z=0$.

Similarly, the coordinates being rectangular, we can exhibit the line-coordinates as a limiting form of the general system.

We have $p=AP \cos \phi$, $q=BQ \sin \phi$, $r=-\rho$. Let us define the line-coordinates as

$$l=\frac{p}{CA}=\frac{p}{b}, \quad m=\frac{q}{CB}=\frac{q}{a}, \quad n=r=-\rho.$$

Then, since the ratios $PA:CA$ and $QB:CB$ both $\to 1$, we get in the limit

$$l=\cos \phi, \quad m=\sin \phi, \quad n=-\rho.$$

The equation of the straight line is then

$$lx+my+n=0 \quad \text{or} \quad x \cos \phi+y \sin \phi=\rho,$$

and the identical relation connecting the line-coordinates $l, m, n$ is

$$l^2+m^2=1.$$

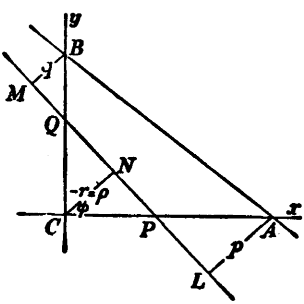




FIG. 83B.

It is easily verified that this is the limiting form of the general relation

$$\Sigma a^2p^2-2\Sigma qrbc \cos A=4\Delta^2,$$

when $a\to\infty$ and $b\to\infty$.

**32. Use of triangular paper.** When areal coordinates are used in numerical work, for the plotting of lines and curves, a triangularly ruled paper may be used with advantage. Three lines of the network are chosen as lines of reference forming a triangle $ABC$. The line $BC$ is marked 0, the line through $A$ parallel to this is marked 1, and the other parallel lines with the same spacing are numbered 2, 3, ... beyond $A$ and $-1$, $-2$, ... on the other side of $BC$. Similarly the series of lines parallel to $CA$ and $AB$ are marked with the positive and negative integers. The coordinates $x$, $y$, $z$ of a point of intersection $P$ of three lines of the network are the numbers marked on the three lines which pass through $P$. $A$, $B$, $C$ have the coordinates (1, 0, 0), (0, 1, 0), (0, 0, 1) respectively. Intermediate lines may be marked ·1, ·2, ... , ·9, etc., and then we have points with coordinates such as (0·3, $-1$·2, 2·5) or (3, $-12$, 25).

We may pass from one point to another by a succession of steps parallel to the lines of the network. A step $u$ parallel to $BC$, $CA$, or $AB$ changes $(x, y, z)$ into $(x, y-u, z+u)$, $(x+u, y, z-u)$, or $(x-u, y+u, z)$ respectively, leaving the sum of the coordinates unaltered, and for $A$ the sum is equal to 1, hence the coordinates $x$, $y$, $z$ of any point satisfy the fixed relation $x+y+z=1$.

To plot a point whose coordinates are given, *e.g.* (7, $-3$, 9), first divide each number by $x+y+z$, in this case 13, and we get (0·54, $-0$·23, 0·69).

If the equation of the line at infinity is $ax+by+cz=0$ the lines parallel to $BC$ are graduated $a^{-1}$, $2a^{-1}$, $3a^{-1}$, ... , those parallel to $CA$ : $b^{-1}$, $2b^{-1}$, $3b^{-1}$, ... , and those parallel to $AB$ : $c^{-1}$, $2c^{-1}$, $3c^{-1}$, ... . Then the coordinates of any point $(x, y, z)$ are connected by the identical relation $ax+by+cz=1$. In this case the centroid of the triangle of reference is $(a^{-1}, b^{-1}, c^{-1})$.

**33. Application of three dimensions.** A simple representation of areal coordinates is afforded by an application of three-dimensional geometry. In three dimensions a cartesian frame of reference consists of three axes $Ox$, $Oy$, $Oz$. The coordinates of a point $P$ are the segments $x$, $y$, $z$ cut off along the axes by planes through $P$ parallel to the coordinate-planes $yOz$, etc. The equation of a plane is of the first degree in $x$, $y$, $z$. In particular the equation $x+y+z=1$ represents a plane which cuts the axes in points $X$, $Y$, $Z$, making equal intercepts $OX=OY=OZ=1$. The cartesian three-dimensional coordinates $x$, $y$, $z$ of any point $P$ on this plane, which are connected by the equation $x+y+z=1$, are also its metrical areal coordinates referred to the triangle $XYZ$. If $PX$ cuts $YZ$ in $U$, and the line through $P$ parallel to $XO$ cuts the plane $YOZ$ in $L$, then

$$x=\frac{LP}{OX}=\frac{PU}{XU}=\frac{\triangle PYZ}{\triangle XYZ}.$$

## EXAMPLES XII.

1. Show that the equation in areals of the line joining the mid-points of the sides $AB$ and $AC$ of the triangle of reference is $-x+y+z=0$.

2. If the straight line $lx+my+nz=0$ cuts the triangle of reference in $X, Y, Z$, and the coordinates are areals, prove that the ratios of the segments

$$YZ : ZX : XY = l(m-n) : m(n-l) : n(l-m).$$

3. Find the conditions that the general equation of the second degree in areal coordinates should represent two parallel straight lines.

4. Prove that the centroid, the circumcentre, and the orthocentre of a triangle are collinear, and find the equation of this line (the *Euler line*).

5. $L, X$ ; $M, Y$ ; $N, Z$ are points on the sides $BC, CA, AB$ of the triangle $ABC$. $ZM$, $XN$, $YL$ are concurrent in $P$, and are parallel respectively to $BC, CA, AB$. $ZL$ and $MX$ meet in $A'$, $XM$ and $NY$ in $B'$, and $YN$ and $LZ$ in $C'$. Prove that $AA'$, $BB'$, $CC'$ are concurrent in $P$.

6. Prove that the locus of the mean point of the three points in which a line parallel to $lx+my+nz=0$ cuts the sides of the triangle of reference is a straight line, and find its equation in areal coordinates.

7.* Triangles are described externally on the sides of the triangle $ABC$ with angles as in the figure, and the coordinates are trilinears.

(i) Find the equation of $AA'$.

(ii) Express the condition that $AA'$, $BB'$, $CC'$ should be concurrent.

(iii) If the angles $\lambda_2$ and $\lambda_3$, $\mu_3$ and $\mu_1$, $\nu_1$ and $\nu_2$ are interchanged, show that the same condition will secure that $AA'$, $BB'$, $CC'$ in the new figure are concurrent.

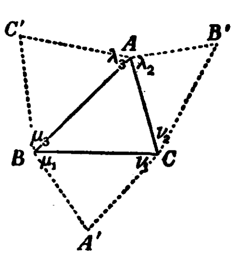


Fig. 84.

(iv) If

$\mu_3=\nu_2=90°-A$, $\nu_1=\lambda_3=90°-B$, $\lambda_2=\mu_1=90°-C$,

prove that $AA'$, $BB'$, $CC'$ are concurrent in the orthocentre.

(v) If $\mu_1=\nu_1=90°-A$, $\nu_2=\lambda_2=90°-B$, $\lambda_3=\mu_3=90°-C$,

prove that the point of concurrence is the centre of the nine-point circle.

(vi) If $\mu_3=\nu_2=A$, $\nu_1=\lambda_3=B$, $\lambda_2=\mu_1=C$, prove that the point of concurrence is the Lemoine point.

(vii) If $\mu_1=\nu_1=A$, $\nu_2=\lambda_2=B$, $\lambda_3=\mu_3=C$, prove that the point of concurrence is the centroid.

(viii) If $\lambda_2=\lambda_3=\lambda$, $\mu_3=\mu_1=\mu$, $\nu_1=\nu_2=\nu$, prove that $AA'$, $BB'$, $CC'$ are concurrent in the point $\{\sin\lambda/\sin(A+\lambda), \ldots\}$.

(ix) In (viii) find $\lambda, \mu, \nu$ so that the point of concurrence may be the circumcentre.

(x) In (viii) find $\lambda, \mu, \nu$ so that the point of concurrence may be the orthocentre.

(xi) If the triangles $A'BC$, $B'CA$, $C'AB$ are all equilateral, prove that $AA'$ $BB'$, $CC'$ meet in the point $\{\operatorname{cosec}(A+60°), \ldots\}$, and show that at this

*These examples are taken from a paper by A. G. Burgess, Edinburgh, *Proc. Math. Soc.*, 32 (1914), p. 58.

point the sides of the triangle $ABC$ each subtend an angle of 120°. (This point is called the *inner isogonic point.* If the equilateral triangles are described inwards, the point of concurrence is called the *outer isogonic point.*)

(xii) If in (viii) $\lambda=\mu=\nu=\theta$, show that the point of concurrence is

$$\{\operatorname{cosec}(A+\theta), \ldots\}.$$

(xiii) If $\mu_1=\nu_1=A\pm\theta$, $\nu_2=\lambda_2=B\pm\theta$, $\lambda_3=\mu_3=C\pm\theta$, prove that the point of concurrence is $\{\sin(A\mp\theta), \ldots\}$. (If $\theta=60°$, these two points are called the *isodynamic points.*)

8. The lines joining $A$, $B$, $C$ to a point $P$ cut the opposite sides of the triangle $ABC$ in $L$, $M$, $N$, and $PL=l\,.\,AL$, $PM=m\,.\,BM$, $PN=n\,.\,CN$. Prove that $l$, $m$, $n$ are proportional to the areal coordinates of $P$.

9. $LPL'$, $MPM'$, $NPN'$ are lines through $P$ parallel to the sides $BC$, $CA$, $AB$ of the triangle of reference and terminated by the other sides. If $x$, $y$, $z$ are the areal coordinates of $P$ show that

$$LL' : MM' : NN' = a(y+z) : b(z+x) : c(x+y).$$

10. Prove that the trilinear coordinates of the two Brocard points are $(b/c, c/a, a/b)$ and $(c/b, a/c, b/a)$. [One Brocard point $P$ has the property that $\angle PBC=PCA=PAB$, and the other $\angle PCB=PAC=PBA$.]

11. If $L, M, N$ are the feet of the perpendiculars from $O$, whose trilinear coordinates are $\alpha$, $\beta$, $\gamma$, on the sides of the triangle $ABC$, prove that

$$MN^2=\beta^2+\gamma^2+2\beta\gamma\cos A.$$

Hence show that the pedal triangle $LMN$ is equilateral when $O$ is either of the points $\{\sin(A\pm 60°), \ldots\}$ (the isodynamic points).

12. Show that the trilinear equation of the line joining the circumcentre and the symmedian point is $\alpha\sin(B-C)+\beta\sin(C-A)+\gamma\sin(A-B)=0$.

13. Show that the trilinear equation of the line joining the two Brocard points is $\Sigma\alpha(a^4-b^2c^2)/a=0$.

14. Show that the trilinear equation of the line joining the in-centre and the circumcentre is $\Sigma\alpha(\cos B-\cos C)=0$.

15. Prove that, if $\theta$ varies, the point $\{\sin(A+\theta), \sin(B+\theta), \sin(C+\theta)\}$ describes a straight line. If the coordinates are trilinears, name the points corresponding to $\theta=0$, $\pm 60°$, $90°$.

16. Show that, if $\theta$ is variable, the line

$$x\sin(\alpha+\theta)+y\sin(\beta+\theta)+z\sin(\gamma+\theta)=0$$

passes through a fixed point.

17. Show that, if $\theta$ is variable, the line

$$\alpha\sin(A-\theta)+\beta\sin(B-\theta)+\gamma\sin(C-\theta)=0$$

in trilinear coordinates is parallel to a fixed direction.

18. Show that $\lambda x=a+pt$, $\lambda y=b+qt$, $\lambda z=c+rt$ are freedom-equations of a straight line through the points $(a, b, c)$ and $(p, q, r)$.

19. $ABC$ is any triangle; $A'$, $Q$, $R$ are any points on $BC$, $CA$, $AB$ respectively, and $l$ is any line through $A$; $A'Q$ and $A'R$ cut $l$ in $C'$ and $B'$ respectively, $BC'$ cuts $B'C$ in $P$. Prove that $P$, $Q$, $R$ are collinear.

20. Four fixed points are joined by three pairs of lines intersecting respectively in the points $A, B, C$; and $P$ is any other point. Through $A$ is drawn the harmonic conjugate of $AP$ with regard to the fixed pair of lines through $A$, and lines are similarly drawn through $B$ and $C$. Show that these three lines are concurrent. (Pembroke, 1899.)

21. A line cuts the sides of the diagonal triangle of a complete quadrilateral in $P, Q, R$; and $P', Q', R'$ are the harmonic conjugates of $P, Q, R$ respectively with regard to the pairs of opposite vertices of the complete quadrilateral. Prove that $P', Q', R'$ are collinear. Deduce the collinearity of the mid-points of the diagonals of the complete quadrilateral.

22. A straight line meets the sides $BC, CA, AB$ of a triangle in $L, M, N$. On $AL, BM, CN$ are taken any three points $A', B', C'$; $B'C', C'A', A'B'$ meet $BC, CA, AB$ in $P, Q, R$. Show that $A'P, B'Q, C'R$ are concurrent.

(Math. Tripos I., 1909.)

23. $ABC$ and $A'B'C'$ are two triangles such that $AA', BB', CC'$ are concurrent (*triangles in perspective*). Show that the points of intersection of $BC$ and $B'C'$, $CA$ and $C'A'$, $AB$ and $A'B'$ are collinear. (Desargues' Theorem.)

24. A straight line cuts the sides of the triangle $ABC$ in $A', B', C'$. $U, V, W$ are the harmonic conjugates of $A', B', C'$ with respect to the vertices of the triangle in pairs. The lines joining any point $O$ to $U, V, W$ cut the opposite sides of the triangle $UVW$ in $X, Y, Z$. Prove that $AX, BY, CZ$ are concurrent.

25. In the last example, if the lines joining $O$ to $A, B, C$ cut the corresponding sides of the triangle $UVW$ in $L, M, N$, prove that $UL, VM, WN$ are concurrent.

26. A straight line cuts the sides of the triangle $ABC$ in $X, Y, Z$. Any line is drawn through $X$ cutting $CA$ in $B_1$ and $AB$ in $C_1$. Similarly, lines are drawn through $Y$ and $Z$ cutting the sides of the triangle in $C_2, A_2$ and $A_3, B_3$. $BB_1$ and $CC_1$ cut in $L$, $CC_2$ and $AA_2$ in $M$, $AA_3$ and $BB_3$ in $N$. Prove that $AL, BM, CN$ are concurrent.

27. Two sets of concurrent lines through the vertices of the triangle $ABC$ cut the opposite sides in $P, Q, R$ and $P', Q', R'$. $BQ$ and $CR'$ cut in $U$, $BQ'$ and $CR$ in $U'$; $CR$ and $AP'$ in $V$, $CR'$ and $AP$ in $V'$; $AP$ and $BQ'$ in $W$, $AP'$ and $BQ$ in $W'$. $BV$ and $CW'$ meet in $L$, $CW$ and $AU'$ in $M$, $AU$ and $BV'$ in $N$. Show that $AL, BM, CN$ are concurrent.

# CHAPTER XIII.

## THE CONIC IN HOMOGENEOUS COORDINATES.

**1. Equation of a conic in homogeneous coordinates.** Since in every system of homogeneous coordinates a straight line has always an equation of the first degree, and a conic has the property that it is cut by any straight line in two points, real, coincident, or imaginary, the equation of a conic is of the second degree. The general homogeneous equation of the second degree in $x$, $y$, $z$ is

$$f(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0.$$

**2. Intersection of a straight line and a conic.** Let $lx + my + nz = 0$ be any straight line. To find the points in which this line cuts the conic we have to solve the two equations simultaneously. We thus determine the ratios $x : y : z$ for a common point. In order to effect the solution it is convenient to eliminate one of the variables, say $z$. We then get a homogeneous equation of the second degree in $x$, $y$, which represents the two lines joining $C$ to the points of intersection.

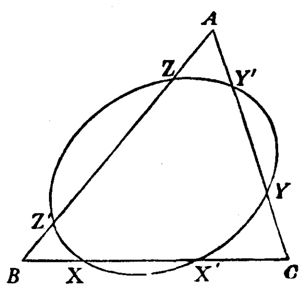


FIG. 85.

Ex. $P$ is a given point on the conic $yz = x^2$, and $y = \mu z$ is a variable line through the fixed point $A$ cutting the conic in $U$, $V$. Prove that $PU$, $PV$ are harmonic conjugates with regard to two fixed lines through $P$.

Let $P \equiv (x_1, y_1, z_1)$, and put $Y \equiv x_1 y - y_1 x$, $Z \equiv x_1 z - z_1 x$ so that $x$, $Y$, $Z$ are the coordinates referred to the triangle $PBC$. Then the equation of the conic become

$$(Y + y_1 x)(Z + z_1 x) = x_1^2 x^2.$$

Since $P$ lies on the conic, $y_1 z_1 = x_1^2$, and the equation reduces to

$$YZ + x(Yz_1 + Zy_1) = 0.$$

The equation of the variable line becomes

$$Y - \mu Z = x(\mu z_1 - y_1).$$

Eliminating $x$, we have

$$YZ(\mu z_1 - y_1) + (Y - \mu Z)(Yz_1 + Zy_1) = 0,$$

*i.e.*
$$z_1 Y^2 - \mu y_1 Z^2 = 0.$$

This represents a pair of lines through $P$ which are harmonic conjugates with regard to $Y = 0$ and $Z = 0$, *i.e.* with regard to $PC$, $PB$.

**3. Carnot's theorem.** If the side $BC$ of the triangle of reference cuts the conic in $X$, $X'$ (Fig. 85), the equation of the pair of lines $AX$, $AX'$ is found by putting $x=0$ in the equation of the conic, and is therefore

$$by^2+2fyz+cz^2=0.$$

Let the coordinates be areals and let $y:z$ and $y':z'$ be the solutions of this equation. Then

$$BX:XC=\triangle ABX:\triangle AXC=z:y.$$

Hence we have $$\frac{BX}{CX}\cdot\frac{BX'}{CX'}=\frac{zz'}{yy'}=\frac{b}{c}.$$

Similarly $$\frac{CY}{AY}\cdot\frac{CY'}{AY'}=\frac{c}{a}\quad\text{and}\quad\frac{AZ}{BZ}\cdot\frac{AZ'}{BZ'}=\frac{a}{b}.$$

Hence $$\frac{BX\,.\,BX'}{CX\,.\,CX'}\cdot\frac{CY\,.\,CY'}{AY\,.\,AY'}\cdot\frac{AZ\,.\,AZ'}{BZ\,.\,BZ'}=1.$$

This is known as *Carnot's Theorem*. It gives a relation between six points which lie on a conic, and the converse theorem may be used as a test in order that six given points may lie on a conic.

**Examples.**

1. If $S$, $S'$ are any two points in the plane of a triangle, and if the lines joining $S$, $S'$ to the vertices meet the opposite sides in $X, Y, Z, X', Y', Z'$, prove that these six points lie on a conic.

2. If $X$, $Y$, $Z$ are points on the sides $BC$, $CA$, $AB$ of a triangle such that $AX$, $BY$, $CZ$ are concurrent, prove that a conic can be constructed touching the sides of the triangle at $X$, $Y$, $Z$.

The resemblance of Carnot's theorem to the theorem of Menelaus may be noted. If the conic degenerates to two straight lines Carnot's theorem is at once deduced from Menelaus', and if one of the lines becomes the line at infinity each of the ratios $BX':CX'$, $CY':AY'$, $AZ':BZ'$ become unity; then Carnot's theorem reduces to that of Menelaus.

**4. Tangential equation.** To find the condition that the line

$$lx+my+nz=0$$

should be a tangent to the conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

eliminate $z$, and we get the equation of the lines joining $C$ to the two points of intersection,

$$(an^2-2gnl+cl^2)x^2+2(hn^2+clm-gmn-fnl)xy+(bn^2-2fmn+cm^2)y^2=0.$$

These will coincide if

$$(hn^2-gmn-fnl+clm)^2=(an^2-2gnl+cl^2)(bn^2-2fmn+cm^2).$$

Multiplying out, and cancelling $n^2$, this reduces to

$$(bc-f^2)l^2+(ca-g^2)m^2+(ab-h^2)n^2+2(gh-af)mn$$
$$+2(hf-bg)nl+2(fg-ch)lm=0,$$

or, with the usual notation,

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0.$$

This is therefore the tangential or line-equation of the conic. We may denote it briefly by

$$F(l, m, n)=0.$$

Further, the line $lx+my+nz=0$ will cut the conic in real or imaginary points according as

$$F(l, m, n) \lesseqgtr 0.$$

**5. Joachimsthal's ratio equation.** Any point on the line joining the two points $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ can be represented by the coordinates $(x_1+\lambda x_2, y_1+\lambda y_2, z_1+\lambda z_2)$, $\lambda$ being a variable parameter. The actual value of the position-ratio corresponding to any value of $\lambda$ can be found only when the particular metrical system of coordinates is specified, but generally it is not necessary to deal with these actual values.

To find the points in which the line joining the points $P\equiv(x', y', z')$ and $Q\equiv(x, y, z)$ cuts the conic, let the coordinates of one of the points be $(x+\lambda x', y+\lambda y', z+\lambda z')$. Then, substituting in the equation of the conic, we have

$$\lambda^2 f(x', y', z')+\lambda\left(x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'}+z\frac{\partial f}{\partial z'}\right)+f(x, y, z)=0.$$

the two roots $\lambda_1$ and $\lambda_2$ of this equation correspond to the two points $X$, $Y$, in which $PQ$ cuts the conic.

*Cor.* 1. If $(PQ, XY)$ is harmonic, the roots are equal and of opposite sign (see Chap. XII, § 13, Ex., or § 5), *i.e.* $\lambda_1+\lambda_2=0$. Hence the locus of $Q$, the harmonic conjugate of $P$ with regard to the conic, *i.e.* the polar of $P$, is

$$x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'}+z\frac{\partial f}{\partial z'}=0.$$

If $P$ lies on the conic this is the equation of the tangent at $P$.

*Cor.* 2. If $Q$ lies on one of the tangents from $P$ to the conic the roots $\lambda_1$ and $\lambda_2$ are equal; hence the equation of the two tangents from $P$ to the conic is

$$f(x, y, z)f(x', y', z')=\frac{1}{4}\left(x\frac{\partial f}{\partial x'}+y\frac{\partial f}{\partial y'}+z\frac{\partial f}{\partial z'}\right)^2.$$

*Conics specially related to the triangle of reference.*

**6. To find the equation of a conic circumscribing the triangle of reference.** The general equation of a conic is

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

If this passes through the point (1, 0, 0), we have $a=0$. Hence, if it passes through the three vertices, $a=0$, $b=0$, $c=0$. The general equation of a conic circumscribing the triangle of reference is therefore

$$fyz+gzx+hxy=0.$$

In an exactly similar way the line-equation of a conic touching the sides of the triangle of reference is

$$Fmn+Gnl+Hlm=0.$$

**7. To find the equation of a conic touching the sides of the triangle of reference.** If $x=0$ is a tangent, the equation

$$by^2+cz^2+2fyz=0$$

must have equal roots; therefore

$$bc=f^2.$$

Similarly, if $y=0$ and $z=0$ are tangents,

$$ca=g^2 \quad \text{and} \quad ab=h^2.$$

Hence $a$, $b$, $c$ are all of the same sign. We may suppose them to be positive and put $a=p^2$, $b=q^2$, $c=r^2$. Then $f=\pm qr$, $g=\pm rp$, $h=\pm pq$. The signs here cannot be all $+$, or two $-$ and one $+$, since in either of these cases the conic would reduce to two coincident straight lines. We may take therefore as the equation

$$p^2x^2+q^2y^2+r^2z^2-2qryz-2rpzx-2pqxy=0,$$

where $p$, $q$, $r$ may be positive or negative.

This equation may be written

$$(px+qy-rz)^2-4pqxy=0,$$

which may be resolved into four irrational factors of the form

$$\sqrt{px}\pm\sqrt{qy}\pm\sqrt{rz}=0.$$

The corresponding line-equation of this conic is

$$pmn+qnl+rlm=0.$$

**Examples.**

1. If a conic touches $BC$, $CA$, $AB$ in $X$, $Y$, $Z$, then $AX$, $BY$, $CZ$ are concurrent.

Take the equation

$$p^2x^2+q^2y^2+r^2z^2-2qryz-2rpzx-2pqxy=0.$$

Putting $z=0$, we find the equation of the line joining $A$ to the point of contact of the conic with $BC$,

$$px=qy;$$

and the three lines $px=qy=rz$ are concurrent in the point $(1/p, 1/q, 1/r)$.

2. If a conic circumscribes the triangle $ABC$, and if the tangents at $A$, $B$, $C$ cut the opposite sides in $X$, $Y$, $Z$, prove that $X$, $Y$, $Z$ are collinear.

**8. To find the equation of a conic referred to two tangents and the chord of contact.** Let $x=0$ and $y=0$ be the tangents and $z=0$ the chord of contact; then, putting $x=0$, we must have equal roots $z^2=0$; therefore $b=0$ and $f=0$. Similarly, since $y=0$ is a tangent at $A$, we must have $a=0$ and $g=0$. Hence the equation reduces to

$$cz^2+2hxy=0,$$

*i.e.*

$$xy=kz^2.$$

**Ex.** Interpret the line-equation $lm=n^2$.

This can be applied to the case of a hyperbola expressed in cartesian coordinates, with the asymptotes as axes, for in this case the triangle of reference consists of the two asymptotes and the line at infinity, the latter being the chord of contact of the asymptotes regarded as tangents. Hence the equation is

$$xy=k$$

**9. Geometrical meaning of the vanishing of any coefficient in the general equation of the second degree.** If $a=0$, the equation is satisfied by the coordinates $(1, 0, 0)$. This is therefore the condition that the conic should pass through the vertex $A$.

If $f=0$, the equation of the two lines joining $A$ to the points of intersection with $x=0$ is $by^2+cz^2=0$. But this represents a pair of lines which are harmonic conjugates with regard to $y=0$ and $z=0$. Hence $f=0$ is the condition that the line $BC$ should be cut harmonically by the conic.

Similarly for the equation in line-coordinates, $A=0$ is the condition that the conic should touch the side $BC$, and $F=0$ is the condition that the tangents from $A$ to the conic should be harmonic conjugates with regard to $AB$ and $AC$.

**10. Equation of a conic referred to a self-conjugate triangle.** If the triangle of reference is self-conjugate with regard to the conic, each of its sides must be cut harmonically by the conic. Hence $f$, $g$, and $h$ all vanish. The equation of the conic is therefore

$$ax^2+by^2+cz^2=0.$$

It is easily verified that the polar of each vertex is the opposite side of the triangle.

In choosing a self-conjugate triangle the first vertex may be taken quite arbitrarily, and has two degrees of freedom. The second vertex must then lie on the polar of the first, and has one degree of freedom. The third vertex is then fixed. Hence there are $\infty^3$ self-conjugate triangles, and $\infty^3$ ways in which the equation of any conic may be reduced to this form.

We have already had examples of the reduction of the general equation to this form when the conic is referred to its principal axes. In this case the self-conjugate triangle consists of the two axes and the line at infinity.

**11.** In order that the conic should be real, $a$, $b$, $c$ cannot be all of the same sign. Let $a$ and $b$ be positive, and $c$ negative, and write the equation

$$p^2x^2+q^2y^2-r^2z^2=0.$$

We may write this

$$p^2x^2=(rz-qy)(rz+qy),$$

showing that $rz \pm qy=0$ are tangents, with $x=0$ as chord of contact. Similarly $px \pm rz=0$ are real tangents with $y=0$ as chord of contact, but the tangents whose chord of contact is $z=0$ are imaginary. Hence one vertex $C$ lies within the conic.

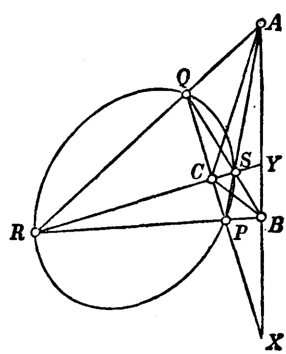


FIG. 86.

**12.** A self-conjugate triangle may be constructed as follows. Take any four points $P$, $Q$, $R$, $S$ on the conic. These form a complete quadrangle whose harmonic or diagonal triangle $ABC$ is self-conjugate with regard to the conic. For if $PQ$ and $RS$ meet $AB$ in $X$ and $Y$, the two ranges $(PQ, CX)$ and $(RS, CY)$ are harmonic, and therefore $XY$ or $AB$

is the polar of $C$. Similarly each side of the triangle $ABC$ is the polar of the opposite vertex. In a similar way we may construct a self-conjugate triangle as the harmonic triangle of a complete quadrilateral circumscribed about the conic.

Ex. The tangents at four points $P, Q, R, S$ on a conic form a complete quadrilateral circumscribed about the conic. Show that its harmonic triangle coincides with that of the complete quadrangle $PQRS$.

Taking $ABC$, the harmonic triangle of the complete quadrangle $PQRS$, as triangle of reference, the equation of the conic may be written

$$ax^2+by^2+cz^2=0.$$

The coordinates may be chosen so that $S\equiv(1, 1, 1)$. Then $a+b+c=0$, and we have $P\equiv(-1, 1, 1)$, $Q\equiv(1, -1, 1)$, $R\equiv(1, 1, -1)$.

The tangents at $P, Q, R, S$ have line-coordinates $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$, $(a, b, c)$. These are of the same form as the point-coordinates of $P, Q, R, S$; hence the harmonic triangle is again $ABC$.

**13.** Two conics intersect in four points, forming a complete quadrangle inscribed in each. The harmonic triangle of this quadrangle is therefore self-conjugate with regard to both conics. By taking this triangle as the triangle of reference it is possible to express the equations of any two conics in the form

$$a_1x^2+b_1y^2+c_1z^2=0,$$
$$a_2x^2+b_2y^2+c_2z^2=0.$$

This is often very useful when we are dealing with a system of conics, but it requires modification if the four common points are not distinct.

**Examples.**

1. A point $P$ describes a straight line. The polars of $P$ with respect to two fixed conics intersect in $Q$. Prove that the locus of $Q$ is a conic.

Let the equations of the conics be

$$a_1x^2+b_1y^2+c_1z^2=0,$$
$$a_2x^2+b_2y^2+c_2z^2=0,$$

and let $(x', y', z')$ be any point on the fixed line $lx+my+nz=0$. The polars of this point with respect to the two conics are

$$a_1x'x+b_1y'y+c_1z'z=0,$$
$$a_2x'x+b_2y'y+c_2z'z=0,$$

and we have also
$$lx'+my'+nz'=0.$$

Eliminating $x', y', z'$ between these three equations, we get

$$l(b_1c_2-b_2c_1)yz+m(c_1a_2-c_2a_1)zx+n(a_1b_2-a_2b_1)xy=0.$$

The locus is therefore a conic which passes through the vertices of the common self-conjugate triangle of the two conics. The pole of $lx+my+nz=0$ with respect to the first conic is $(l/a_1, m/b_1, n/c_1)$, and these coordinates satisfy the equation of the locus. Hence the locus passes also through the poles of the given line with respect to the two conics.

2. State the reciprocal theorem.

**14. Conjugate triangles with regard to a conic.** Two triangles are conjugate with respect to a conic when each vertex of one is the pole of the corresponding side of the other. Take one of the triangles $ABC$ as triangle of reference, and let the equation of the conic be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

Then the sides of the conjugate triangle $A'B'C'$ are

$$ax+hy+gz=0,$$
$$hx+by+fz=0,$$
$$gx+fy+cz=0.$$

These cut the corresponding sides $x=0$, $y=0$, $z=0$ in the points

$$(0, -g, h), \quad (f, 0, -h), \quad (-f, g, 0),$$

which are collinear on the line $x/f+y/g+z/h=0$. Hence *two triangles which are conjugate with regard to a conic are in perspective.* This line is the axis of perspective, and it may be shown that the centre of perspective is $(1/F, 1/G, 1/H)$.

Ex. Prove the converse theorem: If two triangles are in perspective there is a unique conic with respect to which they are conjugate.

**15. Equation of a conic passing through four given points.** Let the four points be the intersections of the two lines $\alpha=0$, $\beta=0$, with the two lines $\gamma=0$, $\delta=0$. Then the equation

$$\alpha\beta=k\gamma\delta$$

represents a conic through the four points.

The equation in § 8 is a particular case of this when two of the lines, $\gamma$ and $\delta$, become coincident. The equation

$$\alpha\beta=k\gamma^2$$

then represents a conic passing through the two pairs of coincident points $\alpha\gamma$ and $\beta\gamma$, *i.e.* having $\alpha$ and $\beta$ as tangents and $\gamma$ as chord of contact.

More generally, if $S$ and $S'$ are any two conics,

$$S=kS'$$

represents a conic through the four points of intersection of $S$ and $S'$;

$$S=k\alpha\beta$$

represents a conic through the four points of intersection of the lines $\alpha$, $\beta$ with the conic $S$.

**16. Contact of conics.** There are several important special forms of this equation.

(1) Let $\alpha$ and $\beta$ coincide. Then the four points of intersection coincide in pairs, and the equation $\quad S=k\alpha^2$

represents a conic touching $S$ at the two points in which it is cut by $\alpha$, *i.e.* a conic having *double contact* with $S$, with $\alpha$ as chord of contact.

(2) Let $\beta$ be a tangent to $S$ at $(x', y', z')$; then the equation

$$S=k\alpha\left(x\frac{\partial S}{\partial x'}+y\frac{\partial S}{\partial y'}+z\frac{\partial S}{\partial z'}\right)$$

represents a conic touching $S$ at $(x', y', z')$ and cutting it again in the two points where it is cut by $\alpha$.

(3) Let $\beta$ be the tangent to $S$ at $P$, and let $\alpha$ also pass through $P$; then the equation

$$S=(lx+my+nz)\left(x\frac{\partial S}{\partial x'}+y\frac{\partial S}{\partial y'}+z\frac{\partial S}{\partial z'}\right),$$

where $lx'+my'+nz'=0$, represents a conic meeting $S$ in three coincident points at $P$ and one other point. This is called *contact of the second order.* It is the same sort of contact as that of the circle of curvature with a conic.

(4) Let $\alpha$ and $\beta$ coincide with the tangent at $P$; then the equation

$$S=k\left(x\frac{\partial S}{\partial x'}+y\frac{\partial S}{\partial y'}+z\frac{\partial S}{\partial z'}\right)^2$$

represents a conic meeting $S$ in four coincident points at $P$, and no other point. This is called *contact of the third order.* Since five points determine a conic no higher contact is possible unless the conics coincide altogether.

**17. Degenerate conics.** When $\Delta=0$ the conic, as a locus, degenerates to two straight lines. What then becomes of the conic when considered as an envelope?

The tangential equation is

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & 0 \end{vmatrix}=0.$$

Now, by a known theorem in determinants, when the determinant

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}=0,$$

the bordered determinant becomes a perfect square in $l, m, n$. In fact, since $\Delta=0$, $GH=AF$, $CA=G^2$, and $AB=H^2$; therefore the equation reduces to

$$(Al+Hm+Gn)^2/A=0.$$

Now, an equation of the first degree in $l, m, n$ represents a pencil of lines passing through one point. Hence, when the conic degenerates, as a locus, to two straight lines, it degenerates, as an envelope, to two coincident points. This double point is just the point of intersection of the two lines, and its coordinates are $(A, H, G)$ or $(H, B, F)$ or $(G, F, C)$, all of which are equivalent.

Suppose now that the conic degenerates further, as a locus, to two coincident straight lines. Then $A$, $B$, $C$, $F$, $G$, $H$ all vanish, and the tangential equation appears to be quite indeterminate. Let us observe the conic in the act of degenerating to the two coincident straight lines $z^2=0$. Take the equation

$$\epsilon(y-px)(y-qx)+z^2=0,$$

which represents a proper conic degenerating to the two coincident straight lines as $\varepsilon \to 0$. The tangential equation of this conic is

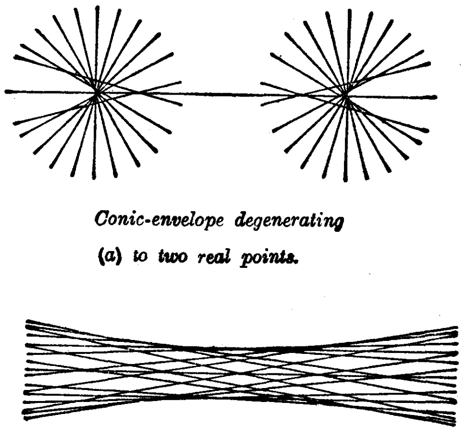
$$4l^2 + 4pqm^2 - \varepsilon(p-q)^2 n^2 + 4(p+q)lm = 0.$$

which reduces to $$l^2 + (p+q)lm + pqm^2 = 0,$$

*i.e.* $$(l+pm)(l+qm) = 0,$$

and this represents two pencils of lines whose vertices are $(1, p, 0)$, $(1, q, 0)$ *i.e.* two points on the line $z=0$.

Thus, when the conic degenerates, as a locus, to two coincident straight lines, it degenerates, as an envelope, to two points which may be real or imaginary. This may be explained geometrically as follows. Consider a



*Conic-envelope degenerating*

*(a) to two real points.*

*(b) to two imaginary points.*

FIG. 87.

hyperbola with its axis and vertices fixed, but let its asymptotes gradually coincide. As the hyperbola ultimately collapses into two coincident straight lines, coinciding with the transverse axis, its tangents will come to pass through one or other of two fixed points, the vertices. If the hyperbola collapses in the other way into two coincident lines coinciding with the conjugate axis, the real tangents all come to coincide with this line. But through any point in the interior of the hyperbola there pass two imaginary tangents, and as the hyperbola collapses these will come to pass respectively through the two imaginary points in which the hyperbola ultimately cuts the double line.

**Examples.**

1. Determine the line-equations of the conics whose point-equations are

(i) $x^2 - 2y^2 - 2z^2 + 5yz - zx - xy = 0$,

(ii) $(x+y+z)^2 + \varepsilon(2y+z)(y+2z) = 0$, $\varepsilon \to 0$.

(iii) $(x+y+z)^2 + \varepsilon(x^2+y^2+z^2) = 0$, $\varepsilon \to 0$,

(iv) $(x-y)^2 + \varepsilon(x+y-2z)(x-2y+z) = 0$, $\varepsilon \to 0$.

2. Determine the point-equations of the conics whose line-equations are

$$\text{(i) } l^2+m^2=0, \qquad \text{(ii) } (al+bm+cn)(a'l+b'm+c'n)=0,$$
$$\text{(iii) } (al+bm+cn)^2+\varepsilon(l^2+m^2+n^2)=0, \ \varepsilon\to 0.$$

## EXAMPLES XIII.

1. The tangents at $A, B, C$ on a conic form a triangle $A', B', C'$. Prove that the triangles $ABC$ and $A'B'C'$ are in perspective.

2. Two conjugate points $P, P'$ lie respectively on two sides of a triangle inscribed in a conic. Prove that $PP'$ passes through the pole of the third side. (Trinity, 1910.)

3. Prove that the locus of isogonal conjugates of points on a straight line is a conic circumscribing the triangle of reference.

4. Prove that the polars, with respect to two conics, of a point on a given straight line intersect on a conic circumscribing the common self-conjugate triangle. (Corpus, 1914.)

5. Through a point $O$ any two lines are drawn to cut in $P, Q$ and $P', Q'$ any conic which touches two fixed lines through $O$ at given points. Prove that $PP'$ meets $QQ'$ on a fixed line. (Selwyn, 1914.)

6. $OA, OB$ are tangents to a conic at $A$ and $B$. The bisector of the angle $AOB$ cuts $AB$ in $M$, and $PQ$ is any chord through $M$. Prove that $OM$ bisects the angle $POQ$.

7. $ABC$ and $A'B'C'$ are two triangles conjugate with regard to a conic, and therefore in perspective. If the axis of perspective cuts the conic in $X$ and $Y$, show that there is one conic touching the given conic at $X$ and $Y$, and having $ABC$ as a self-conjugate triangle. Prove also that the tangents to this conic at $X$ and $Y$ meet in the centre of perspective.

8. The tangents at the fixed points $B, C$ on a conic meet in $A$, and any other tangent to the conic cuts $AB, AC$ in $L, M$ respectively. Prove that, if $BM$ and $CL$ intersect in $P$, the locus of $P$ is a conic having double contact with the given conic at $B$ and $C$.

9. Show that the parabola which osculates the rectangular hyperbola $xy=c^2$ at the point $(h, k)$ is $(xk-yh)^2=4c^2(xk+yh-2c^2)$. (Corpus, 1907.)

10. Two triangles are each self-conjugate with regard to a conic. Prove that they can both be inscribed in a conic, and both circumscribed about another conic.

11. Prove that the conics $7y^2-4xy+4x+2y-17=0$ and $x^2+2y^2-4=0$ touch each other at two distinct points, and find the coordinates of the intersection of the tangents at these points. (Queens', 1910.)

12. Conics are drawn passing through two fixed points $A, B$ and touching two fixed lines. Prove that the locus of the point of intersection of the tangents at $A$ and $B$ is a pair of straight lines separating the given lines harmonically, and also that the chords of contact with the fixed lines pass through one of two fixed points separating $A, B$ harmonically. (Selwyn, 1910.)

13. Prove that if two complete quadrangles have the same diagonal points their eight vertices lie on a proper conic unless they lie four by four on two straight lines. (Trinity, 1907.)

14. A conic touches four fixed straight lines $a, b, c, d$. $U$ and $V$ are two fixed points on $d$, and the tangents from $U$ and $V$ to the conic meet in $P$. Prove that the locus of $P$ is a conic passing through $U$, $V$ and the vertices of the triangle $abc$.

15. Two conics $S_1$ and $S_2$ intersect in $A, B, C, O$. Two lines through $O$ cut $S_1$ in $P_1, Q_1$ and $S_2$ in $P_2, Q_2$. If $P_1Q_1$ passes through a fixed point $T_1$, prove that $P_2Q_2$ also passes through a fixed point $T_2$.

As a particular case, prove that if from the point of intersection of the Steiner ellipse and the circumcircle lines are drawn to the ends of a diameter of the ellipse cutting the circle in $P, Q$, the chord $PQ$ will always pass through the Lemoine point.

16. A quadrangle $ABCD$ is inscribed in a conic; $AB$ and $CD$ intersect in $E$, and $AD$, $BC$ in $F$. Prove that if $P$ be any point on the conic the harmonic conjugate of $EP$ with respect to $EB$, $EC$ meets the harmonic conjugate of $FP$ with respect to $FA$, $FB$ on the tangent at $P$. (Trinity, 1910.)

17. A conic is inscribed in a triangle $ABC$, its equation in homogeneous coordinates being $(lx)^{\frac{1}{2}}+(my)^{\frac{1}{2}}+(nz)^{\frac{1}{2}}=0$. The lines from $A, B, C$ to the points of contact with the opposite sides meet the conic in $a, b, c$; and the tangents at $a, b, c$ form another triangle $A'B'C'$. Show that the equation of the conic through $A, B, C, A', B', C'$ is $(lx)^{-1}+(my)^{-1}+(nz)^{-1}=0$. (Math. Tripos II., 1912.)

18. Two conics have three-point contact at $P$ and intersect again at $Q$. $PR$ is the tangent at $P$, $QR$ the harmonic conjugate of $QP$ with regard to the two tangents at $Q$. Show that $R$ lies on the second common tangent of the two conics. (Math. Tripos I., 1909.)

19. A complete quadrilateral, of which one side is variable, is circumscribed about a fixed conic. Show that the line joining the mid-points of its diagonals passes through a fixed point.

20. Prove that the equation of the isoptic locus of the parabola $y^2=4ax$ for the angle $\alpha$ is $y^2-4ax=(x+a)^2\tan^2\alpha$, and that this represents a hyperbola, of eccentricity $\sec\alpha$, having the same focus and directrix as the parabola.

21. Write down the equation of the circle which osculates the conic

$$ax^2+by^2+2hxy+2gx=0$$

at the origin, and deduce that the radius of curvature at the origin is $g/b$.

22. Determine all the common tangents of the conics $y^2+z^2+2yz+2xy=0$, $x^2+y^2+9z^2-6yz-6zx-8xy=0$. (Math. Tripos II., 1913.)

23. Given any conic $S$ circumscribing a triangle $ABC$, prove that there is one conic $S'$ inscribed in the triangle and having (imaginary) double contact with the given conic. Show that the chord of contact is the line through the points of intersection of the tangents to $S$ at $A, B, C$ with the opposite sides, and its pole is the point of intersection of the lines joining $A, B, C$ to the points of contact of $S'$ with the opposite sides. Show further that there is one conic (virtual) which has double contact with $S$ and $S'$ at the same points, and with respect to which the triangle $ABC$ is self-conjugate.

24. Prove that chords of a conic $\Sigma$, which subtend a constant angle $\alpha$ at a given point of the conic, envelop a conic, and that, for different values of the angle $\alpha$, the envelopes all touch $\Sigma$ at the same two points.

# CHAPTER XIV.

## THE LINE AT INFINITY AND THE CIRCULAR POINTS.

**1.** It has been seen that homogeneous coordinates are not very well suited for the investigation of metrical theorems. The formulae for the distance between two points, the angle between two lines, conditions for perpendicularity, etc., are complicated and unwieldy. Homogeneous coordinates supply, in fact, a beautiful analytical instrument for dealing with projective geometry, but for metrical geometry the appropriate instrument is afforded by cartesian coordinates. With homogeneous coordinates and in projective geometry there is no distinction between the different types of conics: circle, ellipse, parabola, hyperbola. The only distinction is between a proper conic, a virtual conic, and a degenerate conic (*i.e.* two straight lines, real or imaginary, with a double point; or two coincident straight lines, with a pair of points, real, coincident, or imaginary).

Further, in projective geometry there is no distinction between one point and another in relation to a conic, except that between exterior and interior points, which depends only on the reality or otherwise of the tangents from the point to the conic. For example, the foci are not distinguished from any other interior point. Nor is there any distinction between one line and another, except in regard to the reality or otherwise of the points of intersection with the conic.

Now it is always possible, in a more or less cumbrous way, to deal with metrical properties by using a particular system of homogeneous coordinates, trilinears or areals. Thus, by working out a complicated expression for the distance between two points, we could write down the equation of a circle. Again, using Joachimsthal's section-formulae, we could form the condition that a chord through a certain point is always bisected there, and thus obtain the coordinates of the centre. The asymptotes could be found by writing down the equation of the tangent at any point and then letting the coordinates of the point tend to infinity. These processes, however, can all be replaced by the procedure of projective geometry, with the aid of the straight line at infinity and the two special points at infinity called the circular points.

We shall show in this chapter how these figures can be applied to certain metrical problems in connection with the conic when referred to homogeneous coordinates. First, we shall consider some results which require only a consideration of the line at infinity.

**2. Condition for a parabola, ellipse, or hyperbola.** Conics are classified according to the nature of their intersection with the line at infinity. A hyperbola cuts it in real points, an ellipse in imaginary points, and a parabola in coincident points. We can therefore determine the nature of the conic

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0,$$

as follows. Let the equation of the line at infinity be

$$l_0x+m_0y+n_0z=0.$$

Then the condition that the conic should be a parabola is that the line whose line-coordinates are $(l_0, m_0, n_0)$ should be a tangent, *i.e.*

$$Al_0^2+Bm_0^2+Cn_0^2+2Fm_0n_0+2Gn_0l_0+2Hl_0m_0=0.$$

Then, referring to Chap. XIII. § 4, we see that the points of intersection will be real or imaginary according as

$$(hn_0^2-gm_0n_0-fn_0l_0+cl_0m_0)^2 \gtrless (an_0^2-2gn_0l_0+cl_0^2)(bn_0^2-2fm_0n_0+cm_0^2).$$

Hence the conic will be a hyperbola or an ellipse according as

$$Al_0^2+Bm_0^2+Cn_0^2+2Fm_0n_0+2Gn_0l_0+2Hl_0m_0 \lessgtr 0.$$

Ex. The lines $PA$, $PB$, $PC$ cut the sides $BC$, $CA$, $AB$ of the triangle $ABC$ in $L$, $M$, $N$. Find the locus of $P$ so that the conic which touches the sides of the triangle at $L$, $M$, $N$ may be a parabola.

Let $P\equiv(x_1, y_1, z_1)$. Then $L\equiv(0, y_1, z_1)$, $M\equiv(x_1, 0, z_1)$, $N\equiv(x_1, y_1, 0)$. The line-equation of the conic is of the form

$$fmn+gnl+hlm=0,$$

and its point-equation is

$$f^2x^2+g^2y^2+h^2z^2-2ghyz-2hfzx-2fgxy=0.$$

Since this is to touch $BC$ in $L$, $g:h=z_1:y_1$; hence $fx_1=gy_1=hz_1$, and the equation of the conic is

$$y_1z_1mn+z_1x_1nl+x_1y_1lm=0.$$

If the coordinates are areals, the condition that this should be a parabola is

$$y_1z_1+z_1x_1+x_1y_1=0.$$

The locus of $P$ is therefore a conic circumscribing the triangle, and having the tangents at each vertex parallel to the opposite side. This is the minimum circumscribed ellipse (Ex. IV. B, 21).

**3. The centre.** The centre is the pole of the line at infinity, and its coordinates are therefore (Chap. XI. § 11)

$$(Al_0+Hm_0+Gn_0,\ Hl_0+Bm_0+Fn_0,\ Gl_0+Fm_0+Cn_0).$$

Ex. Find the locus of the centre of a conic which passes through the incentre and the three excentres of the triangle.

Taking trilinear coordinates, the coordinates of the four points are $(1, \pm 1, \pm 1)$. The equation of a conic through these is $p\alpha^2+q\beta^2+r\gamma^2=0$, where $p+q+r=0$. If $a$, $b$, $c$ are the lengths of the sides of the triangle of reference, the coordinates of the centre of this conic are

$$\alpha:\beta:\gamma=aqr:brp:cpq=\frac{a}{p}:\frac{b}{q}:\frac{c}{r}.$$

Therefore

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=0,$$

which represents the circumcircle of the triangle (see § 7).

**4. Diameters.** A diameter is a line through the centre. The line $lx+my+nz=0$ will be a diameter if

$$l(Al_0+Hm_0+Gn_0)+m(Hl_0+Bm_0+Fn_0)+n(Gl_0+Fm_0+Cn_0)=0.$$

The diameter $(l, m, n)$ is therefore conjugate to the line at infinity $(l_0, m_0, n_0)$.

**5. The asymptotes.** The asymptotes are the tangents to the conic at its points of intersection $H$, $K$ with the line at infinity. If $u$ is the line at infinity, the equation

$$f(x, y, z)-ku^2=0$$

represents a conic touching the given conic $f$ at $H$, $K$. The equation of the asymptotes is therefore obtained by determining $k$ so that this equation may represent two straight lines.

*Cor.* The conics $f=0$ and $f-\lambda u^2=0$ have asymptotes of the form

$$f-k_1u^2=0, \quad f-k_2u^2=0.$$

Both pairs of lines cut $u=0$ in the same two points, the points at infinity on $f=0$. Hence the two pairs of asymptotes are parallel, and the conics are *similar and similarly placed*, or *homothetic*.

Ex. The conic $yz+zx+xy=0$, in areal coordinates, is the minimum circumscribed ellipse, and the conic $x^2+y^2+z^2-2yz-2zx-2xy=0$ is the maximum inscribed ellipse. Show that these conics are homothetic.

We have $u\equiv x+y+z$, $f_2\equiv u^2-4(yz+zx+xy)\equiv u^2-4f_1$; therefore, etc.

**6.** We have next to consider the circular points at infinity. These have been defined and obtained as the points of intersection of any circle with the line at infinity. In order to obtain the coordinates of the circular points in homogeneous coordinates, we shall find first the equation of a circle, then, as we know the equation of the line at infinity, the coordinates in question can be found by solving the two equations simultaneously. A circle involves metrical determinations; hence we must take a definite system of coordinates, say trilinears, and, as any circle will suffice, we shall choose the circumcircle of the triangle of reference.

**7. Equation of the circumcircle in trilinears.** Let $P\equiv(\alpha, \beta, \gamma)$ be any point on the circumcircle, and invert the circle with centre of inversion $P$. The circle inverts into a straight line, so that $A'$, $B'$, $C'$, the inverses of $A$, $B$, $C$, are collinear (Fig. 88). Then, since $PB \,.\, PB'=PC \,.\, PC'$, $B$, $B'$, $C'$, $C$ are concyclic and $\angle PCB=PB'C'$. Hence the triangles $PBC$ and $PC'B'$ are similar. If $PN$, the perpendicular on $A'B'$, $=p$, we have then

$$PX : BC=p : B'C',$$

*i.e.*

$$\frac{a}{\alpha}=\frac{B'C'}{p}.$$

Similarly

$$\frac{b}{\beta}=\frac{C'A'}{p} \quad\text{and}\quad \frac{c}{\gamma}=\frac{A'B'}{p}.$$

The signs of $\alpha$, $\beta$, $\gamma$ are so arranged that they are all + for a point within the triangle, and will be the same as the signs of $B'C'$, $C'A'$, $A'B'$. Hence, since $B'C'+C'A'+A'B'=0$, we have

$$\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=0.$$

Hence the equation of the circumcircle in trilinear coordinates is

$$a\beta\gamma+b\gamma\alpha+c\alpha\beta=0.$$

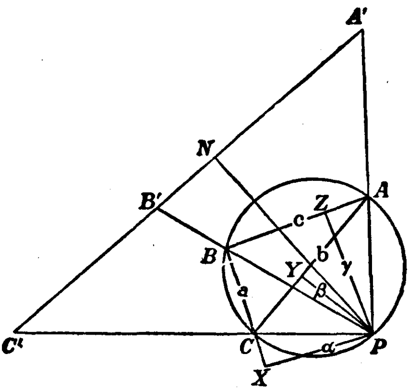



FIG. 88.

To find the equation of the circumcircle in areal coordinates, we have to write $x/a$, $y/b$, $z/c$ instead of $\alpha$, $\beta$, $\gamma$, and we get

$$a^2yz+b^2zx+c^2xy=0.$$

**8. Coordinates of the circular points in trilinears.** The circular points are the points of intersection of any circle with the line at infinity.

Take the circumcircle $a\beta\gamma+b\gamma\alpha+c\alpha\beta=0$, and the line at infinity $a\alpha+b\beta+c\gamma=0$. The ratio $\beta:\gamma$ is given by the equation

$$a^2\beta\gamma=(b\gamma+c\beta)(b\beta+c\gamma),$$

or, since $b^2+c^2-a^2=2bc\cos A$,

$$\beta^2+2\beta\gamma\cos A+\gamma^2=0,$$

*i.e.* $$\beta:\gamma=-\cos A\pm i\sin A=-e^{\mp iA}.$$

Hence $$\alpha:\beta:\gamma=e^{\pm iB}:e^{\mp iA}:-1.$$

Hence the tangential equation of the circular points in trilinears is

$$(le^{iB}+me^{-iA}-n)(le^{-iB}+me^{iA}-n)$$
$$\equiv l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C=0.$$

Similarly the tangential equation in areals is

$$a^2l^2+b^2m^2+c^2n^2-2mnbc\cos A-2nlca\cos B-2lmab\cos C=0.$$

**9. General equation of a circle.** To find the general equation of a circle, we have only to form the equation of a conic which passes through the points of intersection of the line at infinity with the circumcircle. Hence in trilinear coordinates the equation of any circle is of the form

$$a\beta\gamma + b\gamma\alpha + c\alpha\beta - (a\alpha + b\beta + c\gamma)(p\alpha + q\beta + r\gamma) = 0,$$

and in areal coordinates

$$a^2yz + b^2zx + c^2xy - (x + y + z)(px + qy + rz) = 0.$$

**10. Geometrical meaning of the constants in the equation of a circle.** Let the equation of the circle in areal coordinates be

$$(x + y + z)(px + qy + rz) - (a^2yz + b^2zx + c^2xy) = 0.$$

Let the circle cut the sides $a$, $b$, $c$ of the triangle in $X_1$, $X_2$; $Y_1$, $Y_2$; $Z_1$, $Z_2$; and let $t_1$, $t_2$, $t_3$ be the lengths of the tangents from $A$, $B$, $C$ to the circle. Then $t_1^2 = AZ_1 \,.\, AZ_2$, $t_2^2 = BZ_1 \,.\, BZ_2$, etc. $t_1^2$, $t_2^2$, $t_3^2$ are the powers of the vertices $A$, $B$, $C$ with regard to the circle. Let the metrical areal coordinates of $Z_1$ and $Z_2$ be $(x_1, y_1, 0)$, $(x_2, y_2, 0)$. These satisfy the two equations

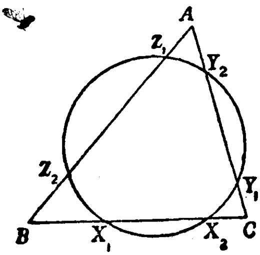


FIG. 89.

$$x + y = 1,$$

$$-c^2xy + (x + y)(px + qy) = 0.$$

Eliminating $x$, we have

$$c^2y^2 - (c^2 + p - q)y + p = 0.$$

Therefore $\quad y_1y_2 = p/c^2.$

But $\quad y_1 = \dfrac{\Delta CAZ_1}{\Delta ABC} = \dfrac{AZ_1}{c}$ and $y_2 = \dfrac{AZ_2}{c}.$

Hence $\quad p = c^2y_1y_2 = AZ_1 \,.\, AZ_2 = t_1^2.$

Similarly $\quad q = t_2^2$ and $r = t_3^2.$

Hence *the general equation of a circle in areal coordinates is*

$$(x + y + z)(t_1^2x + t_2^2y + t_3^2z) - (a^2yz + b^2zx + c^2xy) = 0,$$

where $a$, $b$, $c$ are the lengths of the sides of the triangle of reference $ABC$, and $t_1$, $t_2$, $t_3$ are the lengths of the tangents from $A$, $B$, $C$ to the circle.

Ex. The equation of a circle in trilinear coordinates being

$$\Sigma a\alpha \Sigma p\alpha - \Sigma a\beta\gamma = 0,$$

prove that the squares of the lengths of the tangents from $A$, $B$, $C$ to the circle are $bcp$, $caq$, $abr$.

**11.** The result of the last section enables us to write down the equation of any circle in areal coordinates, when we know its intercepts on the sides of the triangle.

**Examples.**

1. *The inscribed circle.*

With the usual notation, where $2s = a + b + c$, the intercepts on the sides are

$$AZ_1 = AZ_2 = s - a, \quad BZ_1 = BZ_2 = s - b, \quad CX_1 = CX_2 = s - c.$$

Hence the equation is

$$a^2yz+b^2zx+c^2xy-(x+y+z)\{(s-a)^2x+(s-b)^2y+(s-c)^2z\}=0.$$

The equations of the escribed circles can be found in a similar way. They can be written down from the last equation by changing the sign of $a$, $b$, or $c$.

2. *The nine-point circle.*

Here $AZ_1=\frac{1}{2}c$, $AZ_2=b\cos A$, so that the power of $A$ is $\frac{1}{2}bc\cos A$. Similarly for $B$ and $C$. Hence the equation is

$$a^2yz+b^2zx+c^2xy-\tfrac{1}{2}(x+y+z)(xbc\cos A+yca\cos B+zab\cos C)=0,$$

which reduces to $\quad \Sigma a^2yz-\Sigma x^2bc\cos A=0.$

**12. Power of a point P with respect to the circle whose equation in areal coordinates is**

$$\varphi(\mathbf{x}, \mathbf{y}, \mathbf{z})\equiv(\mathbf{x}+\mathbf{y}+\mathbf{z})(\mathbf{px}+\mathbf{qy}+\mathbf{rz})-(\mathbf{a}^2\mathbf{yz}+\mathbf{b}^2\mathbf{zx}+\mathbf{c}^2\mathbf{xy})=0.$$

Join $PA$ cutting the circle in $U$ and $V$. Then the power of $P$ is $PU\,.\,PV$. Let $U$ and $V$ divide $PA$ in the ratio $\lambda:1$. Let the metrical areal coordinates of $P$ be $(x_0, y_0, z_0)$. $A\equiv(1, 0, 0)$, and therefore the coordinates of $U$ are proportional to $(x_0+\lambda, y_0, z_0)$. Substitute in the equation of the circle, and we have, since $x_0+y_0+z_0=1$,

$$(\lambda+1)(p\lambda+px_0+qy_0+rz_0)-\{\lambda(b^2z_0+c^2y_0)+a^2y_0z_0+b^2z_0x_0+c^2x_0y_0\}=0,$$

which gives the quadratic equation for $\lambda$,

$$p\lambda^2+\lambda(p+px_0+qy_0+rz_0-b^2z_0-c^2y_0)+\varphi(x_0, y_0, z_0)=0.$$

The roots of this equation are the ratios

$$\lambda_1=\frac{UP}{AU},\quad \lambda_2=\frac{VP}{AV}.$$

Now $\qquad \lambda_1\lambda_2=\varphi(x_0, y_0, z_0)/p \quad\text{and}\quad AU\,.\,AV=p;$

hence $\qquad PU\,.\,PV=\varphi(x_0, y_0, z_0).$

**13. Line-equation of a circle with given centre and radius.** In questions which involve the centre of a circle it is best to form the line-equation of the circle. When the coordinates of the centre $O$ are given, the line-equation is easily found, for we have simply to write down the equation of a conic touching the lines $OI$ and $OJ$ at $I$ and $J$ respectively. If the joint-equation of the circular points is $\omega\omega'=0$, and the line-equation of the centre is $w=0$, the line-equation of a circle with centre $O$ is (cf. Chap. XIII. § 8)

$$w^2-\lambda\omega\omega'=0.$$

We can establish this result also independently, and show at the same time that $\lambda$ is proportional to the square of the radius.

From Chap. XII. § 28, we have, if $r$ is the distance of the point $(\alpha, \beta, \gamma)$ from the line $(l, m, n)$ in trilinear coordinates,

$$r^2(a\alpha+b\beta+c\gamma)^2(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)$$
$$=4\Delta^2(l\alpha+m\beta+n\gamma)^2.$$

Hence, since a circle is the envelope of a line which is at a constant distance from a fixed point, this is the line-equation of a circle with centre $(\alpha, \beta, \gamma)$

and radius $r$. If $(\alpha_0, \beta_0, \gamma_0)$ are the metrical trilinear coordinates, so that $a\alpha_0 + b\beta_0 + c\gamma_0 = 2\Delta$, and we put $l\alpha + m\beta + n\gamma \equiv w$, and

$$\Sigma l^2 - 2\Sigma mn \cos A \equiv \omega\omega',$$

the equation of the circle reduces to

$$r^2\omega\omega' = w^2.$$

**14. Condition for perpendicularity.** By means of the circular points it is possible to reduce any metrical problem which involves the measurement of angles to a problem in projective geometry, to which we may apply homogeneous coordinates.

We shall first prove that *two perpendicular straight lines are harmonic conjugates, or apolar, to the pair of lines joining their point of intersection to the circular points.*

Taking rectangular cartesian coordinates, let the two lines be represented by the equation
$$ax^2 + 2hxy + by^2 = 0.$$

The equation of the two lines joining their point of intersection $O$ to the circular points, $I$ and $J$, is
$$x^2 + y^2 = 0.$$

The condition for apolarity is (Chap. II. § 21)

$$a + b = 0,$$

but this is just the condition for perpendicularity.

The metrical property, perpendicularity, is thus expressed by means of the projective property, apolarity.

**15. The circular points as a degenerate conic.** In rectangular cartesian coordinates the line $lx + my + nz = 0$ passes through one of the circular points, if $l \pm im = 0$, *i.e.* if
$$l^2 + m^2 = 0, \quad \text{.............................................(1)}$$

and this is thus the line-equation of the two circular points. The circular points therefore appear as a degenerate conic-envelope with (1) as its line-equation. If we form the point-equation in the usual way, we find

$$z^2 = 0, \quad \text{.................................................(2)}$$

which represents the line at infinity taken twice.

Hence the two figures form together a single degenerate conic, the line at infinity being the locus and the circular points the envelope. This conic is called the *Absolute*, its points, the points at infinity, are called *absolute points*, and its tangents, the lines through one of the circular points, *absolute lines*.

From this point of view we can restate the theorem of § 14. Let $l_1x + m_1y + n_1z = 0$ and $l_2x + m_2y + n_2z = 0$ be the rectangular cartesian equations of two perpendicular lines; then

$$l_1l_2 + m_1m_2 = 0.$$

But this is just the condition that the two lines should be conjugate with regard to the conic whose line-equation is $l^2 + m^2 = 0$. Hence *two perpendicular lines are conjugate with regard to the absolute.*

**16. Condition that the two lines ($l_1, m_1, n_1$) and ($l_2, m_2, n_2$) should be perpendicular, in trilinear coordinates.** The line-equation of the circular points in trilinears is

$$l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C=0.$$

The two lines will be at right angles if they are conjugate with regard to this degenerate conic; hence the condition is

$$l_1l_2+m_1m_2+n_1n_2-(m_1n_2+m_2n_1)\cos A-(n_1l_2+n_2l_1)\cos B$$
$$-(l_1m_2+l_2m_1)\cos C=0.$$

**17. Angle between two straight lines.** We shall prove next that *the angle between two straight lines can be expressed in terms of the cross-ratio of the pencil formed by the two straight lines and the two absolute lines through their point of intersection.*

Let $y=x\tan\theta$, $y=x\tan\theta'$ be the equations of the two lines $u$, $u'$ referred to rectangular axes with origin at their point of intersection $O$. The equations of the two lines $OI$ and $OJ$, which we shall denote by $i$ and $j$ are $y=ix$, $y=-ix$. Then the cross-ratio

$$(ij,\ uu')=\frac{i-\tan\theta}{-i-\tan\theta}\div\frac{i-\tan\theta'}{-i-\tan\theta'}.$$

Now $$\frac{i-\tan\theta}{i+\tan\theta}=\frac{\cos\theta+i\sin\theta}{\cos\theta-i\sin\theta}=e^{2i\theta}.$$

Therefore $$(ij,\ uu')=e^{2i(\theta-\theta')},$$

and therefore $$\phi=\theta'-\theta=\tfrac{1}{2}i\log(ij,\ uu').$$

**18. Condition for a rectangular hyperbola.** A conic will be a rectangular hyperbola when its asymptotes are at right angles, and are therefore harmonic conjugates with regard to the absolute lines through the centre. This may be expressed by saying that the circular points $I, J$ are conjugate with regard to the conic.

Let the equation of the conic in trilinear coordinates be

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0,$$

and let $a_0$, $b_0$, $c_0$ and $A_0$, $B_0$, $C_0$, denote the magnitudes of the sides and angles of the triangle of reference.

Let the coordinates of the circular points be $I\equiv(\alpha_1,\beta_1,\gamma_1)$, $J\equiv(\alpha_2,\beta_2,\gamma_2)$; then the condition that these two points should be conjugate with regard to the conic is

$$a\alpha_1\alpha_2+b\beta_1\beta_2+c\gamma_1\gamma_2+f(\beta_1\gamma_2+\beta_2\gamma_1)+g(\gamma_1\alpha_2+\gamma_2\alpha_1)+h(\alpha_1\beta_2+\alpha_2\beta_1)=0.$$

Now the line-equation of the point-pair $I, J$ is either

$$l^2+m^2+n^2-2mn\cos A_0-2nl\cos B_0-2lm\cos C_0=0,$$

or $$(l\alpha_1+m\beta_1+n\gamma_1)(l\alpha_2+m\beta_2+n\gamma_2)=0.$$

Identifying these two equations, we have

$$\alpha_1\alpha_2=\lambda,\quad \beta_1\gamma_2+\beta_2\gamma_1=-2\lambda\cos A_0,$$
$$\beta_1\beta_2=\lambda,\quad \gamma_1\alpha_2+\gamma_2\alpha_1=-2\lambda\cos B_0,$$
$$\gamma_1\gamma_2=\lambda,\quad \alpha_1\beta_2+\alpha_2\beta_1=-2\lambda\cos C_0.$$

Hence the condition reduces to

$$a+b+c-2f\cos A_0-2g\cos B_0-2h\cos C_0=0.$$

Similarly, if the coordinates are areals the condition is

$$aa_0^2+bb_0^2+cc_0^2-2fb_0c_0\cos A_0-2gc_0a_0\cos B_0-2ha_0b_0\cos C_0=0.$$

**Examples.**

1. Prove that every conic which passes through the in- and excentres of a triangle is a rectangular hyperbola.

The trilinear equation of any conic passing through these four points $(1, \pm 1, \pm 1)$ is $p\alpha^2+q\beta^2+r\gamma^2=0$, where $p+q+r=0$. But this is the condition for a rectangular hyperbola.

2. Prove that every circumscribed conic which passes through the orthocentre is a rectangular hyperbola.

The condition that the conic $f\beta\gamma+g\gamma\alpha+h\alpha\beta=0$ should pass through the orthocentre, whose trilinear coordinates are $(\sec A, \sec B, \sec C)$, is

$$f\cos A+g\cos B+h\cos C=0,$$

but this is just the condition for a rectangular hyperbola.

These two examples are particular forms of the same theorem, viz., *If A, B, C, D are four points such that each is the orthocentre of the triangle formed by the other three, every conic which passes through them is a rectangular hyperbola.*

**19.** The condition for a rectangular hyperbola could be found more easily if we were able to write down the condition that a pair of points, whose line-equation only is given, are conjugate with regard to a conic whose point-equation is given. We have simple formulae of this description in the case of two pairs of points on the same straight line, or two pairs of straight lines through the same point, which are given by quadratic equations, viz. the two pairs of lines

$$a_1x^2+2h_1xy+b_1y^2=0,$$
$$a_2x^2+2h_2xy+b_2y^2=0$$

are harmonic conjugates or apolar if

$$a_1b_2-2h_1h_2+a_2b_1=0.$$

We require something analogous of wider application.

The formula in the case of the rectangular hyperbola in areal coordinates suggests the result that *the pair of points represented by the line-equation*

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0$$

*are conjugate with respect to the conic-locus*

$$a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0,$$

*if* $$Aa'+Bb'+Cc'+2Ff'+2Gg'+2Hh'=0.$$

It is left to the reader to prove this by reproducing the proof in § 18 with altered letters. In the same way it can be shown that if the point-equation represents two straight lines, the same condition will secure that these two lines are conjugate with regard to the conic-envelope represented by the line-equation. This relation between the coefficients of a conic-locus and a conic-envelope is of far-reaching importance, not only in these cases where one of the conics is degenerate, but also when both conics are proper conics. The term *apolar* which has been used in the restricted cases is applied also in the general case. The actual geometrical relationship between two conics in the case of apolarity will be considered in Chap. XX.

We can now say that *a rectangular hyperbola is a conic-locus which is apolar to the degenerate conic-envelope consisting of the circular points.*

Further, *two perpendicular straight lines form a degenerate conic-locus which is apolar to the circular points;* hence the condition that the two straight lines

$$a\alpha^2+b\beta^2+c\gamma^2+2f\beta\gamma+2g\gamma\alpha+2h\alpha\beta=0$$

should be at right angles is, in trilinear coordinates,

$$a+b+c-2f\cos A-2g\cos B-2h\cos C=0.$$

**20. The foci of a conic.** We have seen (Chap. XI. § 15) that the foci of a conic are the points of intersection of the tangents to the conic from the circular points. This result may be stated in a slightly different form. Regarding the circular points as a degenerate conic-envelope, the given conic and this degenerate conic have four common tangents, which form a complete quadrilateral. The six vertices of this complete quadrilateral are the circular points $I$, $J$ and the two pairs of foci $F$, $F'$ and $G$, $G'$. The harmonic triangle consists of the line at infinity $IJ$, and the two axes $FF'$ and $GG'$; and its vertices are the centre $C$, and the points at infinity $A$, $B$ on the axes. Since the two conics have real line-equations of the second degree their simultaneous solutions are two pairs of conjugate imaginary sets of values of the line-coordinates $l$, $m$, $n$. Hence the four tangents are two pairs of conjugate imaginary lines. Then, since the point of intersection of a pair of conjugate imaginary lines is real, we have two real foci, $F$, $F'$, the other pair $G$, $G'$ being conjugate imaginary points. All four cannot be real and distinct, for then their joins $FG$, etc., would be real.

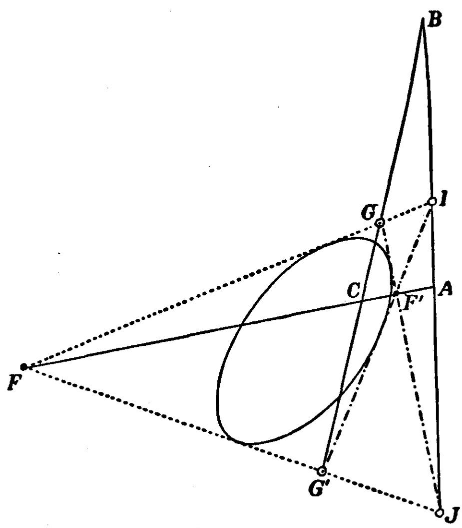


FIG. 90.

If the conic is a parabola it touches the line at infinity $IJ$, and one of the real foci $F'$ becomes the point of contact of the line at infinity with the curve, *i.e.* the point at infinity on the parabola. The two imaginary foci $G$, $G'$ then coincide with $I$ and $J$.

If the conic is a circle it passes through $I$ and $J$. Then all four foci coincide with the pole of the line at infinity, *i.e.* the centre of the circle.

**21.** The equations for the coordinates of the foci of a conic in homogeneous coordinates would be too complicated to be useful, but the converse problem is of importance and can be solved without much difficulty, viz. to find the equation of a conic with given foci.

*To find the line-equation of a conic confocal with a given conic.*

Let the line-equation of the given conic be $\Sigma=0$. Then we have to form the equation of a conic touching the four tangents from the circular points to $\Sigma$, *i.e.* a conic touching the common tangents of $\Sigma$ and the degenerate conic $\omega\omega'=0$. The equation is therefore

$$\lambda\Sigma+\omega\omega'=0,$$

where $\omega\omega'=0$ is the line-equation of the circular points.

The line-equations of the pairs of foci could be found from this by choosing $\lambda$ so that the left-hand side breaks up into factors. The condition for this would give a cubic for $\lambda$. One root $\lambda=0$ will correspond to the pair of circular points, the other roots will give the two pairs of foci.

**22. To find the line-equation of a conic having a given pair of points as foci.** Let the two given points be $F_1\equiv(x_1, y_1, z_1)$, $F_2\equiv(x_2, y_2, z_2)$. These two points form a degenerate conic, whose line-equation is

$$(lx_1+my_1+nz_1)(lx_2+my_2+nz_2)=0. \quad \text{.....................(1)}$$

We have then to write down the equation of a conic touching the four common tangents of the conics $\omega\omega'$ and (1), viz.

$$(lx_1+my_1+nz_1)(lx_2+my_2+nz_2)+\lambda\omega\omega'=0.$$

**Examples.**

1. Show that the real foci of a conic inscribed in a triangle are isogonal conjugates with regard to the triangle.

The trilinear line-equation of a conic with foci $(\alpha_1, \beta_1, \gamma_1)$ and $\left(\frac{1}{\alpha_1}, \frac{1}{\beta_1}, \frac{1}{\gamma_1}\right)$ is

$$(l\alpha_1+m\beta_1+n\gamma_1)(l\beta_1\gamma_1+m\gamma_1\alpha_1+n\alpha_1\beta_1)$$
$$-\lambda(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)=0.$$

If this touches $x=0$ the coefficient of $l^2$ vanishes; therefore $\lambda=\alpha_1\beta_1\gamma_1$. Then the coefficients of $m^2$ and $n^2$ also vanish, and the equation reduces to

$$\Sigma mn\alpha_1(\beta_1{}^2+\gamma_1{}^2+2\beta_1\gamma_1\cos A)=0;$$

therefore the conic touches all three sides.

2. Find the equation of the inscribed conic which has one focus at the circumcentre (and the other consequently at the orthocentre).

The trilinear coordinates of the circumcentre are $(\cos A, \cos B, \cos C)$. Then $\cos^2 B+\cos^2 C+2\cos B\cos C\cos A=1-\cos^2 A=\sin^2 A$, and the equation of the conic is

$$\Sigma mna^2\cos A=0.$$

In areal coordinates the equation becomes

$$\Sigma mn\sin 2A=0.$$

**23. Ex.** Find the locus of the foci of parabolas inscribed in the triangle of reference.

The trilinear equation of a conic having $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ as foci is

$$(l\alpha_1+m\beta_1+n\gamma_1)(l\alpha_2+m\beta_2+n\gamma_2)$$
$$-\lambda(l^2+m^2+n^2-2mn\cos A-2nl\cos B-2lm\cos C)=0. \qquad \text{(1)}$$

For a parabola one focus is at infinity, and therefore

$$a\alpha_2+b\beta_2+c\gamma_2=0. \qquad \text{(2)}$$

The conditions that the conic should touch the three sides of the triangle are

$$\alpha_1\alpha_2=\lambda=\beta_1\beta_2=\gamma_1\gamma_2. \qquad \text{(3)}$$

Substituting the values of $\alpha_2, \beta_2, \gamma_2$ from (3) in (2), we get

$$\frac{a}{\alpha_1}+\frac{b}{\beta_1}+\frac{c}{\gamma_1}=0.$$

Hence the focus $(\alpha_1, \beta_1, \gamma_1)$ lies on the circumcircle of the triangle.

## EXAMPLES XIV.

1. Find the locus of the centre of a conic which touches two given lines at given points.

2. A system of conics is drawn passing through the vertices of the triangle of reference and also through the centroid. Prove that the locus of their centres is a conic touching the sides of the triangle of reference at their mid-points.

3. A conic is inscribed in a triangle $ABC$, touching the sides at $P, Q, R$. The lines $QR, RP, PQ$ meet $BC, CA, AB$ in $L, M, N$ respectively. Prove that $LMN$ are collinear, and show that if this line passes through the centre of the conic, it will, for different conics, envelop a conic touching the sides of $ABC$.
(Pembroke, 1906.)

4. Prove that the line $(l, m, n)$ will be a diameter of the conic $ax^2+by^2+cz^2=0$, the coordinates being areals, if $l/a+m/b+n/c=0$.

5. Prove that in areal coordinates the equation of the diameter of the conic $ax^2+by^2+cz^2=0$ which is conjugate to the direction of the line $(l, m, n)$ is

$$a(m-n)x+b(n-l)y+c(l-m)z=0.$$

6. Prove that the areal coordinates of the centre of the conic

$$(lx)^{\frac{1}{2}}+(my)^{\frac{1}{2}}+(nz)^{\frac{1}{2}}=0$$

are $(m+n, n+l, l+m)$.

7. If the equation $(lx)^{\frac{1}{2}}+(my)^{\frac{1}{2}}+(nz)^{\frac{1}{2}}=0$ in areal coordinates represents a parabola, show that $l+m+n=0$, and that the line $(m-n)x+(n-l)y+(l-m)z=0$ is parallel to the axis.

8. Find the trilinear line-equation of a conic which touches the sides of the triangle of reference at the feet of the altitudes; and prove that the trilinear coordinates of its centre are proportional to the sides.

9. A conic $S$ touches the sides $CA, CB$ of the triangle of reference at $A, B$. Another conic $S'$ has three-point contact with $S$ at $A$ and passes through $B$. Prove that the locus of the centre of $S'$ is a conic touching $CA$ at $A$ and intersecting $S$ in two points on the line whose areal equation is $3x-y+z=0$.
(Math. Tripos II., 1911.)

10. A hyperbola is such that the triangle of reference is self-conjugate with respect to it, and one of its asymptotes passes through a vertex of the triangle. Prove that the locus of its centre is a straight line through this vertex parallel to the opposite side. (Corpus, 1912.)

11. A variable line moves in a plane so that the intercepts made on it by the sides of a fixed triangle bear constant ratios to one another. Show that the line envelops a parabola inscribed in the fixed triangle. (See Ex. XII. 2.) (Pembroke, 1913.)

12. Prove that the areal equation of the ellipse circumscribing the triangle of reference and having its centre at the centroid of the triangle is $yz+zx+xy=0$. (The Steiner ellipse, or minimum circumscribed ellipse.)

13. Prove that the areal equation of the ellipse inscribed in the triangle of reference and having its centre at the centroid of the triangle is $\sqrt{x}+\sqrt{y}+\sqrt{z}=0$. (The maximum inscribed ellipse.)

14. Given in position two points and their polars with respect to a conic, find the locus of the centre of the conic. (Corpus, 1912.)

15. Show that if the polars of $P$ and $Q$ with respect to a conic are parallel respectively to $QR$ and $PR$, the polar of $R$ must be parallel to $PQ$; and that the triangle $PQR$ is self-conjugate with respect to a second conic having the same asymptotes as the former. (Pembroke, 1910.)

16. If an ellipse is either inscribed or circumscribed to a triangle, show that its centre must lie either inside the triangle formed by the mid-points of the sides or in one of the angular areas bounded by two of the sides of this triangle.

17. Prove that the lines $\Sigma\alpha \sin(B-C)=0$ and $\Sigma\alpha/a=0$ are at right angles.

18. Prove that the line joining the circumcentre and symmedian point (Ex. XII. 12) is perpendicular to the join of the Brocard points (Ex. XII. 13).

19. Prove that the Euler line (Ex. XII. 4) is perpendicular to the line $\alpha\cos A+\beta\cos B+\gamma\cos C=0$ (the polar of the orthocentre).

20. Show that the freedom-equations $x=f+\lambda,\ y=g-\lambda\cos C,\ z=h-\lambda\cos B$ represent in trilinear coordinates a line through $(f, g, h)$ perpendicular to $BC$. Show further that if $f, g, h$ are the metrical trilinear coordinates, $\lambda$ is the actual distance of the point $(x, y, z)$ from $(f, g, h)$ measured in the positive direction.

21. From a given point $O$ distances $OA'=p$, $OB'=q$, $OC'=r$ are measured in directions perpendicular to the sides $BC, CA, AB$ of the triangle of reference. Prove that the perpendiculars from $A, B, C$ on $B'C', C'A', A'B'$ are concurrent in a point whose trilinear coordinates are $(1/p : 1/q : 1/r)$.

22. $L, M, N$ are the feet of the perpendiculars from a point $O$ on the sides of the triangle of reference. Show that the perpendiculars from $A, B, C$ on the corresponding sides of the pedal triangle $LMN$ of $O$ are concurrent in a point $O'$, the isogonal conjugate of $O$.

23. $O$ is a point at which the sides of the triangle $ABC$ subtend equal angles (*isogonic point*). Show that the pedal triangle of the isogonal conjugate of $O$ is equilateral (*isodynamic point*).

24. Show that the equation $bc\alpha\beta-ab\beta\gamma+(c^2-a^2)\gamma\alpha=0$ represents a rectangular hyperbola, and find the equations of its asymptotes.

25. If in Ex. XII. 7 (xii) the angle $\theta$ is varied, show that the point of concurrence of $AA'$, $BB'$, $CC'$ describes a rectangular hyperbola passing through $A$, $B$, $C$, the centroid, the orthocentre, and the two isogonic points.

26. Find the areal equation of the circle with respect to which the triangle of reference is self-conjugate (the *polar circle*), and write down its line-equation. Show also that its centre is at the orthocentre of the triangle.

27. Find the point- and line-equations, in areal coordinates, of the escribed circles.

28. Show that if $f\beta\gamma+g\gamma\alpha+h\alpha\beta=0$ represents a parabola in trilinear coordinates, and if it touches at $A$ the circumcircle of the triangle of reference $ABC$, then $f/(b\pm c)^2=g/ab=h/ac$. (Corpus, 1911.)

29. Prove that the parabola whose focus is the vertex $C$ of the triangle of reference and whose directrix is the side $AB$ has for its trilinear equation

$$\alpha^2+\beta^2+2\alpha\beta\cos C=\gamma^2\sin^2 C.$$ (Corpus, 1910.)

30. Prove that, if the conic $ax^2+by^2+cz^2=0$ touches any one of the sides of the triangle formed by joining the middle points of the sides of the triangle of reference, it is a parabola whose focus lies on the nine-point circle of the triangle of reference. (Selwyn, 1910.)

31. Prove that $z^2=4xy$ in areal coordinates represents a parabola whose focus is $(a^2, b^2, 2ab\cos C)$ and axis $x(a^2+3b^2-c^2)-y(3a^2+b^2-c^2)-z(a^2-b^2)=0$.

32. A variable parabola passes through a fixed point and touches two fixed straight lines. Show that the envelope of the diameter of the parabola through the point of contact with one of the lines is a hyperbola, and that the two hyperbolas so enveloped are of the same dimensions. Show further that the envelope of the line joining the points of contact is a hyperbola equal to the conjugate hyperbola of the first two. (Pembroke, 1906.)

33. Find the trilinear equation of the locus of a point such that the feet of the perpendiculars from it on the sides of the triangle of reference are collinear.

34. A straight line cuts the sides of the triangle of reference at $L$, $M$, $N$, and the perpendiculars at these points to the sides of the triangle are concurrent. Find the trilinear equation of the envelope of the straight line.

35. $L$, $M$, $N$ are the feet of the perpendiculars from a point $P$ on the sides of the triangle of reference $ABC$. Find the locus of $P$ if $AL$, $BM$, $CN$ are concurrent, and show that the locus passes through $A$, $B$, $C$, the orthocentre, and the centres of the inscribed and escribed circles.

36. Prove that the inverse of a line through one of the circular points is a line through the other of the circular points, and that the two lines intersect on the circle of inversion.

37. If $x$, $y$, $z$ are superabundant coordinates connected by the identical relation $l_0x+m_0y+n_0z\equiv 1$, show that the equation of any circle can be written

$$\phi(x, y, z)\equiv(l_0x+m_0y+n_0z)(px+qy+rz)-(a^2m_0n_0yz+b^2n_0l_0zx+c^2l_0m_0xy)=0,$$

and that the power of the point $(x_0, y_0, z_0)$ is $\phi(x_0, y_0, z_0)$.

In particular for trilinear coordinates, where $\phi(\alpha, \beta, \gamma)\equiv\Sigma a\alpha\,.\,\Sigma p\alpha-\Sigma a\beta\gamma$ and $\Sigma a\alpha=2\Delta$, prove that the power of the point $(\alpha_0, \beta_0, \gamma_0)$ is

$$\tfrac{1}{4}abc/\Delta^2.\ \phi(\alpha_0, \beta_0, \gamma_0).$$

38. Prove that the areal equation of the point-circle at $(x_0, y_0, z_0)$ is

$$(x+y+z).\Sigma x(c^2y_0^2+b^2z_0^2+2bcy_0z_0\cos A)-(\Sigma x_0)^2.\Sigma a^2yz=0.$$

39. Prove that the areal equation of a circle with centre $(x_1, y_1, z_1)$ and radius $r$ is

$$(x+y+z).\Sigma x(c^2y_1^2+b^2z_1^2+2bcy_1z_1\cos A)-(x_1+y_1+z_1)^2.\Sigma a^2yz-r^2(x+y+z)^2=0.$$

40. Deduce from Ex. 39 that the distance $r$ between the points $(x, y, z)$ and $(x', y', z')$ in areal coordinates is given by

$$r^2(\Sigma x.\Sigma x')^2=\Sigma a^2(yz'-y'z)^2-2\Sigma(zx'-z'x)(xy'-x'y)bc\cos A.$$

41. If $(\xi, \eta, \zeta)$, $\xi', \eta', \zeta')$ are the areal coordinates of the circular points, adjusted so that $\xi\xi'=a^2$, prove that the square of the distance between the points $(x, y, z)$ and $(x', y', z')$ is

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ \xi & \eta & \zeta \end{vmatrix} . \begin{vmatrix} x & y & z \\ x' & y' & z' \\ \xi' & \eta' & \zeta' \end{vmatrix} \div(\Sigma x.\Sigma x')^2,$$

and that the corresponding expression in rectangular coordinates is

$$\begin{vmatrix} x & y & z \\ x' & y' & z' \\ 1 & i & 0 \end{vmatrix} . \begin{vmatrix} x & y & z \\ x' & y' & z' \\ 1 & -i & 0 \end{vmatrix} \div(zz')^2.$$

42. Prove that the areal coordinates of the centre of the circle

$$\Sigma x.\Sigma px-\Sigma a^2yz=0$$

are $a(-pa+qb\cos C+rc\cos B+abc\cos A)$, etc.

43. Show that the areal equation of the circle passing through the points $(x_i, y_i, z_i)(i=1, 2, 3)$ is

$$\begin{vmatrix} x & y & z & \Sigma a^2yz/\Sigma x \\ x_1 & y_1 & z_1 & \Sigma a^2y_1z_1/\Sigma x_1 \\ x_2 & y_2 & z_2 & \Sigma a^2y_2z_2/\Sigma x_2 \\ x_3 & y_3 & z_3 & \Sigma a^2y_3z_3/\Sigma x_3 \end{vmatrix}=0.$$

44. Three lines are drawn through the Lemoine point of a triangle parallel to the sides, meeting them in $Y, Z'$; $Z, X'$; $X, Y'$. Prove that these six points are concyclic (the *Lemoine circle*).

45. If three lines are drawn through the Lemoine point antiparallel to the sides of the triangle, meeting the sides in $Y, Z'$; $Z, X'$; $X, Y'$, prove that these six points are concyclic (the *cosine circle*).

46. $XX'$, $YY'$, $ZZ'$ are pairs of points on the sides $BC$, $CA$, $AB$ of the triangle $ABC$, such that $Y'Z \parallel CB$, $Z'X \parallel AC$, $X'Y \parallel BA$; $Z'X$ and $X'Y$ meet in $P$, $X'Y$ and $Y'Z$ in $Q$, $Y'Z$ and $Z'X$ in $R$. Prove that $AP$, $BQ$, $CR$ are concurrent in a point $K$, and that the six points $XX'YY'ZZ'$ lie on a conic. Show further that the conic is a circle if, and only if, $K$ is the Lemoine point (the *Tucker circles*).

47. Prove that if $K$, in Ex. 46, is a fixed point, the locus of the centre of the conic is the straight line joining $K$ to the centroid. Interpret the case when $K$ is the centroid.

48. Prove that the centre of the circle in Ex. 45 is the mid-point of the join of the circumcentre and the Lemoine point.

49. Prove that the condition that the two circles whose trilinear equations are $\Sigma a\alpha . \Sigma p\alpha = \Sigma a\beta\gamma$ and $\Sigma a\alpha . \Sigma p'\alpha = \Sigma a\beta\gamma$ should cut orthogonally is

$$1 + \Sigma pp' - \Sigma(p + p' + qr' + q'r) \cos A = 0.$$

50. Show that the trilinear equation of the circle whose diameter is the line joining the points $(\alpha_1, \beta_1, \gamma_1)$ and $(\alpha_2, \beta_2, \gamma_2)$ is

$$\Sigma a\alpha . \Sigma \alpha bc\{(\beta_1\beta_2 + \gamma_1\gamma_2) + (\beta_1\gamma_2 + \beta_2\gamma_1) \cos A\} = \Sigma a\alpha_1 . \Sigma a\alpha_2 . \Sigma a\beta\gamma.$$

51. Show that the trilinear equation of the circle with the circumcentre and the Lemoine point as ends of a diameter (the *Brocard circle*) is $abc\Sigma\alpha^2 = \Sigma a^3\beta\gamma$.

52. Prove that the orthoptic circle of any inscribed conic cuts the polar circle of the triangle orthogonally.

53. Prove that the orthoptic circles of a system of conics touching four given lines form a coaxal system.

54. Prove that the directrices (two real and two imaginary) of a conic are common chords of the conic and its orthoptic circle.

55. Prove that the circumcircle of a self-conjugate triangle of a rectangular hyperbola passes through the centre.

# CHAPTER XV.

## CONFOCAL CONICS AND SIMILAR CONICS.

**1. Systems of conics with common foci.** Two conics are said to be confocal when they have both (real) foci in common. Since the foci are the vertices of the complete quadrilateral formed by the tangents to the conic from the circular points, we see that if the real foci are given the imaginary foci are fixed, and *vice versa.* And the conics which have the same foci form a system touching the four fixed lines which join the foci to the circular points.

**2. Confocal ellipses and hyperbolas.** We shall consider first the case in which the two foci $F$, $F'$ are distinct and at a finite distance.

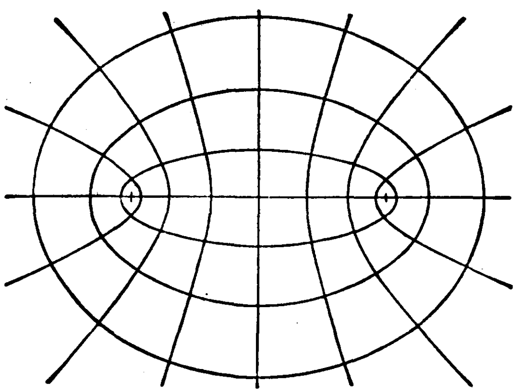


FIG. 91.

The mid-point $O$ of $FF'$ is then the centre of every conic of the system, and $FF'$ is the principal axis. Taking $O$ as origin and $OF$ as axis of $x$ the equation of any conic of the system must be of the form

$$\frac{x^2}{p}+\frac{y^2}{q}=1,$$

where, for a real conic, $p$ at least must be positive, and $p>q$. Now

$$OF^2=c^2=p-q.$$

Hence, if $p$ and $q$ are fixed and $\lambda$ is a variable parameter, the equation

$$\frac{x^2}{p-\lambda}+\frac{y^2}{q-\lambda}=1$$

represents a system of confocal conics with foci at the points $(\pm\sqrt{p-q},\ 0)$

When $\lambda<q<p$, the conic is an ellipse, As $\lambda$ increases, both axes become smaller, but the minor axis more rapidly, until, when $\lambda=q$, the conic flattens out into the two coincident lines $y^2=0$.

When $q<\lambda<p$, the conic is a hyperbola, and as $\lambda$ increases, both axes increase, the conjugate axis more rapidly, until, when $\lambda=p$, the conic flattens out into the two coincident lines $x^2=0$.

When $\lambda>p$, the conic is virtual.

For negative values of $\lambda$, increasing numerically, the axes of the ellipse increase, and their ratio becomes more nearly equal to unity, *i.e.* the ellipse becomes more nearly circular; and as $\lambda\to-\infty$, the ellipse tends to the line at infinity, taken twice. The last result is seen by writing the equation in the homogeneous form

$$\frac{x^2}{p-\lambda}+\frac{y^2}{q-\lambda}=z^2,$$

for this reduces to $z^2=0$.

We have thus a system of confocal ellipses and hyperbolas.

**3.** *Through every point there pass two conics of the system, one ellipse and one hyperbola, and these cut at right angles.*

If the conic passes through the point $(x_1, y_1)$, we have

$$x_1^2(q-\lambda)+y_1^2(p-\lambda)-(p-\lambda)(q-\lambda)=0,$$

which is a quadratic for $\lambda$. Now, on putting $-\infty$, $q$, $p$ for $\lambda$, the left-hand side has the signs $-$, $+$, $-$. Hence, of the roots of this equation one is $<q$, and the other lies between $q$ and $p$, *i.e.* one of the conics is an ellipse and the other a hyperbola.

Let $P$ be the point of intersection of an ellipse and a hyperbola of the system. Then the tangents to the ellipse and the hyperbola, which pass through $P$, are the two bisectors of the angles between $PF$ and $PF'$, and are therefore at right angles.

**4.** A very convenient way of writing the equations of the ellipses and hyperbolas of a confocal system is as follows:

Hyperbolas $$\frac{x^2}{\cos^2\phi}-\frac{y^2}{\sin^2\phi}=c^2,$$

Ellipses $$\frac{x^2}{\cosh^2\psi}+\frac{y^2}{\sinh^2\psi}=c^2;$$

for in the first case, $p=c^2\cos^2\phi$, $q=-c^2\sin^2\phi$, and in the second case, $p=c^2\cosh^2\psi$, $q=c^2\sinh^2\psi$. In each case, $p-q=c^2$.

Ex. Prove that if $x+iy=c\cos(\phi+i\psi)$, the curves $\phi=$const. and $\psi=$const. form a system of confocal hyperbolas and ellipses.

**5. Confocal parabolas.** Let us consider the next case, in which one focus is at infinity. The conics are then parabolas.

The equation of a parabola, with focus at the origin and axis the axis of $x$, is

$$y^2 = 4\lambda(x+\lambda).$$

When $\lambda > 0$ the vertex is to the left, and when $\lambda < 0$ the vertex is to the right. In the intermediate case, when $\lambda = 0$, the parabola flattens out into the two coincident lines $y^2 = 0$. As $\lambda \to \pm\infty$, the parabola tends to the line at infinity taken twice. The last result is easily seen by writing the equation in the homogeneous form

$$\frac{y^2}{\lambda^2} = 4z\left(\frac{x}{\lambda} + z\right).$$

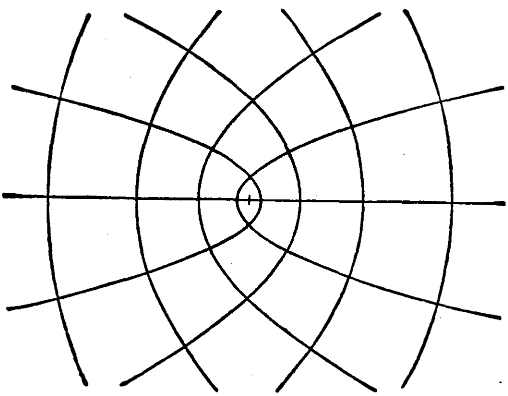


FIG. 92.

6. *Through every point there pass two parabolas of the confocal system, one belonging to each group, and these cut at right angles.*

If the parabola passes through the point $(x_1, y_1)$, we have

$$4\lambda^2 + 4\lambda x_1 - y_1^2 = 0.$$

This is a quadratic for $\lambda$, and the roots are one positive and the other negative.

Further, the tangents to the two parabolas which pass through $P$ are the bisectors of the angles between $PO$ and the line through $P$ parallel to the axis, and they are therefore at right angles.

Ex. Prove that if $x + iy = (\varphi + i\psi)^2$ the curves $\varphi = \text{const.}$ and $\psi = \text{const.}$ are two groups of parabolas forming a confocal system.

**7. Line-equation of a conic of a confocal system.** The line-equation of the conic

$$\frac{x^2}{a^2 - \lambda} + \frac{y^2}{b^2 - \lambda} = 1,$$

*i.e.* the condition that the line

$$lx + my + n = 0$$

should be a tangent, is $(a^2-\lambda)l^2+(b^2-\lambda)m^2=n^2$,

or $\lambda(l^2+m^2)=a^2l^2+b^2m^2-n^2$.

Since this equation is of the first degree in $\lambda$, it follows that *one and only one conic of a confocal system can be drawn to touch a given line.*

A similar result holds for confocal parabolas, for the line-equation of the parabola
$$y^2=4\lambda(x+\lambda)$$
is
$$\lambda(l^2+m^2)=nl.$$

Referring to Chap. XIV. § 21, the general line-equation of a system of conics confocal with the conic
$$\Sigma \equiv Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0$$
is
$$\lambda(l^2+m^2)=\Sigma,$$
for this represents a conic touching the four lines common to the conic $\Sigma$ and the degenerate conic $l^2+m^2=0$, *i.e.* touching the four tangents to $\Sigma$ from the circular points.

If $C=0$, $\Sigma$ is a parabola, and all the conics of the system are parabolas.

Ex. Find the equation of the conics confocal with $2xy=c^2$.

The line-equation of this is $2c^2lm-n^2=0$. Hence the line-equation of the confocal system is
$$2c^2lm-n^2+\lambda c^2(l^2+m^2)=0, \quad \text{......(1)}$$
and the point-equation is
$$\lambda x^2+\lambda y^2-2xy=(\lambda^2-1)c^2. \quad \text{......(2)}$$
To find the foci we have to determine $\lambda$ so that the left-hand side of (1) may factorize. This gives $\lambda=\pm 1$. The factors corresponding to $\lambda=1$ are
$$(cl+cm+n)(cl+cm-n).$$
Hence the real foci are $(c, c)$ and $(-c, -c)$. $\lambda=-1$ gives the imaginary foci.

**8. Line-equation of a conic with given foci.** If $(x_1, y_1, z_1)$ and $(x_2, y_2, z_2)$ are the homogeneous cartesian coordinates of the foci, so that their line-equations are
$$x_1l+y_1m+z_1n=0,$$
$$x_2l+y_2m+z_2n=0,$$
the equation of the system of conics with these points as foci is
$$\lambda(l^2+m^2)=(x_1l+y_1m+z_1n)(x_2l+y_2m+z_2n).$$
If either $z_1=0$ or $z_2=0$, all the conics are parabolas. If both $z_1=0$ and $z_2=0$, the conic degenerates to a point-pair lying on the line at infinity.

**9. Tangential properties of confocals.** *The locus of the point of intersection of two perpendicular tangents, one to each of two given confocal conics, is a circle.*

Let the equations of the confocal conics be
$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad \text{and} \quad \frac{x^2}{a^2-\lambda}+\frac{y^2}{b^2-\lambda}=1.$$
The equations of two perpendicular tangents are
$$x\cos\phi+y\sin\phi=(a^2\cos^2\phi+b^2\sin^2\phi)^{\frac{1}{2}},$$
$$x\sin\phi-y\cos\phi=\{(a^2-\lambda)\sin^2\phi+(b^2-\lambda)\cos^2\phi\}^{\frac{1}{2}}.$$

Squaring and adding, we get

$$x^2+y^2=a^2+b^2-\lambda.$$

As particular, or limiting, cases of this theorem, we have the following:

(1) Let the two conics coincide. Then the locus is the orthoptic circle.

(2) Let one of the conics reduce to a point-pair, viz. the foci; then the locus is the auxiliary circle of the other conic.

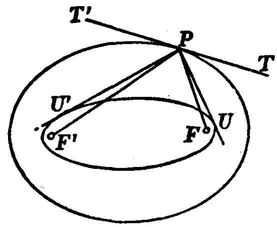



FIG. 93.

**10.** *The tangents from any point P on a conic to a confocal conic are equally inclined to the tangent at P.* For

$$\angle FPT=\angle F'PT',$$

$$\angle FPU=\angle F'PU';$$

therefore $\angle UPT=\angle U'PT'$.

**11. Graves' Theorem.** These tangential properties of confocal conics are analogous to the focal properties of a conic, where the focal lines through a point are replaced by the tangents to a confocal. It appears, in fact, that the ordinary focal properties are limiting cases of these more general theorems, since the two focal lines are just the pair of tangents to the confocal conic when it has degenerated to a point-pair. One of the most remarkable extensions of focal properties is the generalization of the method of generating an ellipse by a thread passing round the foci.

*A thread, with its ends joined, is passed round an ellipse and drawn tight by a pencil point at P; as P moves it will trace an ellipse confocal with the given ellipse.*

Let $P$, $P'$ be two positions of the tracing point very close to one another. Let $S$, $T$ and $S'$, $T'$ be the points of contact of the tangents from $P$, $P'$, and let $PS$ and $P'S'$ meet in $M$, $PT$ and $P'T'$ in $N$.

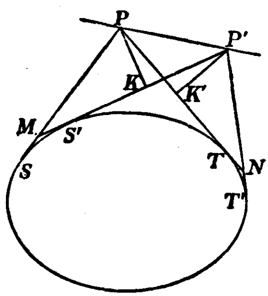



FIG. 94.

We have then the condition

$$SP+PT+\text{arc } TT'=\text{arc } SS'+S'P'+P'T'.$$

Now in the limit, as $P'$ approaches $P$,

$$\text{arc } SS'=2SM \quad\text{and}\quad \text{arc } TT'=2TN.$$

Draw $PK\perp S'P'$ and $P'K'\perp PT$. Then, ultimately, $MK=MP$ and $NP'=NK'$. We have then

$$(SM+MP)+(PK'+K'T)+(TN+NT')$$
$$=(SM+MS')+(S'K+KP')+(P'N+NT'),$$

*i.e.* $$MK+PK'+K'N=MK+KP'+K'N.$$

Hence $$PK'=P'K.$$

It follows, therefore, that $PP'$ is equally inclined to $PM$ and $PN$, and therefore equally inclined to $FP$ and $F'P$. $PP'$ is therefore the tangent to an ellipse with foci $F$, $F'$, and hence the locus of $P$ is this ellipse. A similar proof holds for the parabola and the hyperbola.

12. *The locus of the pole of a given straight line with regard to the conics of a confocal system is a straight line.* The pole of the line $lx+my+n=0$, with regard to the conic

$$\lambda(l^2+m^2)=a^2l^2+b^2m^2-n^2,$$

is $$x:y:z=(a^2-\lambda)l:(b^2-\lambda)m:-n.$$

Eliminating $\lambda$, we have

$$mx-ly=-(a^2-b^2)lm/n,$$

which represents a line at right angles to the given line.

In particular, if the given line is a tangent to one of the conics of the system, its pole with regard to that conic is the point of contact $P$, and the locus of the poles with regard to all the conics is the normal at $P$. Hence, if $S$ and $S'$ are confocal conics, and $t$ is a tangent to $S$ at $P$, the pole of $t$ with regard to $S'$ lies on the normal to $S$ at $P$. As there is always one conic of the system touching the given line, we can express the theorem thus: *The locus of poles of a given straight line with regard to the conics of a confocal system is the normal at the point of contact to the conic which touches the given straight line.*

If $l'$, $m'$, $n'$ are the coordinates of the line associated with $l$, $m$, $n$, we have $$l':m':n'=mn:-nl:(a^2-b^2)lm.$$

Hence $$-ll'=mm'=nn'/(a^2-b^2).$$

The relation between the two lines is therefore symmetrical. Hence we may say that *the tangents at a point P to the two conics of the system which pass through P have the property that each is the locus of poles of the other with respect to the conics of the system.*

Ex. From any point $Q$ on the tangent at $P$ to a conic tangents $QT$, $QT'$ are drawn to a confocal conic; prove that $TP$, $T'P$ are equally inclined to the tangent at $P$.

The pole of $PQ$ lies on the normal $PK$; it also lies on $TT'$, the polar of $Q$. Hence it is the point $K$.

Let $TT'$ meet $PQ$ on $L$. Then $(LK, TT')$ is harmonic, and $P(LK, TT')$ is harmonic. But $PK \perp PL$; therefore $PK$ and $PL$ are the bisectors of the angles between $PT$ and $PT'$.

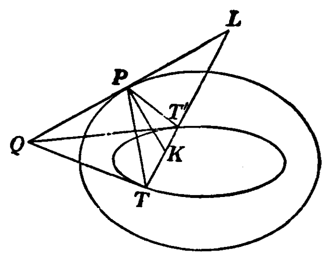



FIG. 95.

13. **Corresponding points.** Consider first two circles with centres $O$ and $O'$, and fix a diameter $AB$ and $A'B'$ on each. Then calling $A$ and $A'$ corresponding points, any other pair of points $P$ and $P'$ on the two circles are called *corresponding points*, when the angles $AOP$ and $A'O'P'$ are equal, the angles being measured in a definite sense round each circle.

If the circles are now distorted into ellipses with major axes $AB$ and $A'B'$ by diminishing the ordinates in given ratios, the new positions $Q$ and $Q'$ of $P$ and $P'$ are called corresponding points on the two ellipses. The points $Q$ and $Q'$ have then equal eccentric angles, and if the semi-axes are $a$, $b$ and $a'$, $b'$ and the coordinates of $Q$ and $Q'$ referred to the principal axes are $x$, $y$ and $x'$, $y'$, then

$$x/a = x'/a' \quad \text{and} \quad y/b = y'/b'.$$

Similarly corresponding points on two hyperbolas $x^2/a^2 - y^2/b^2 = 1$ and $x'^2/a'^2 - y'^2/b'^2 = 1$ are such that $x/a = x'/a'$ and $y/b = y'/b'$, and such points have the same parametric value when the equations are expressed in the form $x = a \cosh \psi$, $y = b \sinh \psi$.

And lastly, corresponding points on two parabolas $y^2 = 4ax$ and $y'^2 = 4a'x'$ (the axes of reference not necessarily being the same for the two curves) are such that $x/a = x'/a'$ and $y/a = y'/a'$. Such points have the same parametric value when the equations are expressed in the form $x = at^2$, $y = 2at$, or, if the origin is at the focus and the equation is

$$y^2 = 4\lambda^2(x + \lambda^2),$$

when the freedom-equations are $x = t^2 - \lambda^2$, $y = 2t\lambda$.

When the two conics are similar, so that their axes are in the same ratio, corresponding points are similarly situated on the two curves.

**14.** *A system of confocal ellipses and hyperbolas intersect one another in series of corresponding points.* The equations of an ellipse and a hyperbola of a confocal system are

$$\frac{x^2}{\cosh^2\psi} + \frac{y^2}{\sinh^2\psi} = c^2 \quad \text{and} \quad \frac{x^2}{\cos^2\varphi} - \frac{y^2}{\sin^2\varphi} = c^2.$$

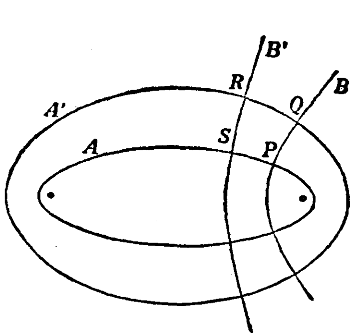


FIG. 96.

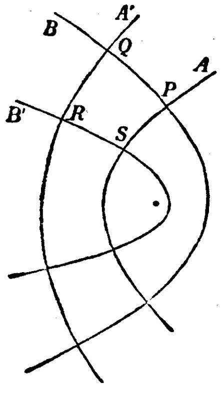


FIG. 97

Both these equations are satisfied by

$$x = c \cosh \psi \cos \varphi,$$
$$y = c \sinh \psi \sin \varphi.$$

Hence, keeping $\varphi$ constant and letting $\psi$ vary, the ellipses are cut by the fixed hyperbola in a series of points $P$, $Q$, ... with the constant eccentric angle $\varphi$; similarly, keeping $\psi$ constant and letting $\varphi$ vary, the hyperbolas are cut by the fixed ellipse in a series of points $P$, $S$, ... with the constant parameter $\psi$.

*A system of confocal parabolas intersect one another in a series of corresponding points.*

The equations of two intersecting parabolas of a confocal system are

$$y^2=4\psi^2(x+\psi^2) \quad \text{and} \quad y^2=-4\varphi^2(x-\varphi^2).$$

Both these equations are satisfied by

$$x=\varphi^2-\psi^2,$$
$$y=2\varphi\psi.$$

Hence, keeping $\varphi$ constant and letting $\psi$ vary, the first system of parabolas are cut by the fixed parabola of the second system in a series of points $P$, $Q$, ... with the constant parameter $\varphi$; and similarly for the other system.

**Examples.**

1. *The distance between two points, one on each of two confocal conics, is equal to the distance between their corresponding points.*

Taking the case of two ellipses $A$, $A'$ (Fig. 96), let their semi-axes be $a$, $b$ and $a'$, $b'$, so that $a^2-b^2=a'^2-b'^2$; and let the eccentric angles of the two points $P$, $R$ be $\varphi$, $\varphi'$.

Then $$PR^2=(a\cos\varphi-a'\cos\varphi')^2+(b\sin\varphi-b'\sin\varphi')^2$$

and $$QS^2=(a'\cos\varphi-a\cos\varphi')^2+(b'\sin\varphi-b\sin\varphi')^2$$

$$PR^2-QS^2=(a^2-a'^2)(\cos^2\varphi-\cos^2\varphi')+(b^2-b'^2)(\sin^2\varphi-\sin^2\varphi')$$
$$=(a^2-a'^2)(\cos^2\varphi-\cos^2\varphi'+\sin^2\varphi-\sin^2\varphi')=0.$$

Therefore $$PR=QS.$$

This proves the theorem also for the two hyperbolas $B$, $B'$, since $S$ and $Q$ are the points which correspond respectively to $P$ and $R$. The theorem may be proved in a similar way for two confocal parabolas.

A corresponding theorem to this in three dimensions is known as Ivory's Theorem.

2. Prove that if $\psi$, $\psi'$ and $\varphi$, $\varphi'$ are the parameters for the two ellipses and hyperbolas which intersect in $P$, $Q$, $R$, $S$,

$$PR^2=QS^2=c^2\{\cosh(\psi+\psi')-\cos(\varphi+\varphi')\}\{\cosh(\psi-\psi')-\cos(\varphi-\varphi')\}.$$

3. Prove similarly for confocal parabolas that

$$PR^2=QS^2=\{(\psi+\psi')^2+(\varphi+\varphi')^2\}\{(\psi-\psi')^2+(\varphi-\varphi')^2\}.$$

**15. Elliptic coordinates.** The position of any point $P$ can be determined by the parameters $\lambda$, $\mu$ of the two conics which pass through $P$ and belong to a given confocal system of ellipses and hyperbolas. This gives us a sort of coordinate system in which the two systems of lines of the coordinate network are confocal ellipses and hyperbolas. As these lines are curved and not straight lines as in cartesian coordinates, the coordinates are called *curvilinear coordinates.* Ordinary polar coordinates are also partly curvilinear. The particular system of curvilinear coordinates that we are to consider is called *elliptic coordinates*. (There is an analogous system based on confocal parabolas called *parabolic coordinates*.)

Take one particular conic of the system which we shall choose as an ellipse

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1;$$

then any conic of the confocal system is represented by

$$\frac{x^2}{a^2+\lambda}+\frac{y^2}{b^2+\lambda}=1.$$

To any point $(x, y)$ there corresponds one set of values of the parameters $\lambda$, $\mu$ of the conics which pass through this point. These are the roots of the equation

$$\lambda^2+\lambda(a^2+b^2-x^2-y^2)+a^2b^2-b^2x^2-a^2y^2=0.$$

Therefore

$$\left.\begin{aligned}\lambda+\mu&=-a^2-b^2+x^2+y^2,\\ \lambda\mu&=a^2b^2-b^2x^2-a^2y^2.\end{aligned}\right\}\quad\text{.........(1)}$$

From these we get

$$\left.\begin{aligned}(a^2-b^2)x^2&=\ \ (a^2+\lambda)(a^2+\mu),\\ (a^2-b^2)y^2&=-(b^2+\lambda)(b^2+\mu).\end{aligned}\right\}\quad\text{.........(2)}$$

To every set of values of $(\lambda, \mu)$ there correspond four points, the points of intersection of the two conics. Since $\lambda$, $\mu$ are given as the roots of a quadratic equation they are interchangeable, and the point $(x, y)$ is represented either by $(\lambda, \mu)$ or by $(\mu, \lambda)$.

Elliptic coordinates were devised at first by Lamé (about 1840), not for plane geometry but for the corresponding system of confocal ellipsoids and hyperboloids in space, and it is in three dimensions that they are most useful. They afford a convenient system of curvilinear coordinates for treating the geometry of curves on the surface of an ellipsoid analogous to the determination of a point on an ellipse by its eccentric angle, or a point on a sphere by its latitude and longitude, and are also of use in physical problems such as the attraction of an ellipsoid or the distribution of temperature within an ellipsoid.

## SIMILAR CONICS.

**16. Similar figures.** Two plane figures are said to be similar and similarly placed, or homothetic, when, corresponding to a point $O$ in the plane of the first figure, it is possible to find a point $O'$ in the plane of the second figure, such that radii $OP$ drawn from $O$ to the points of the first figure are proportional to parallel radii $O'P'$ drawn from $O'$ to points of the second figure. The ratio $OP : O'P'$ is called the *ratio of similitude*. A pair of points $P$, $P'$ or $O$, $O'$ are called *corresponding points*, and the lines $PQ$, $P'Q'$ joining any two points and their corresponding points are called *corresponding lines*.

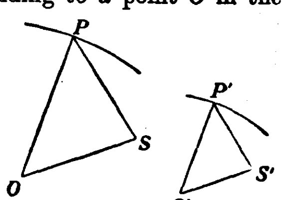


FIG. 98.

When one such pair of points $O$, $O'$ exists, there is an infinity of such pairs. For take any point $S$ in the plane of the first figure and draw $O'S' \parallel OS$, so that $\angle S'O'P' = \angle SOP$, and $O'S' : OS = O'P' : OP$. Then the triangles $OSP$ and $O'S'P'$ are similar, and $SP : S'P' = OP : O'P' =$ the ratio of similitude.

If corresponding radii are not parallel, but pairs of corresponding radii are proportional and include equal angles, the two figures are similar without being homothetic. A mere rotation of one of them will suffice to make them homothetic.

If one of two homothetic figures is rotated in its plane through two right angles, the figures are again homothetic, but in the one case corresponding radii are drawn in the same sense, and in the other case in opposite senses. In either case, since pairs of corresponding lines of the two figures are parallel, and therefore intersect on the straight line at infinity, two homothetic figures are in perspective. The centre of perspective, which is the point of concurrence of lines joining pairs of corresponding points $PP'$, $QQ'$, etc., is called the *homothetic centre.*

**17. Similar conics.** *If two conics are similar their centres are corresponding points.*

Since, by supposition, the conics are similar, there exists a pair of corresponding points $O$, $O'$ such that, if $POQ$ and $P'O'Q'$, $pOq$ and $p'O'q'$ are corresponding chords through $O$, $O'$,

$$OP : O'P' = OQ : O'Q' = Op : O'p' = Oq : O'q' = k.$$

Let $D$, $D'$, $d$, $d'$ be the lengths of the diameters parallel to these chords

Then $$D^2 : d^2 = OP \,.\, OQ : Op \,.\, Oq = k^2 \,.\, O'P' \,.\, O'Q' : k^2 \,.\, O'p' \,.\, O'q'$$
$$= O'P' \,.\, O'Q' : O'p' \,.\, O'q' = D'^2 : d'^2.$$

Hence corresponding diameters are proportional, and therefore the centres are corresponding points.

**18. Condition that two conics should be homothetic.** Let the equations of the two conics, referred to parallel axes through their centres, be

$$ax^2 + 2hxy + by^2 = 1,$$
$$a'x'^2 + 2h'x'y' + b'y'^2 = 1.$$

Transforming to polar coordinates, these equations become

$$r^2(a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta) = 1,$$
$$r^2(a'\cos^2\theta + 2h'\cos\theta\sin\theta + b'\sin^2\theta) = 1.$$

The ratio of the squares of parallel radii, corresponding to the angle $\theta$, is therefore

$$\frac{a'\cos^2\theta + 2h'\cos\theta\sin\theta + b'\sin^2\theta}{a\cos^2\theta + 2h\cos\theta\sin\theta + b\sin^2\theta}.$$

This ratio must be a constant, $k^2$, for all values of $\theta$; hence

$$\frac{a'}{a} = \frac{h'}{h} = \frac{b'}{b} = k^2.$$

These are therefore two necessary conditions that the conics should be homothetic.

**19.** A peculiarity arises when $k^2$ is a negative constant. In this case the equations of the two conics are of the form

$$ax^2+2hxy+by^2=1,$$
$$ax'^2+2hx'y'+by'^2=-c^2,$$

and the one conic is homothetic to the *conjugate* of the other conic. The ratio of similitude is in this case purely imaginary.

Ex. Deduce that two conics are homothetic when the terms of the second degree in their equations are the same.

**20.** *Two homothetic conics have their corresponding asymptotes parallel*, for the terms of the second degree in the two equations are the same, and these, equated to zero, represent lines parallel to the asymptotes. Hence two homothetic conics cut the line at infinity in the same two points. The equation of a conic passing through the points of intersection of the conic, $S=0$, with the line at infinity, $u=0$, is of the form

$$S-uv=0,$$

where $v=0$ represents any straight line. In cartesian coordinates $u$ does not contain $x$ or $y$; hence this equation has the terms of the second degree always the same.

Further, if the conics are concentric their asymptotes coincide, and since a conic touches its asymptotes at the points at infinity, we have to conceive of two similar, similarly situated, and concentric conics as having double contact at infinity. The equation of a system of such conics is

$$S-\lambda u^2=0.$$

**Examples.**

1. In areal coordinates the conic $S_1 \equiv yz+zx+xy=0$ is the minimum circumscribed ellipse of the triangle of reference, and the conic

$$S_2 \equiv x^2+y^2+z^2-2yz-2zx-2xy=0$$

is the maximum inscribed ellipse. Show that these conics are homothetic and concentric.

We have $u\equiv x+y+z$, $S_2\equiv u^2-4(yz+zx+xy)\equiv u^2-4S_1$; therefore, etc.

2. Prove that the conditions in areal coordinates that two general conics should be homothetic are

$$\frac{b'+c'-2f'}{b+c-2f}=\frac{c'+a'-2g'}{c+a-2g}=\frac{a'+b'-2h'}{a+b-2h},$$

and that the four conditions that they should be homothetic and concentric may be written

$$\left\Vert \begin{matrix} a & b & c & f & g & h \\ a' & b' & c' & f' & g' & h' \\ 1 & 1 & 1 & 1 & 1 & 1 \end{matrix} \right\Vert_3 =0.$$

**21. Condition that two conics should be similar.** If the conics

$$ax^2+2hxy+by^2+\ldots=0 \quad \text{and} \quad a'x^2+2h'xy+b'y^2+\ldots=0$$

are similar, but not similarly placed, we can rotate one of them so as to make them homothetic. Let the equation of the second conic after rotation be

$$a_1x^2+2h_1xy+b_1y^2+\ldots=0.$$

We are now to have $$\frac{a_1}{a}=\frac{h_1}{h}=\frac{b_1}{b}=\lambda, \quad\text{..........(1)}$$

but, the axes being rectangular,

$$a_1+b_1=a'+b',$$
$$a_1b_1-h_1^2=a'b'-h'^2;$$

and from (1),

$$a_1+b_1=\lambda(a+b),$$
$$a_1b_1-h_1^2=\lambda^2(ab-h^2).$$

Therefore $$\frac{a'+b'}{a+b}=\lambda=\frac{\sqrt{(a'b'-h'^2)}}{\sqrt{(ab-h^2)}},$$

*i.e.* $$\frac{a'+b'}{\sqrt{(a'b'-h'^2)}}=\frac{a+b}{\sqrt{(ab-h^2)}}. \quad\text{..........(2)}$$

The geometrical interpretation of this analytical condition is that the asymptotes contain equal (or supplementary) angles. Since the eccentricity of a conic is determined by an equation involving only the ratio $(ab-h^2)/(a+b)^2$ (see Chap. X. § 13), this condition is often stated loosely thus: *two conics are similar when they have the same eccentricity.* The complete statement, however, is: the necessary and sufficient condition that two conics should be similar is that the two eccentricities of the one should be equal to the two eccentricities of the other; in order that the two conics should be really of the same shape, and have a real ratio of similitude, it is necessary that the eccentricities of the real foci in each case should be equal.

*Cor.* 1. All parabolas are similar conics.

*Cor.* 2. All rectangular hyperbolas are similar conics.

## EXAMPLES XV.

1. Two mutually perpendicular straight lines are so related with respect to a conic that each passes through the pole of the other. Show that they are similarly related with respect to any confocal conic. (Trinity, 1913.)

2. Prove that the conics $x^2-y^2-4x+2y+2=0$ and $x^2+3y^2-4x-6y+4=0$ are confocal.

3. Show that the locus of the intersection of tangents to a variable ellipse of a confocal family at points having given eccentric angles is a hyperbola. (Math. Tripos II., 1913.)

4. Prove that the locus of the point of intersection of two perpendicular tangents, one to each of two confocal parabolas, is a straight line perpendicular to the axis.

5. Prove that the general equation of a conic confocal with the conic $S$ is $\Delta S+\lambda C\Phi+\lambda^2=0$, where $\Phi=0$ is the equation of the orthoptic circle in its normal form.

6. Show that the general equation of conics whose foci are the given points $(a, b)$ $(a', b')$ is

$$\{(x-a)(b-b')-(y-b)(a-a')\}^2+2\lambda\{(x-a)(x-a')+(y-b)(y-b')\}-\lambda^2=0.$$

7. If $\phi$ is the angle between the tangents to the conic $\lambda = k$ of the confocal system $x^2/(a^2+\lambda)+y^2/(b^2+\lambda)=1$ from the point whose elliptic coordinates referred to the ellipse $x^2/a^2+y^2/b^2=1$ are $\lambda$, $\mu$, prove that

$$\tan^2\phi = \frac{-4(\lambda-k)(\mu-k)}{(\lambda+\mu-2k)^2}$$

8. If $\lambda$, $\mu$ are the values of the parameters for the parabolas of the confocal system $y^2=4\lambda(x+\lambda)$ which pass through $P$, prove that the angle $\phi$ between the tangents from $P$ to the parabola $y^2=4k(x+k)$ is given by $\tan^2\phi = -(\lambda-k)(\mu-k)$.

9. A pair of tangents to any confocal of $x^2/a^2+y^2/b^2=1$ pass respectively through the fixed points $(0, c_1)$ and $(0, c_2)$; show that the intersection of the tangents lies on the circle

$$(x^2+y^2-a^2+b^2)(c_1+c_2)=2y(c_1c_2-a^2+b^2).$$ (Pembroke, 1913.)

10. Two conics are concentric and coaxial; a point is taken such that the tangents from it to one of the conics intersect the other conic in four points, of which one of the joining chords (other than the tangents themselves) is a diameter. Prove that the locus of the point is a concentric and coaxial conic, which when the two conics are confocal is also confocal with them. (Pembroke, 1900.)

11. The three sides of a varying triangle touch the parabola $y^2=4ax$, and two of the vertices lie on the confocal parabola $y^2=4(a+\lambda)(x+\lambda)$; prove that the third vertex lies on the confocal $y^2=4(a+\mu)(x+\mu)$, where $a\mu=4\lambda(a+\lambda)$. (Pembroke, 1913.)

12. If a focal chord of an ellipse, which is parallel to a semi-diameter $CP$, is equal to the transverse axis of the confocal hyperbola through $P$, prove that $P$ is one of four points on the ellipse, and the hyperbola cuts off from the focal chord a length equal to the transverse axis of the ellipse. (Selwyn, 1913.)

13. A variable line is such that the lengths $p_1$, $p_2$ of the perpendiculars on it from two fixed points satisfy the equation $\lambda(p_1^2+p_2^2)+2\mu p_1p_2=d^2$, where $\lambda$, $\mu$, $d$ are constants; find the envelope.

14. Given a triangle $ABC$, prove that there are four conics (one ellipse and three hyperbolas) which can be inscribed in the triangle and have the property that a confocal conic can be circumscribed about the triangle. Prove the following properties of this group of conics: (i) at each vertex the circumscribed ellipse and one of the hyperbolas touch the exterior bisector of the angle, and the other two hyperbolas touch the interior bisector; (ii) the points of contact of the inscribed conics are points of contact of the in- and e-scribed circles; (iii) the three normals to each of the four inscribed conics at its points of contact are concurrent.

15. An angle of constant size moves so that one arm passes through the focus and the other is a tangent to a fixed parabola. Find the locus of its vertex.

16. An angle of constant size moves so that one arm passes through a focus and the other is a tangent to a fixed ellipse. Prove that the locus of its vertex is a circle touching the ellipse at two points where the tangent is inclined at the given angle to the focal radius.

17. Tangents are drawn from a point $P$ to two confocal parabolas, and they form a constant angle $\alpha$. Show that the locus of $P$ is a hyperbola which has double contact with each of the two parabolas.

18. Prove that the centre of similitude of two confocal parabolas is the common focus.

19. Prove that two similar, similarly situated, and concentric conics intercept between them equal segments on any line which they cut.

20. Show that a tangent to the inner of two concentric homothetic ellipses and terminated by the outer is bisected at the point of contact.

21. Show that any tangent to the inner of two homothetic and concentric ellipses cuts the other in points whose eccentric angles have a constant difference.

22. Prove the theorem corresponding to the last example for the hyperbola.

23. Prove that any tangent to the inner of two concentric homothetic conics cuts off a constant area from the other.

24. Prove that the distance between two parallel tangents at corresponding points of two concentric homothetic ellipses is proportional to the perpendicular from the centre on the tangent to one of them.

25. Prove that the theorems of Exs. 19, 20 and 23 are true for two equal coaxial parabolas whose vertices are in the same direction. State the theorems corresponding to Exs. 21 and 24.

26. Given three conics, homothetic in pairs, prove that their six homothetic centres lie in sets of three on four straight lines.

27. Two variable conjugate semi-diameters of an ellipse $S$ cut a fixed concentric ellipse $S'$ in points $P, Q$; show that the envelope of $PQ$ is an ellipse homothetic and concentric with $S$, and find the condition that it may coincide with $S$. (Cf. Ex. X. 7.)

28. Two variable conjugate semi-diameters of an ellipse cut a fixed concentric circle in points $P, Q$; prove that the envelope of $PQ$ is a similar ellipse. (Selwyn, 1910.)

29. Prove that the locus of the Frégier point (see Chap. VIII. § 9) corresponding to a variable point on a fixed ellipse is a homothetic and concentric ellipse.

30. Distances $PQ$ are cut off on the inward drawn normals at points $P$ on an ellipse, equal to $k$ times the semi-diameter conjugate to the diameter through $P$. Find the locus of $Q$. Examine particularly the cases $k=\pm 1$, $k=2ab/(a^2+b^2)$.

31. Find the equation of the circle of which the chord of the ellipse

$$ax^2+by^2=1$$

intercepted on the line $lx+my=1$ is a diameter. Prove that this circle will touch the ellipse at a third point if the given chord touches a certain similar, similarly situated, and concentric ellipse. (Pembroke, 1910.)

32. An ellipse $S'$ of given area has contact of the third order (four-point contact) with a given ellipse $S$. Show that the locus of the centre of $S$ is an ellipse similar and similarly situated to $S$. (King's, 1912.)

33. A variable circle touches an ellipse, and the chord joining the other two points of intersection touches a similar coaxial ellipse. Prove that the locus of the centre of the variable circle is a coaxial ellipse. (Trinity, 1899.)

34. If the conic $ux^2+vy^2+wz^2=0$ in areal coordinates touches at a finite point the conic similar and similarly situated, but which passes through the angular points of the triangle of reference, show that $u+v+w=0$, and that the conics are hyperbolas. (Pembroke, 1901.)

# CHAPTER XVI.

## PENCILS AND RANGES OF CONICS.

**1. Pencil of conics; four-point system.** A conic is completely and uniquely determined by five points. If four points only are given the conic has still one degree of freedom, and an infinity of conics can be drawn to pass through the four points. This is analogous to a system of lines passing through one fixed point, and we call it a *pencil of conics.* It is also called a *four-point system of conics*, and the four fixed points are called *base-points.*

If $S_1$ and $S_2$ are any two conics of a pencil, they intersect in four points, and all the conics of the system pass through these four points. A pencil of conics is therefore completely determined by any two conics of the system.

The equation of any conic passing through the points of intersection of $S_1$ and $S_2$ is

$$S_1 + \lambda S_2 = 0.$$

This equation exhibits the conics of the pencil as depending *linearly* on a single variable parameter $\lambda$.

Let the four base-points be $A, B, C, D$. Then the pairs of lines $BC, AD$; $CA, BD$; $AB, CD$ can be regarded as degenerate conics of the system. If the equations of the lines $AB, BC, CD, DA$ are denoted by $\alpha=0$, $\beta=0$, $\gamma=0$, $\delta=0$, then two conics of the system are $S_1 \equiv \alpha\gamma = 0$ and $S_2 \equiv \beta\delta = 0$. Hence the equation of the pencil can be written

$$\alpha\gamma - \lambda\beta\delta = 0,$$

where $\lambda$ is the parameter on which the system linearly depends.

Ex. Find the equation of the pencil of conics through the four points $(\pm 1, \pm 1)$ in cartesian coordinates.

It is only necessary to find the equations of two distinct conics through these four points. The pairs of lines $x = \pm 1$ and $y = \pm 1$ both pass through all four points; hence we can take

$$S_1 \equiv (x-1)(x+1) \equiv x^2 - 1, \quad S_2 \equiv (y-1)(y+1) \equiv y^2 - 1,$$

and the equation of the system can be written

$$x^2 - 1 = \lambda(y^2 - 1).$$

**2.** *Through any given point there passes one conic of a given pencil,* for the given point together with the four base-points completely and uniquely determine the conic.

Otherwise: if the conic $S_1+\lambda S_2=0$ is to pass through a given point $(x', y')$, we have $S_1'+\lambda S_2'=0$, an equation of the first degree to determine $\lambda$. This property is therefore a result of the fact that the parameter $\lambda$ is involved linearly.

*There are two conics of a given pencil which touch any given line,* for the condition that a conic should touch a line is an equation of the second degree in the coefficients, and gives therefore an equation of the second degree in $\lambda$. If the line passes through one of the base-points the two conics will coincide, for the point of contact must coincide with the base-point, otherwise the line would meet the conic in more than two points. If the line passes through two of the base-points the conic will degenerate to this line and the line through the other two base-points.

**3. Harmonic triangle.** We shall find it most convenient to use homogeneous coordinates. We have seen that, by taking as triangle of reference the harmonic triangle of the complete quadrangle determined by four points, the trilinear coordinates of the four points become $(\pm p, \pm q, \pm r)$, and by taking more general homogeneous coordinates the coordinates can be reduced to the simple form $(\pm 1, \pm 1, \pm 1)$.

Now, if the conic $ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0$ passes through the four points $(\pm 1, \pm 1, \pm 1)$, we have

$$a+b+c+2f+2g+2h=0,$$
$$a+b+c+2f-2g-2h=0,$$
$$a+b+c-2f+2g-2h=0,$$
$$a+b+c-2f-2g+2h=0.$$

Subtracting each equation from the first, we get

$$f=0, \quad g=0, \quad h=0,$$

and $$a+b+c=0.$$

Hence the equation $$ax^2+by^2+cz^2=0,$$

with the condition $$a+b+c=0,$$

represents a pencil of conics through the four points $(\pm 1, \pm 1, \pm 1)$. The form of the equation shows that the triangle of reference is self-conjugate with regard to every conic of the system. Hence *all the conics of a pencil have a common self-conjugate or harmonic triangle.*

This is obvious also geometrically, since the harmonic triangle of the quadrangle $ABCD$ is self-conjugate with regard to any conic through $A, B, C, D$.

**4. Polar properties.** The polar of the point $(x', y', z')$ with regard to a conic of the pencil is $$ax'x+by'y+cz'z=0.$$

Let $(l, m, n)$ be the line-coordinates of this line, so that $l=ax'$, $m=by'$, $n=cz'$; then, since $a+b+c=0$,

$$\frac{l}{x'}+\frac{m}{y'}+\frac{n}{z'}=0.$$

Hence *the polars of a fixed point* $(x', y', z')$ *with regard to all the conics of a pencil pass through the fixed point* $(1/x', 1/y', 1/z')$. If $(l, m, n)$ is a fixed line, and $(x', y', z')$ a variable point, we can interpret the same equation as follows: *the poles of a fixed straight line with regard to all the conics of a pencil lie on a conic which circumscribes the common self-conjugate triangle.*

As a particular case of this we have the theorem of the following paragraph.

**5. Locus of centres of conics of a pencil.** Let the equation of the line at infinity be

$$l_0x + m_0y + n_0z = 0.$$

Then the coordinates of the centre of the conic

$$ax^2 + by^2 + cz^2 = 0$$

are $(l_0/a, m_0/b, n_0/c)$. But $a + b + c = 0$;

therefore the coordinates of the centre are connected by the equation

$$\frac{l_0}{x} + \frac{m_0}{y} + \frac{n_0}{z} = 0.$$

*The centre-locus is therefore a conic circumscribing the harmonic triangle of the pencil.*

**Ex.** Prove that the polars of a point on the centre-locus with regard to all the conics of a pencil are parallel.

**6. Nine-points conic.** If $I$, $I_1$, $I_2$, $I_3$ are the four base-points, the equation of $AI$ is $y = z$. Let $AI$ cut the centre-locus in $M$, and the line at infinity in $M'$. We have the following lines:

$BI \,: x - z = 0,$

$BI_1 : x + z = 0,$

$BM : (m_0 + n_0)x + l_0z = 0,$

$BM' : l_0x + (m_0 + n_0)z = 0.$

Hence the range $(II_1, MM')$ is harmonic, *i.e.* $M$ is the harmonic conjugate of the point at infinity with respect to $I$, $I_1$. Hence $M$ is the mid-point of $II_1$. Hence we have the theorem: *The locus of centres of conics of a pencil with base-points $I$, $I_1$, $I_2$, $I_3$ is a conic circumscribing the harmonic triangle of the quadrangle $II_1I_2I_3$, and also passing through the mid-points of the six segments $II_1$, $II_2$, $II_3$, $I_2I_3$, $I_3I_1$, $I_1I_2$.* The fact that the centre-locus passes through the points $A$, $B$, $C$ can also be shown in this way: the pairs of lines $II_1$, $I_2I_3$, etc., are degenerate conics of the system, and their centres are the points $A$, $B$, $C$.

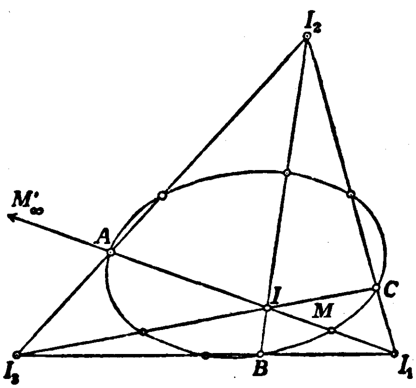


FIG. 99.

When the coordinates are trilinears, $I$, $I_1$, $I_2$, $I_3$ are the in- and ex-centres of the triangle $ABC$. They form an *orthocentric quadrangle*, *i.e.* each of the four points is the orthocentre of the triangle formed by the other three. The locus of the centres is the circle $ABC$, and we have the theorem of the nine-points circle.

**7.** *The asymptotes of any conic of a pencil are harmonic conjugates with regard to the asymptotes of the centre-locus, and are therefore parallel to a pair of conjugate diameters of the centre-locus.*

Consider the lines joining the vertex $C$ to the points of intersection of the line at infinity with the locus of the centres. We get the equation of the pair of lines by eliminating $z$, viz.

$$(l_0x+m_0y)(m_0x+l_0y)-n_0^2xy=0,$$

and similarly for any conic of the pencil we have

$$n_0^2(ax^2+by^2)+c(l_0x+m_0y)^2=0.$$

Applying the condition for a harmonic pencil we have

$$l_0m_0(n_0^2a+cl_0^2)+l_0m_0(n_0^2b+cm_0^2)$$
$$-(l_0^2+m_0^2-n_0^2)\,cl_0m_0=l_0m_0n_0^2(a+b+c)=0.$$

**8. Special conics of a pencil.** When the centre-locus is a circle its asymptotes pass through the circular points, and therefore the asymptotes of every conic of the pencil, being harmonic conjugates with regard to the lines through the circular points, are at right angles. Hence *when the base-points are orthocentric, all the conics of the pencil are rectangular hyperbolas.*

*A pencil of conics in general contains one rectangular hyperbola*, for its points at infinity are uniquely determined as the pair of points which are harmonic conjugates at the same time with regard to the circular points and the two points at infinity on the centre-locus. The asymptotes of this rectangular hyperbola are therefore parallel to rectangular conjugate diameters of the centre-locus, *i.e.* parallel to the axes of the centre-locus. If there are two rectangular hyperbolas in the system, these two pairs of points must coincide, and all the conics of the system are rectangular hyperbolas; the centre-locus is a circle, and the base-points are orthocentric.

*A pencil of conics in general contains two parabolas*, for the centre-locus has two points at infinity, real, coincident, or imaginary, according as the centre-locus is a hyperbola, a parabola, or an ellipse. The axes of the two parabolas are therefore parallel to the asymptotes of the centre-locus.

**9. Range of conics; four-line system.** A conic is completely and uniquely determined by five tangents, for the condition that a conic should touch a given line $(l, m, n)$ is expressed by the equation

$$Al^2+Bm^2+Cn^2+2Fmn+2Gnl+2Hlm=0,$$

which is an equation of the first degree in the coefficients $A$, $B$, etc., of the line-equation. Five such equations will determine the ratios $A:B:C:F:G:H$ uniquely.

An infinity of conics can be drawn to touch four given lines. If $\Sigma_1$ and $\Sigma_2$ are any two conics, expressed by their line-equations, they have four common tangents, whose line-coordinates are the solutions of the two simultaneous equations $\Sigma_1 = 0$, $\Sigma_2 = 0$. The equation

$$\Sigma_1 + \lambda\Sigma_2 = 0$$

represents a conic touching these four lines, and exhibits a system of conics the coefficients of whose line-equations depend linearly upon a single variable parameter $\lambda$. This is analogous to a range of points, the coefficients of whose line-equations depend also linearly upon a single variable parameter. A system of conics touching four fixed lines is therefore called a *range of conics*, or a *four-line system of conics*. We may also call it a *pencil of conic-envelopes* as distinguished from a *pencil of conic-loci*.

**10. Harmonic properties.** If the harmonic triangle of the complete quadrilateral formed by the four lines is taken as triangle of reference, the line-coordinates of the lines can be taken to be $(\pm 1, \pm 1, \pm 1)$, and the equation of the system of conics, in line-coordinates, is of the form

$$Al^2 + Bm^2 + Cn^2 = 0,$$

with
$$A + B + C = 0.$$

All the conics of the system have the triangle of reference as a self-conjugate triangle.

The pole of the line $(l', m', n')$ with regard to a conic of the system is

$$Al'l + Bm'm + Cn'n = 0.$$

The coordinates of the pole being $x = Al'$, $y = Bm'$, $z = Cn'$, the locus of the poles of the line $(l', m', n')$ with regard to all the conics of the range is the line

$$\frac{x}{l'} + \frac{y}{m'} + \frac{z}{n'} = 0.$$

Again, if $(x', y', z')$ is a fixed point, the envelope of its polar with regard to the conics is the conic

$$\frac{x'}{l} + \frac{y'}{m} + \frac{z'}{n} = 0,$$

which touches the sides of the harmonic triangle.

In particular, therefore, taking the line at infinity $l_0x + m_0y + n_0z = 0$ as the fixed line, *the locus of the centres of all the conics of the range is a straight line*, $x/l_0 + y/m_0 + z/n_0 = 0$.

If $A, A'$; $B, B'$; $C, C'$ are the pairs of opposite vertices of the complete quadrilateral formed by the base-lines, these pairs of points are degenerate conics of the system, and their centres are the mid-points of the diagonals $AA'$, $BB'$, $CC'$. Hence we have proved the theorem that *the mid-points of the diagonals of a complete quadrilateral are collinear*, and the line on which they lie is the locus of centres of conics touching the four sides.

**11.** *There is one conic of a given range which touches any given straight line*, for the given line together with the four base-lines completely and uniquely determine the conic.

*There are two conics of a given range which pass through any given point*; these will coincide if the given point lies on one of the base-lines, and the base-line is then the tangent at the point. If the given point is the point of intersection of two of the base-lines, the conic will degenerate into two pencils, the vertices being at the given point and the intersection of the other two base-lines.

As a particular case, a range of conics contains in general one parabola, viz. that conic of the system which touches the line at infinity. If the line at infinity is one of the base-lines, all the conics of the range are parabolas. The centre of the parabola of a range is the point at infinity on the locus of centres; hence the axis of the parabola is parallel to the line of centres.

## EXAMPLES XVI.

1. Prove that one rectangular hyperbola, and only one, touches a given conic at the ends of any chord; and that, if the rectangular hyperbola passes through a fixed point $P$, the chord must touch a fixed circle with centre $P$. (Pembroke, 1909.)

2. Find the locus of centres of a system of conics having four-point contact.

3. Show that in the case of a family of conics having three-point contact at $P$ and passing through a fourth point $Q$, the locus of centres touches the conics at $P$, has curvature at $P$ of the opposite sign and of double the magnitude of that of the conics, and has $PR$ as a diameter, where $R$ is the middle point of $PQ$. (Pembroke, 1912.)

4. Prove that the equation of a circle which touches the parabola $y^2=4ax$ and passes through its focus may be written

$$(1+t^2)(y^2-4ax)+(x-ty+at^2)(x+ty+3a)=0.$$

(Peterhouse, 1901.)

5. What is the nature of the curve $5(y^2-4ax)+(x-2y-a)(x+2y+3a)=0$? Show that it has a focal chord of the parabola $y^2=4ax$ as diameter, and find where it meets the parabola again. (Peterhouse, 1913.)

6. Prove that two circles can be drawn through the origin, each having double contact with the conic $(x^2-y^2)\cos 2\alpha+2xy\sin 2\alpha+2=0$, and find the equations of their chords of contact. (Peterhouse, 1914.)

7. If a system of conics is drawn having four-point contact with the conic $ax^2+2hxy+by^2+2fy=0$ at the origin, prove that the orthoptic circles of these conics form a coaxal system whose limiting points are the origin and the point $-hf/(a^2+h^2),\ af/(a^2+h^2)$, and whose radical axis is $hx-ay+\frac{1}{2}f=0$.

8. A conic has three fixed pairs of conjugate lines. Show that its orthoptic circle cuts a fixed circle at right angles. (Math. Tripos II., 1915.)

9. Determine the different kinds of conics represented by the equation $x^2+4\lambda xy+4y^2+2(1+\lambda)x+8y+5+2\lambda=0$, as $\lambda$ changes from a large positive value to a large negative value. Examine in particular the critical cases $\lambda=1, 0, -1, -2$, and illustrate by rough diagrams the transition from one kind of conic to another. (Pembroke, 1911.)

10. Draw diagrams of the following systems of conics. Determine the critical values of $\lambda$ for which the conic becomes two straight lines or a parabola, and state the character of the conic in the various intervals:

(i) $16x^2+(\lambda-1)y^2-16(\lambda+6)x+24y=0$,

(ii) $4x^2+\lambda y^2+4xy+4(9-\lambda)x+16y+80=0$,

(iii) $x^2+2(1+\lambda)xy+y^2-4(1+7\lambda)x+4(1+\lambda)y-56\lambda=0$.

11. Show that the conics whose axes are in a given direction, and which pass through three fixed points, pass through another fixed point. (Math. Tripos I., 1909.)

12. When one of the base-points of a pencil of conics is at infinity, so that there is one parabola in the system and the centre-locus is a parabola, show that the former parabola is double the size of the latter.

13. A system of conics is drawn passing through two fixed points and touching two fixed lines. Show that their chords of contact with these lines will pass through one of two fixed points. (Trinity, 1899.)

14. A conic touches the sides of a parallelogram; show that its foci lie on the rectangular hyperbola through the corners. (Math. Tripos II., 1915.)

15. Prove that the equation of the family of conics inscribed in the rectangle formed by the lines $x\pm a=0$, $y\pm b=0$ is $x^2/a^2+y^2/b^2-1+2\lambda xy/ab+\lambda^2=0$. Prove also that the locus of the foci is $x^2-y^2=a^2-b^2$, and that if two of the conics intersect on this latter locus they do so at right angles. (Pembroke, 1899.)

16. Prove that the axes of the parabolas which have a common focus and pass through two given points are parallel to the asymptotes of the hyperbola which passes through the common focus and has the two given points for foci. (Corpus, 1913.)

17. Show that Ex. 14 is equivalent to the following: Two complete quadrilaterals with a common diagonal are circumscribed about a conic; prove that the eight vertices of the quadrilaterals, other than the vertices $A$, $B$ and $A'$, $B'$ which lie on the common diagonals, lie on a conic which is apolar to both $A$, $B$ and $A'$, $B'$.

18. $ABCD$ is a complete quadrangle, and $AD$, $BD$, $CD$ meet $BC$, $CA$, $AB$ respectively in $X$, $Y$, $Z$. If $P$ is any other point, prove that the six conics $(PYZAD)$, $(PYZBC)$, $(PZXBD)$, $(PZXCA)$, $(PXYCD)$, $(PXYAB)$ have a second common point. (Pembroke, 1909.)

19. If the focus of a conic be given, and if the asymptotes pass each through a fixed point on a straight line through the focus, prove that the locus of the centre is a circle. (Corpus, 1913.)

20. If three conics have a pair of common tangents, show that the points of intersection of the three other pairs of common tangents are collinear. (Trinity, 1899.)

21. $BOB'$, $COC'$ are two chords of a conic, intersecting at $O$; $AOA'$ is a third chord through $O$, and a second conic is drawn touching the first at $A$ and $A'$. If $CB$ meet the second conic in $H$, $K$, and $C'B'$ meet the second conic in $H'$, $K'$, prove that $HOH'$ (or $HOK'$) is a straight line. (Math. Tripos II., 1914.)

22. Show that by choosing suitable coordinates any pencil of conic-envelopes can be represented by the equation $x^2/(a+\lambda)+y^2/(b+\lambda)+z^2/(c+\lambda)=0$, where $\lambda$ is the variable parameter.

23. If the centre-locus of a pencil of conics is a rectangular hyperbola, show that the system contains a circle.

# CHAPTER XVII.

## PARAMETRIC REPRESENTATION.

**1. Rational, algebraic freedom-equations.** We have frequently had occasion to represent a locus by freedom-equations, which express the coordinates of any point on the locus in terms of a variable parameter. Thus, in rectangular coordinates the straight line was represented by the freedom-equations

$$x = a + pt,$$
$$y = b + qt.$$

$(a, b)$ is one point of the line and $q/p$ is the gradient.

The ellipse $x^2/a^2 + y^2/b^2 = 1$ was represented by the freedom-equations

$$x = a \cos \varphi,$$
$$y = b \sin \varphi,$$

the eccentric angle $\varphi$ being taken as parameter.

The parabola $y^2 = 4ax$, on the other hand, was represented by equations of quite a different form,

$$x = at^2,$$
$$y = 2at,$$

in which the parameter $t$ is involved algebraically.

There is a great advantage in choosing the parameter, if possible, so that the equations involve it algebraically and rationally.

In the above freedom-equations of the ellipse the parameter $\varphi$ is involved not algebraically, but trigonometrically. If we write $\cos \varphi = t$, we get the algebraic equations

$$x = at, \quad y = b\sqrt{1 - t^2};$$

but these do not involve the parameter *rationally*. If, however, we put $\tan \frac{1}{2}\varphi = t$, the equations become

$$x = a\frac{1 - t^2}{1 + t^2}, \quad y = b\frac{2t}{1 + t^2};$$

and now the parameter is involved both algebraically and rationally. In a similar way we may reduce the freedom-equations of the hyperbola to the form

$$x = a\frac{1 + t^2}{1 - t^2}, \quad y = b\frac{2t}{1 - t^2}.$$

**2. Algebraic, rational, and integral freedom-equations in homogeneous coordinates.** There is still a fundamental point of distinction between these freedom-equations of the ellipse and the hyperbola and those of the parabola, for in the latter the parameter is involved *integrally*. It is not in general possible, with cartesian coordinates, to choose the parameter so that the freedom-equations of a conic involve the parameter algebraically, rationally *and* integrally, but we can do so if we use homogeneous coordinates. Introduce the third variable $z$ to make a system of homogeneous cartesian coordinates. Then the freedom-equations of the ellipse will be

$$x=a\frac{1-t^2}{1+t^2},\quad y=b\frac{2t}{1+t^2},\quad z=1.$$

Now, since we are dealing with homogeneous equations, it is only necessary to consider the ratios of the coordinates. We may therefore multiply each by any common factor, and thus we may write

$$x=a(1-t^2),$$
$$y=2bt,$$
$$z=1+t^2.$$

These are more correctly written

$$\lambda x=a(1-t^2),$$
$$\lambda y=2bt,$$
$$\lambda z=1+t^2,$$

where $\lambda$ is any multiple different from zero, but we shall usually omit the $\lambda$, it being understood that the equations imply that $x$, $y$, $z$ are only proportional to the expressions on the right.

3. *Any conic can be represented by rational freedom-equations.*

We see then that any of the three types of conics can be represented by algebraic, rational, and integral freedom-equations of the second degree. This can be shown also for any conic as follows. If we take as triangle of reference two tangents and the chord of contact, the equation of any conic can be written in the form

$$xy=kz^2.$$

Write this

$$\frac{y}{kz}=\frac{z}{x}=t.$$

Then we can put

$$x=\lambda,\quad z=\lambda t,\quad y=\lambda kt^2,\qquad \text{or simply}\quad x=1,\quad y=kt^2,\quad z=t.$$

4. *Freedom-equations of the second degree represent a conic.*

Conversely the equations

$$x=a_2t^2+2a_1t+a_0,$$
$$y=b_2t^2+2b_1t+b_0,$$
$$z=c_2t^2+2c_1t+c_0$$

in general represent a conic.

For the intersections of any straight line $lx+my+nz=0$ with the curve are found from the equation

$$(la_2+mb_2+nc_2)t^2+2(la_1+mb_1+nc_1)t+(la_0+mb_0+nc_0)=0.$$

This is a quadratic equation giving two values for $t$, and therefore, in general, two points of intersection.

In particular the parameters of the points in which the curve cuts the sides of the triangle of reference are found by putting $x=0$, $y=0$, $z=0$.

**Examples.**

1. Find the position of the conic

$$x=t^2,$$
$$y=t^2-1,$$
$$z=t(t+1)$$

with respect to the triangle of reference.

$x=0$ cuts the conic in two coincident points $t=0$, and is therefore a tangent. $z=0$ cuts the conic at the same point $t=0$, and also at $t=-1$. $y=0$ cuts the conic at $t=-1$ and $t=1$. Hence the conic touches $BC$ at $B$ and passes through $A$.

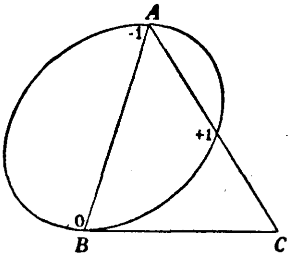


FIG. 100.

2. Find the freedom-equations of a conic circumscribing the triangle of reference.

Let the values of the parameters at $A$, $B$, $C$ be $a$, $b$, $c$; then we have the equations

$$x=p(t-b)(t-c),$$
$$y=q(t-c)(t-a),$$
$$z=r(t-a)(t-b).$$

**5. Condition for a parabola, ellipse, or hyperbola.** If the coordinates are homogeneous cartesians, $z=0$ is the equation of the line at infinity. The conic will therefore be an ellipse, parabola, or hyperbola according as

$$c_1{}^2-c_0c_2 \lesseqgtr 0,$$

for the roots of the equation $c_2t^2+2c_1t+c_0=0$ are the parameters of the points at infinity on the conic.

In particular the freedom-equations

$$x=a_2t^2+2a_1t+a_0,$$
$$y=b_2t^2+2b_1t+b_0,$$

with $z=1$, always represent a parabola. For $z=0$, as a quadratic in $t$, has two roots infinite. The two points at infinity therefore coincide, and $t=\infty$ is the point at infinity.

**Ex.** Find the asymptotes of the hyperbola

$$x=\frac{t^2+t-6}{t^2-1}, \quad y=\frac{t^2+2t}{t^2-1}.$$

The points at infinity are $t=\pm 1$. Let $lx+my+n=0$ be an asymptote; then this cuts the curve in the two coincident points $t=1$ or $t=-1$. We have as the quadratic for $t$,

$$l(t^2+t-6)+m(t^2+2t)+n(t^2-1)=0,$$

*i.e.*

$$(l+m+n)t^2+(l+2m)t-(6l+n)=0.$$

In order that this equation may have the equal roots $t=1$ or $t=-1$, we must have

$$l+m+n=-6l-n,$$

and

$$l+2m=\pm 2(6l+n),$$

whence

$$l:m:n=6:8:-25 \quad \text{or} \quad 2:-12:-1.$$

Therefore the asymptotes are
$$6x+8y=25,$$
$$2x-12y=1.$$

**6. Formation of freedom-equations of a conic.** If the equation of a conic can be written in the form $\alpha\beta=\gamma\delta$, where $\alpha$, $\beta$, $\gamma$, $\delta$ denote expressions of the first degree, we may form freedom-equations by putting
$$\alpha=\gamma t, \quad \beta t=\delta.$$
$x$ and $y$ may then be expressed in terms of $t$.

Ex. $3x^2+2xy-y^2+4x-6y+5=0$.

The terms of the second degree can be factorized, and we have
$$(3x-y)(x+y)+4x-6y+5=0$$
Put
$$3x-y=t$$
$$-t(x+y)=4x-6y+5,$$
whence
$$(4t-14)x=\quad t^2-6t-5,$$
$$(4t-14)y=-t^2-4t-15,$$
or, using homogeneous coordinates,
$$x=\quad t^2-6t-5,$$
$$y=-t^2-4t-15,$$
$$z=\quad 4t-14.$$
In this case the curve is a hyperbola, and the points at infinity are determined by $t=\infty$ and $t=\frac{7}{2}$.

**7. Parametric line-equations.** Just as the point-coordinates $x$, $y$, $z$ may be expressed in terms of a parameter, so also the line-coordinates $l$, $m$, $n$ may be parametrically expressed.

Thus the equations
$$l=a_2t^2+2a_1t+a_0,$$
$$m=b_2t^2+2b_1t+b_0,$$
$$n=c_2t^2+2c_1t+c_0$$
represent an envelope of the second class, for the pencil $xl+ym+zn=0$ through the point $x, y, z$ has two lines in common with the envelope. These are therefore the general tangential freedom-equations of a conic.

Ex. Determine how the conic
$$l=t(t-1),$$
$$m=(t-1)^2,$$
$$n=t(t-2)$$
is situated with respect to the triangle of reference.

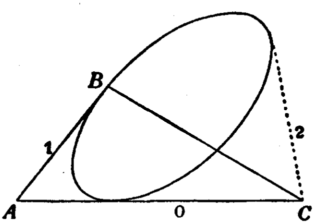


FIG. 101.

When $l=0$ we have $t=0$ or $t=1$, and therefore there are two real and distinct tangents from $A$ to the curve. When $m=0$ we have $t=1$ (twice); therefore the tangents from $B$ both coincide with one of the tangents from $A$, and therefore with $AB$, and $B$ lies on the curve. When $n=0$, $t=0$ or $t=2$; therefore one of the tangents from $C$ coincides with the other tangent from $A$, i.e. with $AC$.

**8. Line-coordinates of the tangent.** The parametric line-equations of a conic can be found from the point freedom-equations by obtaining the line-coordinates of the tangent at a point $t$. We shall first obtain the equation of the chord joining the points $t$ and $t'$. We can write down the equation in the form of a determinant

$$\begin{vmatrix} x & a_2t^2+2a_1t+a_0 & a_2t'^2+2a_1t'+a_0 \\ y & b_2t^2+2b_1t+b_0 & b_2t'^2+2b_1t'+b_0 \\ z & c_2t^2+2c_1t+c_0 & c_2t'^2+2c_1t'+c_0 \end{vmatrix}=0. \quad \text{.............(1)}$$

On expanding this, the factor $t-t'$ cuts out and the coefficients of $x, y, z$ are

$$\left.\begin{aligned} l&=A_0tt'-A_1(t+t')+A_2, \\ m&=B_0tt'-B_1(t+t')+B_2, \\ n&=C_0tt'-C_1(t+t')+C_2, \end{aligned}\right\} \quad \text{.............(2)}$$

where the capital letters denote the cofactors of the corresponding small letters in the determinant

$$\begin{vmatrix} a_0 & 2a_1 & a_2 \\ b_0 & 2b_1 & b_2 \\ c_0 & 2c_1 & c_2 \end{vmatrix}.$$

Now put $t'=t$, and we get the line-coordinates of the tangent at the point $t$,

$$\left.\begin{aligned} l&=A_0t^2-2A_1t+A_2, \\ m&=B_0t^2-2B_1t+B_2, \\ n&=C_0t^2-2C_1t+C_2. \end{aligned}\right\} \quad \text{.............(3)}$$

These are therefore the parametric line-equations of the conic.

**9. Pole and polar in terms of parameters.** In the last paragraph the equations (2) give the line-coordinates of the line joining the points at which the tangents have parameters $t$ and $t'$ on the conic whose parametric line-equations are given by (3), *i.e.* (2) represents the *polar of the point of intersection of the lines whose parameters are $t$ and $t'$.*

In a similar way it can be proved that the coordinates of the *pole of the line joining the points $t$ and $t'$* can be written down from the point freedom-equations by changing $t^2$ into $tt'$, and $2t$ into $t+t'$, thus

$$\left.\begin{aligned} x&=a_2tt'+a_1(t+t')+a_0, \\ y&=b_2tt'+b_1(t+t')+b_0, \\ z&=c_2tt'+c_1(t+t')+c_0. \end{aligned}\right\} \quad \text{.............(4)}$$

**10. Equation of the tangent at t.** Taking the equation (1) of § 8, which represents the chord $tt'$, subtract the second column from the last and cancel $t'-t$; then put $t'=t$ and cancel the factor 2; then multiply the last column by $t$ and subtract from the second. Then we get as the equation of the tangent at $t$,

$$\begin{vmatrix} x & a_2t+a_1 & a_1t+a_0 \\ y & b_2t+b_1 & b_1t+b_0 \\ z & c_2t+c_1 & c_1t+c_0 \end{vmatrix}=0. \quad \text{.............(5)}$$

This equation may be written in a very convenient and general form as follows. In the point freedom-equations introduce another parameter $t'$ to make the quadratic expressions homogeneous, thus

$$x = a_2t^2 + 2a_1tt' + a_0t'^2, \text{ etc.};$$

then we have $\quad \frac{1}{2}\frac{\partial x}{\partial t} = a_2t + a_1t', \qquad \frac{1}{2}\frac{\partial x}{\partial t'} = a_1t + a_0t'$, etc.

Hence the equation of the tangent at $t$ can be written

$$\begin{vmatrix} x & y & z \\ \frac{\partial x}{\partial t} & \frac{\partial y}{\partial t} & \frac{\partial z}{\partial t} \\ \frac{\partial x}{\partial t'} & \frac{\partial y}{\partial t'} & \frac{\partial z}{\partial t'} \end{vmatrix} = 0. \quad \text{.........................................(6)}$$

It can be proved that this holds generally for freedom-equations of any degree.

**11. Centre and asymptotes.** Let the coordinates be cartesian, so that $z=0$ is the equation of the line at infinity. Then the parameters $t_1$, $t_2$ of the two points at infinity on the curve are the roots of the equation

$$c_2t^2 + 2c_1t + c_0 = 0.$$

The centre is the pole of the chord $t_1$, $t_2$, and its coordinates are therefore

$$x = a_2t_1t_2 + a_1(t_1 + t_2) + a_0$$
$$= (a_2c_0 - 2a_1c_1 + a_0c_2)/c_2, \text{ etc.}$$

The asymptotes are the tangents at $t_1$ and $t_2$. Their separate equations may therefore be written down by equations (6) of § 10.

Ex. Find the coordinates of the centre and the equations of the asymptotes of the conic whose freedom-equations in homogeneous cartesian coordinates are

$$x = t^2 + 2t - 3, \quad y = t^2 + 3t, \quad z = t^2 - 1.$$

The parameters of the points at infinity are $\pm 1$. The pole of the chord joining $t=1$, $t'=-1$ is

$$x = tt' + (t+t') - 3 = -4, \quad y = tt' + \tfrac{3}{2}(t+t') = -1, \quad z = tt' - 1 = -2.$$

Hence the coordinates of the centre are $(2, \frac{1}{2})$.

The tangent at $t$ is $\quad \begin{vmatrix} x & 2t+2 & 2t-6 \\ y & 2t+3 & 3t \\ z & 2t & -2 \end{vmatrix} = 0.$

Putting $t=1$, $t=-1$, successively, we find the equations of the two asymptotes $x=2$ and $x-2y=1$.

**12. Foci of a conic.** If the point $F \equiv (\alpha, \beta)$ is a focus of the conic whose rectangular cartesian freedom-equations are

$$x = a_2t^2 + 2a_1t + a_0, \text{ etc.},$$

then the lines $FI$, $FJ$ joining $F$ to the circular points are tangents to the conic. The equation of $FI$ is

$$(x - \alpha z) + i(y - \beta z) = 0.$$

Substitute the values of $x$, $y$, $z$ in terms of $t$, and express the condition for equal roots in the resulting quadratic equation. This will give one equation connecting $\alpha$ and $\beta$. Treating $FJ$ similarly, we get another equation which will only differ in the sign of $i$. These two equations, which are equivalent to the two which we get by equating to zero the real and imaginary parts of one of them, determine the values of $\alpha$ and $\beta$.

Ex. Find the foci of the ellipse

$$x=2t^2-1, \quad y=2t, \quad z=t^2+1.$$

Let $F=(\alpha, \beta)$ be a focus. Then the equation of $FI$ is

$$(x-\alpha z)+i(y-\beta z)=0.$$

Substituting for $x$ and $y$, we have the quadratic equation

$$t^2(2-\alpha-i\beta)+2it-(1+\alpha+i\beta)=0.$$

$FI$ will be a tangent if this equation has equal roots; the condition for this is

$$1=2+(\alpha+i\beta)-(\alpha+i\beta)^2,$$

*i.e.* $$\alpha^2-\beta^2-\alpha-1+i(2\alpha\beta-\beta)=0.$$

Similarly $FJ$ will be a tangent if

$$\alpha^2-\beta^2-\alpha-1-i(2\alpha\beta-\beta)=0.$$

Hence $$\alpha^2-\beta^2-\alpha=1,$$

and $$2\alpha\beta-\beta=0.$$

The solutions are $\alpha=\frac{1}{2}(1\pm\sqrt{5})$, $\beta=0$ for the real foci,

$\alpha=\frac{1}{2}$, $\beta=\pm\frac{1}{2}i\sqrt{5}$ for the imaginary foci.

**13. Indeterminateness of the freedom-equations.** We have seen that the freedom-equations of a given straight line are not unique. Thus

$$\left.\begin{array}{l} x=1-3t \\ y=2+t \end{array}\right\} \quad \text{and} \quad \left.\begin{array}{l} x=-2+6u \\ y=\ \ 3-2u \end{array}\right\}$$

represent the same straight line $x+3y=7$, the parameters $t$ and $u$ being connected by the relation $2u+t=1$.

The reason for the indeterminateness of the freedom-equations in cartesian coordinates is that the *origin* and *scale* of measurement of the parameter are both quite arbitrary. Thus the general freedom-equations of a straight line in cartesian coordinates are

$$x=a+pt,$$
$$y=b+qt.$$

Here we have four constants, $a$, $b$, $p$, $q$, but only two are required to fix the line. Instead of $a$, $b$, we may put the coordinates of any other point on the line, say $a+pt_0$, $b+qt_0$, and we get equivalent freedom-equations

$$x=a+pt_0+pt,$$
$$y=b+qt_0+qt.$$

We might choose $t_0$ so that $a+pt_0=0$, and then one of the four constants disappears. This corresponds to a change of *origin* of $t$. Again, if we put $ku$ instead of $t$, we get the equations

$$x=pku,$$
$$y=b'+qku,$$

and we may choose $k$ so that $pk=1$. The equations then become simply

$$x=u,$$
$$y=b'+\mu u,$$

which involve only the two constants $b'$ and $\mu$. This corresponds to a change of *scale* of $t$.

**14. Relation between the parameters in equivalent representations.** With homogeneous coordinates the general freedom-equations of a straight line are

$$x=a_1t+a_0,$$
$$y=b_1t+b_0,$$
$$z=c_1t+c_0.$$

Here we have six constants, but as only the ratios $x:y:z$ are significant, we need only the five ratios of the constants. We are still free to choose three relations between the constants, since only two independent constants are required, *i.e.* we can change the parameter so that *the parameters of any three given points on the line may have assigned values.* Let the points which have the parameters $t_1$, $t_2$, $t_3$ have now the assigned values $u_1$, $u_2$, $u_3$, and let the freedom-equations become

$$x=p_1u+p_0,$$
$$y=q_1u+q_0,$$
$$z=r_1u+r_0.$$

Then
$$\frac{a_1t_1+a_0}{p_1u_1+p_0}=\frac{b_1t_1+b_0}{q_1u_1+q_0}=\frac{c_1t_1+c_0}{r_1u_1+r_0}, \qquad (1)$$

and two similar sets of equations with $t_2$, $u_2$ and $t_3$, $u_3$. These six homogeneous equations of the first degree in $p_1$, $p_0$, etc., determine the ratios of the new constants uniquely in terms of the ratios of $a_1$, $a_0$, etc. They are equivalent to five equations.

The parameters $t$ and $u$ are now connected by a certain relation which is found by equating any two of the ratios in (1). Thus we have

$$(a_1q_1-b_1p_1)tu+(a_1q_0-b_1p_0)t+(a_0q_1-b_0p_1)u+(a_0q_0-b_0p_0)=0.$$

This relation is of the form $atu-bt+cu-d=0$, ..................(2)

and is linear in both $t$ and $u$; it is therefore called a *lineo-linear* relation. Solving for $u$, we get

$$u=\frac{bt+d}{at+c}. \qquad (3)$$

The characteristic property of this relation, in either the form (2) or the form (3), is that to each value of $t$ corresponds a single value of $u$, and *vice versa,* and the values of $t$ and $u$ are said to be in one-to-one correspondence. (1, 1) correspondences are of very great importance, and will be considered in detail in the following chapter.

**15.** *Any three points on a conic may have arbitrarily assigned parametric values.*

Let $A$, $B$, $C$ be the three points on the conic, and $t_1$, $t_2$, $t_3$ their parameters in any given parametric representation. We have to show that the same conic can be represented in terms of a parameter $u$, so that the points $A$, $B$, $C$, have any arbitrarily assigned parametric values $u_1$, $u_2$, $u_3$.

Take $ABC$ as triangle of reference. The parametric equations of the conic are then

$$\left.\begin{aligned} x &= a(t-t_2)(t-t_3),\\ y &= b(t-t_3)(t-t_1),\\ z &= c(t-t_1)(t-t_2)\end{aligned}\right\}\quad\text{and}\quad\left.\begin{aligned} x &= p(u-u_2)(u-u_3),\\ y &= q(u-u_3)(u-u_1),\\ z &= r(u-u_1)(u-u_2),\end{aligned}\right\}$$

where $a$, $b$, $c$ and $p$, $q$, $r$ have certain definite values.

Now, for any value of $t$ and the corresponding value of $u$ the ratios $x : y : z$ must be the same; hence

$$\frac{b}{c}\,\frac{t-t_3}{t-t_2}=\frac{q}{r}\,\frac{u-u_3}{u-u_2}.$$

Put $t=t_1$ and $u=u_1$, the corresponding value; then

$$\frac{b}{c}\,\frac{t_1-t_3}{t_1-t_2}=\frac{q}{r}\,\frac{u_1-u_3}{u_1-u_2}.$$

Hence, dividing corresponding sides of these two equations, we have

$$\frac{t-t_2}{t-t_3}\Big/\frac{t_1-t_2}{t_1-t_3}=\frac{u-u_2}{u-u_3}\Big/\frac{u_1-u_2}{u_1-u_3}. \quad\text{.............................(1)}$$

Multiplying up and collecting terms, we get an equation which can be written in the determinant form

$$\begin{vmatrix} tu & t & u & 1\\ t_1u_1 & t_1 & u_1 & 1\\ t_2u_2 & t_2 & u_2 & 1\\ t_3u_3 & t_3 & u_3 & 1\end{vmatrix}=0. \quad\text{...............................(2)}$$

The symmetry of this equation shows that it is immaterial which of the three ratios $x : y : z$ we compare. Hence the three given points will have the assigned parametric values when $u$ is determined by this relation.

The equation (2) is of the lineo-linear form

$$\alpha tu-\beta t+\gamma u-\delta=0,$$

and $u$ is expressed in terms of $t$ by the equation

$$u=(\beta t+\delta)/(\alpha t+\gamma).$$

The equation (1) may be written in cross-ratio notation

$$(tt_1,\ t_2t_3)=(uu_1,\ u_2u_3). \quad\text{..................................(3)}$$

Hence we have the result: *When a conic is represented in terms of two different parameters, the parameters are connected by a lineo-linear relation, and the cross-ratio of the parameters of any four points on the conic is independent of the particular parametric representation.*

The significance of the last result will apear in the following chapter.

The general freedom-equations of a conic contain 9 constants, but only their ratios are concerned, and the number reduces to 8; further, by giving three assigned points fixed parametric values, the constants reduce finally to 5. This is the exact number of constants required to determine a conic, and hence we see again that freedom-equations of this type are capable of representing any conic.

Ex. Find freedom-equations for the conic $yz+zx+xy=0$ such that the vertices of the triangles of reference may have the parametric values 0, 1, $\infty$.

The freedom-equations must be of the form

$$x=p(t-1),$$
$$y=qt,$$
$$z=rt(t-1).$$

Substituting in the equation of the conic, we have

$$qrt^2(t-1)+rpt(t-1)^2+pqt(t+1)\equiv 0,$$

*i.e.* $$qrt+rp(t-1)+pq\equiv 0.$$

Hence $$qr+rp=0,$$

and $$-rp+pq=0;$$

therefore $-p=q=r$, and the equations are

$$x=t-1,\quad y=-t,\quad z=-t(t-1).$$

## EXAMPLES XVII.

1. Find the point-equations corresponding to the following freedom-equations:

(i) $x=2t^2-t+1,\ y=t^2+2t,\ z=t^2-t$; (ii) $x=3t^2-t,\ y=t^2+2t-1,\ z=t+2$; (iii) $x=2t^2+t-1,\ y=t^2-3t+4,\ z=3t^2+2t+2.$

2. Find the line-equations corresponding to the freedom-equations in Ex. 1.

3. Find freedom-equations for the following conics:

(i) $x^2-2y^2+3z^2-yz-2zx-xy=0$, (ii) $x^2-3y^2-z^2-4yz-2zx+2xy=0$,

(iii) $x^2+2y^2-3z^2+2yz+2zx+2xy=0$.

4. Determine the coordinates of the centre of each of the following conics in cartesian coordinates:

(i) $x=2t^2+1,\ y=t^2+2t,\ z=t^2-1$; (ii) $x=t,\ y=t^2+t,\ z=t^2-1$; (iii) $x=3t^2+t,\ y=2t+1,\ z=t^2+1$; (iv) $x=t-1,\ y=t^2+2t-1,\ z=t.$

5. Determine the coordinates of the centre of each of the following conics in areal coordinates:

(i) $x=t(t-1),\ y=t-1,\ z=t$; (ii) $x=t^2,\ y=2t-1,\ z=2t^2+1$; (iii) $x=2t^2+1,\ y=-t^2-t+1,\ z=-t^2+t-1.$

6. Determine the nature of each of the conics in Ex. 4, and in the case of a hyperbola find the equations of the asymptotes.

7. Determine the nature, and, in the case of a hyperbola, the asymptotes of the conics in Ex. 5.

8. Show that $\left.\begin{array}{l}x=3t^2+2t-1,\\ y=t^2-3,\\ z=t^2-1\end{array}\right\}$ and $\left.\begin{array}{l}x=t^2-2t-3,\\ y=t^2+2t-1,\\ z=c(t^2-1),\end{array}\right\}$ in cartesian coordinates represent homothetic hyperbolas.

9. Show that $\left.\begin{aligned} x&=2t^2-t-1,\\ y&=t+1,\\ z&=-2t^2+2t \end{aligned}\right\}$ and $\left.\begin{aligned} x&=2t^2-t+1,\\ y&=-2t^2+t,\\ z&=2t-2, \end{aligned}\right\}$ in areal coordinates represent homothetic hyperbolas.

10. Find the foci of the conics whose freedom-equations in rectangular cartesian coordinates are

(i) $x=3t^2-2t,$ $y=2t-1$; (ii) $x=2t^2-4t+5,$ $y=t^2+2t+1$; (iii) $x=-31t^2+36t+21,$ $y=17t^2+48t-22,$ $z=5(t^2+1)$.

11. Show that the freedom-equations of a conic in cartesian coordinates can be reduced to one of the three forms:

$$\begin{aligned} x&=t^2+2a_1t+a_0, & z&=c(t^2-1) && \text{for a hyperbola,}\\ y&=t^2+2b_1t+b_0, & z&=c(t^2+1) && \text{for an ellipse,}\\ & & z&=1 && \text{for a parabola.} \end{aligned}$$

12. Find the equation of the tangent at $t$ to the locus $x=at^2+b,\ y=ct+d$; and find the locus of the intersection of two tangents at right angles.

(Corpus, 1913.)

13. The parameters $t,\ t'$ of the ends of a variable chord of the conic $x=1,\ y=2t,\ z=t^2$ are connected by the equation $tt'+t-t'+2=0$; prove that the envelope of the chords is a conic, and find its line-equation.

14. The parameters of the ends of a chord of the conic in Ex. 13 are connected by the equation $att'-\frac{1}{2}b(t+t')+c=0$; prove that the chord passes through a fixed point, and find its coordinates.

15. Find the equation of the directrix and the length of the latus-rectum of the parabola $x=at^2+2bt,\ y=ct^2$. (Selwyn, 1913.)

16. Prove that the locus of the point, of which the coordinates with regard to rectangular axes are given by $x(t^2+s^2)=at(s^2+1),\ y(t^2+s^2)=as(1-t^2)$, is a circle when $t$ is constant and $s$ varies, and also when $s$ is constant and $t$ varies, and that the two circles are orthogonal. (Pembroke, 1912.)

17. Show that if $a$ and $b$ are constant the freedom-equations $x=a\cos(\theta-\alpha)$, $y=b\cos(\theta-\beta)$ in the variable parameter $\theta$ represent, for all values of $\alpha,\ \beta$, an ellipse inscribed in the rectangle $x\pm a=0,\ y\pm b=0$.

# CHAPTER XVIII.

## CORRESPONDENCE, HOMOGRAPHY, AND INVOLUTION.

**1.** ONE of the most important and fundamental ideas, not only in geometry, but in all branches of mathematics, is that of correspondence. While it would be outside the scope of this book to treat at length the whole theory of correspondence, it is necessary to enter in some detail into the elementary ideas which underlie it. A fuller treatment requires a wider acquaintance with the theory of projective geometry than can be assumed at this stage.

It will be convenient to begin with some simple examples.

(1) Let $l$, $l'$ be two straight lines, and $S$ any point not on either of them. Then if lines through $S$ cut $l$, $l'$ in points $P$, $P'$; $Q$, $Q'$; $R$, $R'$; ..., we have two ranges of points $PQR\ldots$ and $P'Q'R'\ldots$ in correspondence. To every point $P$ on $l$ corresponds a single definite point $P'$ on $l'$ which is the intersection of $l'$ with $SP$; and *vice versa*, to every point $P'$ on $l'$ corresponds a single definite point $P$ on $l$. This type of correspondence is called one-to-one or (1, 1). (In particular the two ranges are said to be in perspective with centre of perspective $S$.)

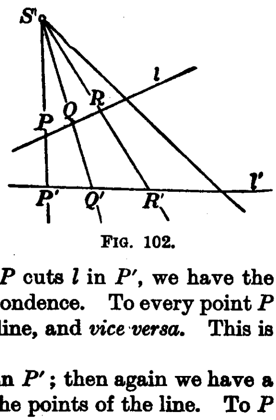


FIG. 102.

(2) Let a fixed line $l$ and a fixed circle (or conic) $C$ be given, and let $O$ be a fixed point on the circle, and $P$ any point on the circle. Then if $OP$ cuts $l$ in $P'$, we have the points of the circle and the points of the line in correspondence. To every point $P$ on the circle corresponds the unique point $P'$ on the line, and *vice versa*. This is also an example of a (1, 1) correspondence.

(3) Let the tangent at $P$ to the fixed circle cut $l$ in $P'$; then again we have a correspondence between the points of the circle and the points of the line. To $P$ on the circle corresponds the unique point $P'$ on the line, but since two tangents can be drawn from $P'$ to the circle there are two points $P_1$ and $P_2$ on the circle corresponding to each point on the line. The correspondence in this case is (2, 1).

(4) In example (1) there is a (1, 1) correspondence between the points $P\ldots$ and the lines $OP\ldots$. Thus there may be a correspondence between different sorts of geometrical elements.

**2.** Consider now the analytical expressions of the correspondences described in § 1.

(1) Two ranges in perspective.

(*a*) Let the two lines $l$, $l'$ be parallel, and take $l$ as axis of $x$, $OS \perp OP$ as axis of $y$ (Fig. 103). Then the points $P$, $P'$ are determined by their abscissae $OP=x$, $O'P'=x'$. If $SO'=k\,.\,SO$, we have

$$x'=kx.$$

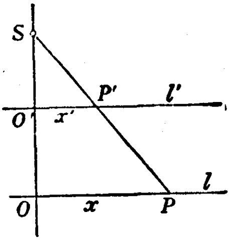



FIG. 103.

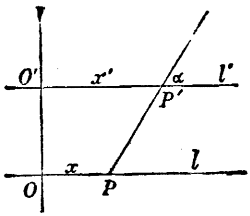



FIG. 104.

(*b*) Next let $S$ be at infinity in the direction $\alpha$ (Fig. 104). Then, if $OO'\cot\alpha=c$,

$$x'=x+c.$$

(*c*) Suppose next that $l$, $l'$ intersect in $O$ (Fig. 105). Draw $SA \parallel P'O$ and $SA' \parallel PO$, and let $OA=a$, $OA'=b$, $AP=x$, $A'P'=x'$. Then $x' : a=b : x$; hence

$$x'=\frac{ab}{x}.$$

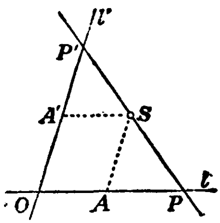



FIG. 105.

(*d*) Taking now the perfectly general case, let the points of the line

$$\left.\begin{matrix} x=a+pt \\ y=b+qt \end{matrix}\right\} \text{ be in perspective with the points of the line } \left.\begin{matrix} x=a'+p't' \\ y=b'+q't' \end{matrix}\right\},$$

the centre of perspective being $S\equiv(\alpha, \beta)$. Then, since $P$, $S$, $P'$ are collinear, we have the relation

$$\begin{vmatrix} a+pt & a'+p't' & \alpha \\ b+qt & b'+q't' & \beta \\ 1 & 1 & 1 \end{vmatrix}=0,$$

which reduces to the form

$$ltt'+mt+nt'+k=0.$$

In each case we have obtained a *lineo-linear* equation in the parameters $t$, $t'$ or $x$, $x'$.

(2) Correspondence between points $P$ on a conic $C$ and points $P'$ on a straight line $l$, when $PP'$ passes through a fixed point $O$ on the conic.

Let $l$ cut the conic in $X$, $Y$; then, taking $OXY$ as triangle of reference, the equation of the conic can be written

$$x=t(t-1), \qquad y=t-1, \qquad z=t.$$

The parametric values for $O$, $X$, $Y$ are thus 1, $\infty$, 0. Let $P'$ be determined by the ratio $x' : y'=t'$. The equation of $OP$ is $x=ty$. Since this line passes through $P'$, $x'=ty'$; hence

$$t'=t,$$

which is the simplest possible representation of (1, 1) correspondence.

(3) Correspondence between points $P$ on a conic and points $P'$ on a straight line $l$, when $PP'$ touches the conic at $P$.

Let $l$ cut the conic in $X$, $Y$. Take $XY$ and the tangents at $X$ and $Y$ as triangle of reference. Then the freedom-equations of the conic may be taken to be

$$x=t^2, \qquad y=-1, \qquad z=2t,$$

giving the equation $z^2+4xy=0$. The equation of the tangent at $t$ is

$$-x+t^2y+tz=0.$$

Putting $z=0$ and $x:y=t'$, we get $\quad t'=t^2,$

the expression of the (2, 1) correspondence.

3. Generalizing from these examples, it will be seen that an $(n, n')$ correspondence between two sets of elements can be represented by an equation connecting the algebraic parameters $t, t'$ of the elements, involving $t$ to power $n$ and $t'$ to power $n'$.

The most important case for us is that of a (1, 1) correspondence, which is represented by a lineo-linear equation of the form

$$\gamma tt' - \alpha t + \delta t' - \beta = 0. \quad \text{.......................(1)}$$

Expressing $t'$ in terms of $t$, we have the equation

$$t' = \frac{\alpha t+\beta}{\gamma t+\delta}. \quad \text{.......................(2)}$$

In this equation only the ratios of the four coefficients have to be considered, and they must be such that the determinant $\begin{vmatrix} \alpha & \beta \\ \gamma & \delta \end{vmatrix}$ or $\alpha\delta-\beta\gamma$ is not zero; for if this were the case, we should have

$$t' = \frac{\alpha(\alpha t+\beta)}{\alpha\gamma t+\beta\gamma} = \frac{\alpha}{\gamma},$$

or else $t=-\beta/\alpha$, and there would be no proper relation between $t$ and $t'$.

4. **Equality of cross-ratios in a (1, 1) correspondence.**

*When a (1, 1) correspondence exists between two sets of numbers $t, t'$, the cross-ratio of any four values of $t$ is equal to the cross-ratio of the four corresponding values of $t'$.*

Let the correspondence-equation connecting the two parameters $t, t'$ be

$$t' = \frac{\alpha t+\beta}{\gamma t+\delta}.$$

We have to prove that the cross-ratio

$$(t_1t_2, t_3t_4) \equiv \frac{t_1-t_3}{t_1-t_4} \Big/ \frac{t_2-t_3}{t_2-t_4} = \frac{t_1'-t_3'}{t_1'-t_4'} \Big/ \frac{t_2'-t_3'}{t_2'-t_4'} \equiv (t_1't_2', t_3't_4').$$

We have

$$t_1'-t_3' = \frac{\alpha t_1+\beta}{\gamma t_1+\delta} - \frac{\alpha t_3+\beta}{\gamma t_3+\delta}$$

$$= \frac{(\alpha\delta-\beta\gamma)(t_1-t_3)}{(\gamma t_1+\delta)(\gamma t_3+\delta)}.$$

Changing the suffix 3 into 4, we have

$$t_1' - t_4' = \frac{(\alpha\delta - \beta\gamma)(t_1 - t_4)}{(\gamma t_1 + \delta)(\gamma t_4 + \delta)}.$$

By division,
$$\frac{t_1' - t_3'}{t_1' - t_4'} = \frac{\gamma t_4 + \delta}{\gamma t_3 + \delta} \cdot \frac{t_1 - t_3}{t_1 - t_4}.$$

Then changing the suffix 1 into 2, we have

$$\frac{t_2' - t_3'}{t_2' - t_4'} = \frac{\gamma t_4 + \delta}{\gamma t_3 + \delta} \cdot \frac{t_2 - t_3}{t_2 - t_4},$$

and by division again,

$$\frac{t_1' - t_3'}{t_1' - t_4'} \Big/ \frac{t_2' - t_3'}{t_2' - t_4'} = \frac{t_1 - t_3}{t_1 - t_4} \Big/ \frac{t_2 - t_3}{t_2 - t_4},$$

*i.e.*
$$(t_1't_2',\ t_3't_4') = (t_1t_2,\ t_3t_4).$$

5. **Homographic ranges.** It is proved in text-books on pure geometry that when two ranges of points on the same or on different lines are in perspective with each other or each with another range, so that there is a (1, 1) correspondence between them, the cross-ratio of four points on the one range is equal to the cross-ratio of the four corresponding points on the other range. Two such ranges are said to be *homographic.*

This theorem can be deduced immediately from the theorem of § 4, when we have established the following relation between the cross-ratio of four points and the cross-ratio of their parameters.

*When a range of points on a line is determined by equations of the form*

$$x = a_1 t + a_0,$$
$$y = b_1 t + b_0,$$

*in terms of a rational algebraic parameter, the cross-ratio of four points on the range is equal to the cross-ratio of their parameters.*

The cross-ratio of the four points

$$(P_1P_2,\ P_3P_4) = \frac{P_1P_3}{P_1P_4} \Big/ \frac{P_2P_3}{P_2P_4} = \frac{M_1M_3}{M_1M_4} \Big/ \frac{M_2M_3}{M_2M_4}$$
$$= \frac{x_1 - x_3}{x_1 - x_4} \Big/ \frac{x_2 - x_3}{x_2 - x_4}.$$

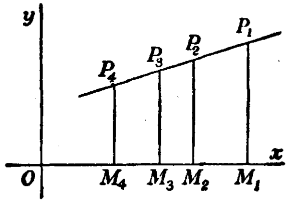



FIG. 106.

But
$$x_1 - x_3 = a_1(t_1 - t_3), \text{ etc.},$$

therefore
$$(P_1P_2,\ P_3P_4) = \frac{t_1 - t_3}{t_1 - t_4} \Big/ \frac{t_2 - t_3}{t_2 - t_4} = (t_1t_2,\ t_3t_4).$$

Further, by Chap. XVII. § 14, if the range is expressed in terms of any other parameter $t'$, the parameters $t$ and $t'$ are connected by a lineo-linear relation, and therefore $(t_1't_2',\ t_3't_4') = (t_1t_2,\ t_3t_4)$. Hence the cross-ratio of the four points is always equal to the cross-ratio of the corresponding parameters.

**6. Cross-ratio of four points on a conic.** (1) We shall prove first that *in whatever way a conic may be represented by rational algebraic freedom-equations of the form*

$$x = a_2t^2 + 2a_1t + a_0,$$
$$y = b_2t^2 + 2b_1t + b_0,$$
$$z = c_2t^2 + 2c_1t + c_0,$$

*the cross-ratio of the parameters of four given points on the conic is always the same.*

If the conic is represented in terms of another parameter $u$, we have seen (Chap. XVII. § 14) that the parameters are connected by a lineo-linear relation, and therefore, by § 4,

$$(t_1t_2,\ t_3t_4) = (u_1u_2,\ u_3u_4).$$

The cross-ratio of the parameters of four points on a conic is therefore a perfectly definite number which does not depend upon the particular mode of representation.

We have next to see what is the geometrical meaning of this cross-ratio. Take four points $P_1$, $P_2$, $P_3$, $P_4$ on the conic, and let $X$ be any other point on the conic. Taking as triangle of reference the tangents at $X$ and any other point $Z$ on the curve, and the chord of contact $XZ$, we can write the freedom-equations of the conic in the form

$$x = t^2, \quad y = t, \quad z = 1.$$

Let the parameters of $P_1$, $P_2$, $P_3$, $P_4$ be $t_1$, $t_2$, $t_3$, $t_4$. Join $X$ to the four points; then the equations of these four lines are

$$y = t_1z, \quad y = t_2z, \quad y = t_3z, \quad y = t_4z.$$

Then (by Chap. XII. § 13) the cross-ratio of the pencil

$$X(P_1P_2,\ P_3P_4) = \frac{t_1 - t_3}{t_1 - t_4} \bigg/ \frac{t_2 - t_3}{t_2 - t_4},$$

but this is just equal to the cross-ratio of the parameters. Hence

(2) *Four fixed points on a conic subtend at any other point on the conic a pencil with constant cross-ratio, and when the conic is represented in terms of an algebraic parameter the cross-ratio of the four points is equal to the cross-ratio of their parameters.*

By the principle of duality, or by interpreting the equations in line-coordinates, we have the reciprocal theorem :

(3) *Four fixed tangents to a conic cut any other tangent in a range with constant cross-ratio, and when the conic is represented in terms of an algebraic parameter the cross-ratio of the four tangents is equal to the cross-ratio of their parameters.*

**7. Cross-ratio of the base-points of a pencil of conics.** Consider a pencil of conics through the four points $A$, $B$, $C$, $D$. Taking the harmonic triangle of this quadrangle as triangle of reference, the coordinates of the four points will be $(\pm p,\ \pm q,\ \pm r)$, and the equation of a conic of the pencil is

$$ax^2 + by^2 + cz^2 = 0,$$

with the condition $\qquad ap^2 + bq^2 + cr^2 = 0.$

Hence, eliminating $c$, we can write the equation of the conic

$$r^2(ax^2+by^2)=(ap^2+bq^2)z^2$$

or

$$a(r^2x^2-p^2z^2)=b(q^2z^2-r^2y^2).$$

Hence we can write

$$\frac{a(rx-pz)}{b(qz-ry)}=\frac{qz+ry}{rx+pz}=t,$$

and the conic is thus expressed in terms of the parameter $t$.

The parameters of the four given points are then

$$t_1=\infty,\quad t_2=0,\quad t_3=-\frac{ap}{bq},\quad t_4=\frac{q}{p};$$

hence the cross-ratio

$$(t_1t_2,\, t_3t_4)=-\frac{bq^2}{ap^2}.$$

If the points are taken in different orders, the other values of the cross-ratio are

$$-(cr^2)/(bq^2), \text{ etc.}$$

**Examples.**

1. Prove that the cross-ratio of the pencil subtended by the points of intersection of the two conics

$$S\equiv ax^2+by^2+cz^2=0,$$
$$S'\equiv a'x^2+b'y^2+c'z^2=0$$

at any point on $S$ are $-(ap)/(bq)$, etc., where

$$p:q:r=bc'-b'c:ca'-c'a:ab'-a'b.$$

2. Prove that the cross-ratio of the pencil subtended by the points of intersection of the two conics

$$S\equiv fyz+gzx+hxy=0,$$
$$S'\equiv f'yz+g'zx+h'xy=0$$

at any point on $S$ are $-(fp)/(gq)$, etc., where

$$p:q:r=gh'-g'h:hf'-h'f:fg'-f'g.$$

3. Prove that through four given points there pass three conics on which the four points form a harmonic set, and determine whether these conics are all real.

Let the four points be $(\pm p,\ \pm q,\ \pm r)$; then the equation of any conic through these four points is

$$ax^2+by^2+cz^2=0,$$

with the condition

$$ap^2+bq^2+cr^2=0.$$

The cross-ratio $-(bq^2)/(cr^2)=-1$; therefore $bq^2=cr^2$ and $ap^2=-2bq^2$. By equating the various ratios to $-1$, we get three conics

$$-2x^2/p^2+y^2/q^2+z^2/r^2=0,$$
$$x^2/p^2-2y^2/q^2+z^2/r^2=0,$$
$$x^2/p^2+y^2/q^2-2z^2/r^2=0.$$

(i) If the four points are all real, let $p=q=r=1$, and we get three real conics

$$-2x^2+y^2+z^2=0,\quad x^2-2y^2+z^2=0,\quad x^2+y^2-2z^2=0.$$

(ii) If the points are two pairs of conjugate imaginaries, let $p=i$, $q=r=1$; then the three conics are

$$2x^2+y^2+z^2=0,\quad -x^2-2y^2+z^2=0,\quad -x^2+y^2-2z^2=0.$$

The last two are real, but the first is virtual.

(iii) If the points are two real and two conjugate imaginaries, we must proceed otherwise, since the harmonic triangle has two sides conjugate imaginary lines. Let the sides be $z=0$ and $x \pm iy=0$. Then the conic

$$(a-ib)(x+iy)^2+(a+ib)(x-iy)^2+cz^2=0$$

has this triangle self-conjugate and its coefficients all real; it will pass through the four points given by

$$x+iy=\pm(p+iq), \quad x-iy=\pm(p-iq), \quad z=\pm 1$$

if

$$2a(p^2-q^2)+4bpq+c=0.$$

The four points are $(p, q, \pm 1)$, $(q, -p, \pm i)$. Proceeding as before, we get three conics. One will be found to be

$$(p^2-q^2)(x^2-y^2)-(p^2+q^2)^2z^2+4pqxy=0,$$

which is real; the other two will be found to be conjugate imaginary conics.

4. Prove that through four given points there pass two conics on which the four points form an equianharmonic set. Show that they are conjugate imaginary if the points are either all real or all imaginary, but are real if two of the points are real and two conjugate imaginary points.

If the points are $(\pm 1, \pm 1, \pm 1)$ the two conics are $x^2+\omega y^2+\omega^2z^2=0$ and $x^2+\omega^2y^2+\omega z^2=0$, which are conjugate imaginaries. If the points are $(\pm 1, \pm 1, \pm i)$, the two conics are $\omega x^2+\omega^2y^2-z^2=0$ and $\omega^2x^2+\omega y^2-z^2=0$, again conjugate imaginaries. If the points are two real and two imaginary, let the sides of the harmonic triangle be $z=0$, $x+\omega y=0$, $x+\omega^2y=0$, and let the points be given by $x+\omega y=\pm(1+\omega)$, $x+\omega^2y=\pm(1+\omega^2)$, $z=\pm 1$. Then the two conics are $2x^2-y^2+z^2-2xy=0$ and $-x^2+2y^2+z^2-2xy=0$, which are real.

**8. Conic generated by two homographic pencils.** The converse of the theorem in § 6 affords one of the most important applications of (1, 1) correspondence.

*If A and B are two fixed points, and if to every ray drawn through A corresponds one and only one ray through B; and similarly to every ray through B corresponds one and only one ray through A, the locus of the point of intersection of corresponding rays is a conic through A and B.*

Consider two pencils with vertices $A\equiv(1, 0, 0)$ and $B\equiv(0, 1, 0)$. The equations of corresponding rays are

$$y=\lambda z \quad \text{and} \quad x=\mu z.$$

But if they are in (1, 1) correspondence, the parameters $\lambda$, $\mu$ are connected by an equation of the form

$$a\lambda\mu+b\lambda+c\mu+d=0.$$

Hence the locus of points of intersection of pairs of corresponding rays is

$$axy+byz+czx+dz^2=0,$$

which represents a conic passing through $A$ and $B$.

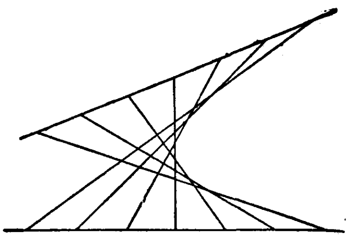


FIG. 107.

Hence *the locus of points of intersection of corresponding rays of two homographic pencils is a conic passing through the vertices of the two pencils.*

By the principle of duality, we have the reciprocal theorem: *the envelope* *of the lines joining pairs of corresponding points of two homographic ranges is a conic touching the lines on which the ranges lie.* (Fig. 107.)

We shall now exhibit the use of the above theorems in solving geometrical problems.

**Examples.**

1. A variable triangle is such that its sides pass through three fixed points and two of its vertices lie on fixed straight lines; find the locus of the third vertex. (Colin Maclaurin, 1722.)

Let the sides $QR$, $RP$, $PQ$ of the triangle $PQR$ pass respectively through the fixed points $A$, $B$, $C$, and let the vertices $Q$, $R$ lie on the fixed lines $OY$, $OZ$. Draw any line $u$ through $B$. This cuts $OZ$ in a unique point $R$, and $RA$ cuts $OY$ in a unique point $Q$; lastly, $QC$ is a unique line $u'$ corresponding to $u$; and similarly it may be shown that $u$ is a unique line corresponding to $u'$. Hence we have two pencils through $B$ and $C$ in (1, 1) correspondence, and therefore the locus of $P$, the point of intersection of corresponding rays, is a conic passing through $B$ and $C$.

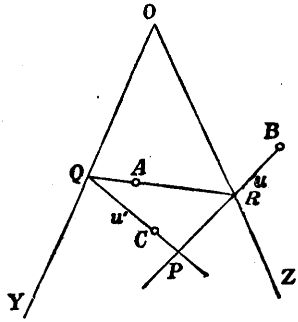



FIG. 108.

2. Show that $O$ lies on the locus in Ex. 1.

3. $A$ and $B$ are two fixed points on a given conic, and $P$ is a variable point on the fixed line $l$. $PA$, $PB$ meet the conic in $X$, $Y$ respectively; $AY$ cuts $BX$ in $Q$. Prove that the locus of $Q$ is a conic.

Draw any ray $u$ through $A$ meeting the conic in $Y$. Join $BY$ cutting $l$ in $P$, and join $AP$ meeting the conic in $X$. Then $BX$ corresponds uniquely to $AY$, and similarly it can be shown that $AY$ corresponds uniquely to $BX$. Hence there is a (1, 1) correspondence between the rays $AY$ and $BX$. The locus of their point of intersection is therefore a conic passing through $A$ and $B$.

4. If in Ex. 3 $l$ cuts the conic in $L$ and $M$, show that $L$ and $M$ lie on the locus.

**9.** A very important case arises when to $AB$, regarded as a ray through $A$, corresponds $BA$, regarded as a ray through $B$. Then, if $P$ is any point on the line $AB$, since $AP$ and $BP$ are always corresponding rays, $P$ is a point on the locus. The conic-locus in this case therefore contains the whole line $AB$; the remaining part of the locus must therefore consist of another straight line. Hence: *If $A$ and $B$ are fixed points, and if a* (1, 1) *correspondence exists between the rays through $A$ and $B$ of such a nature that the ray $AB$ through $A$ corresponds to the ray $BA$ through $B$, the locus of points of intersection of corresponding rays is a straight line* (*which, with the line $AB$, makes up the complete conic-locus*).

Reciprocally: *If $a$ and $b$ are fixed lines, and if a* (1, 1) *correspondence exists between the points on $a$ and $b$ of such a nature that the point of intersection $ab$, regarded as a point on $a$, corresponds to the point $ba$, regarded as a point on $b$, the envelope of the lines joining corresponding points is a point* (*which, with the point $ab$, makes up the complete conic-envelope*).

Ex. $OX, OY$ are two fixed lines, on which lie the fixed points $A, B$ respectively. Circles through $A, B$ cut $OX, OY$ in $P, Q$ respectively. Prove that the line $PQ$ passes through a fixed point.

$P$ and $Q$ are in (1, 1) correspondence, and when $P$ is at $O$, $Q$ also coincides with $O$. Hence $O$ on $OX$ corresponds to $O$ on $OY$. The conic-envelope generated by $PQ$ therefore consists of the point $O$ and another fixed point $U$. To find $U$, consider two particular positions of the circle. If the circle passes through $O$, $PQ$ becomes the tangent at $O$. Another particular circle consists of the line $AB$ and the line at infinity; in this case $PQ$ is therefore the line at infinity. Hence $U$ is the point of intersection of the line at infinity with the tangent to the circle which passes through $O$. Hence all the lines $PQ$ are parallel, as is also easily proved by elementary geometry.

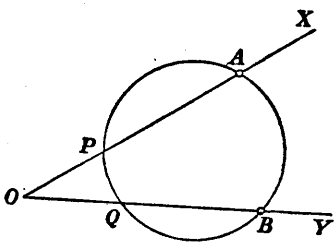



FIG. 109.

**10.** The theorem (2) of § 6, which is the fundamental projective theorem for a conic, can be stated from another point of view. We know that five points completely determine a conic. If a sixth point is to lie on the conic it must satisfy some condition. We have found then that the condition that six points $ABCDEF$ should lie on a conic, is that the cross-ratios $E(ABCD)$ and $F(ABCD)$ must be equal. This condition can be expressed in terms of collinearity, and leads to

**Pascal's Theorem.** *If a hexagon is inscribed in a conic the points of intersection of pairs of opposite sides are collinear.*

Let the pairs of opposite vertices be denoted by $A, A'$; $B, B'$; $C, C'$; so that the order of the vertices is $AB'CA'BC'$; and let $B'C$ and $BC'$ meet in $X$, $C'A$ and $CA'$ in $Y$, $A'B$ and $AB'$ in $Z$. Then we have to prove that $X, Y, Z$ are collinear.

We have two pencils with equal cross-ratios $\quad B(A'B'C'A) = C(A'B'C'A).$

Cut these by the transversals $B'A$ and $C'A$ respectively, and let the points

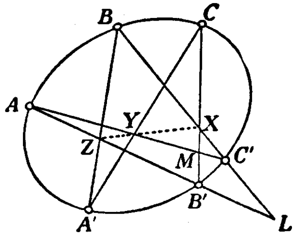



FIG. 110.

$$\binom{BC'}{B'A} = L, \quad \binom{CB'}{C'A} = M;$$

then $\qquad (ZB'LA) = (YMC'A).$

But these two ranges have the point $A$ in common. They are therefore, by a known theorem, in perspective, and $ZY$, $B'M$, $LC'$ are concurrent. But $B'M$ and $LC'$ intersect in $X$; therefore $ZY$ passes through $X$.

The line $XYZ$ is called the *Pascal line* of the hexagon $AB'CA'BC'$. By taking the points in different orders we can obtain 60 Pascal lines. A large number of theorems relating to the configuration formed by these lines has been discovered by Steiner, Kirkman, Cayley, Salmon, and others.

Some information regarding these is to be found in the Notes to Salmon's *Conic Sections*.

By the principle of duality we obtain from Pascal's theorem :

**Brianchon's Theorem.** *If a hexagon is circumscribed about a conic the lines joining pairs of opposite vertices are concurrent.*

**11.** The following theorem is proved very simply by an application of Pascal's and Brianchon's theorems :

*If two triangles are inscribed in the same conic they are both circumscribed about another conic.*

Let $ABC$ and $A'B'C'$ be two triangles inscribed in the same conic $S$. Then, considering the hexagon $ABCC'B'A'$, we have by Pascal's theorem the three collinear points

$$\begin{pmatrix} AB \\ C'B' \end{pmatrix} = P, \ \begin{pmatrix} BC \\ B'A' \end{pmatrix} = P', \ \begin{pmatrix} CC' \\ AA' \end{pmatrix} = O,$$

*i.e.* $AA'$, $CC'$, $PP'$ are concurrent. Hence, by the converse of Brianchon's theorem, the hexagon $APC'A'P'C$ circumscribes a conic, but the sides of this hexagon are those of the two triangles $ABC$, $A'B'C'$ ; therefore, etc.

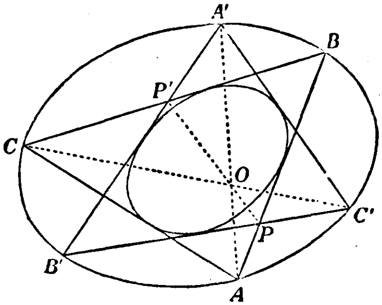


FIG. 111.

This gives us two conics $S$ and $S'$ so related that there is a pair of tringles inscribed in $S$ and circumscribed about $S'$. We shall now prove the theorem :

*If two conics are so related that one triangle can be inscribed in one and circumscribed about the other, an unlimited number of triangles can be so constructed.*

Let $ABC$ be one given triangle, and take any point $A'$ on $S$. Draw the tangents from $A'$ to $S'$ cutting $S$ in $B'$ and $C'$. Then, since the triangles $ABC$ and $A'B'C'$ are both inscribed in the same conic $S$, a conic can be constructed to touch their six sides. But the conic $S'$ touches five of the sides, and since a conic is completely determined by five tangents, $S'$ must be the conic which touches the six sides. Hence $B'C'$ is also a tangent to $S'$.

Ex. *If a parabola is inscribed in a triangle its focus lies on the circumscribed circle.*

Let $ABC$ be the given triangle, $I$ and $J$ the circular points. Then we have a triangle $ABC$ circumscribed about one conic, the parabola, and inscribed in another conic, the circumscribed circle, which passes through $I$ and $J$. Draw the tangents from $I$ and $J$ to the parabola intersecting in the focus $F$. Then the triangle $IJF$ is circumscribed about the parabola, and is therefore inscribed in the circle. Therefore $F$ lies on the circumscribed circle.

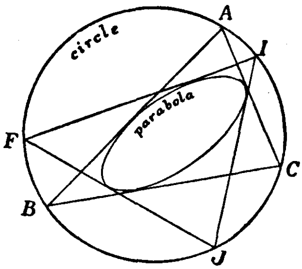


FIG. 112.

**12. Correspondence between pairs of points on the same line.** When two homographic ranges lie on the same line the question arises: can a point coincide with its correspondent?

Let the equation connecting the parameters $t$, $t'$ of corresponding points be

$$att'+bt+ct'+d=0.$$

Then, if the two corresponding points, $t$ and $t'$, coincide, we have, putting $t'=t$,

$$at^2+(b+c)t+d=0.$$

This is a quadratic in $t$; hence there are in general two points which coincide with their correspondents. These two points are called the *double-points* of the homography. They may be real, coincident, or imaginary.

13. *The cross-ratio of any pair of corresponding points and the two double points of the homography is constant.*

Let the parameters be so chosen that the parameters of the double points are 0 and $\infty$. The equation of the homography is then

$$bt+ct'=0,$$

and the cross-ratio $(0\,\infty\,,\,tt')=t/t'=-c/b$, which is constant.

14. Suppose a homography to be defined in the following way. Let $S$, $S'$ be two fixed points outside the line $l$, and let $l'$ be another fixed line. Then, to obtain the point $P'$ from the point $P$, join $SP$ cutting $l'$ in $Q$, and join $S'Q$ cutting $l$ in $P'$. In this case the double-points of the homography are $X$ and $Y$, the points of intersection of $l$ with $l'$ and $SS'$.

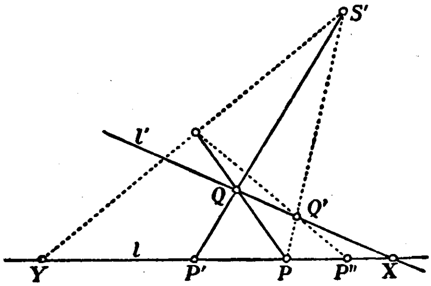


FIG. 113.

Although the two ranges lie on the same line, it is essential that they should be kept distinct in all operations, and it must be made quite clear to which of the two ranges a given point $P$ belongs. If in the above example $P$ belonged to the second range instead of the first, its correspondent would be constructed differently; first join $S'P$ cutting $l'$ in $Q'$, then join $SQ'$ cutting $l$ in $P''$.

Thus, when $P$ is considered as belonging to the first range, its correspondent is $P'$; but when it is considered as belonging to the second range, its correspondent is $P''$.

We can say that to $P$ on the first range corresponds $P'$ on the second, and to $P'$ on the second corresponds $P$ on the first; but we cannot say that to the point $P$ corresponds the point $P'$ and *vice versa*, without mentioning to which range each point belongs, for there is then an ambiguity owing to the fact that the relation between pairs of points is not symmetrical.

15. There is a very important case in which the correspondent of any point is the same, whichever range the point is supposed to belong to. Let us see what is the condition for this.

Let the equation connecting the parameters be

$$t' = \frac{\alpha t + \beta}{\gamma t + \delta},$$

so that to $P(t)$ on the first range corresponds $P'(t')$ on the second. Now to $P'(t')$ considered as being on the first range will in general correspond a third point $P''(t'')$ on the second, but if $P'$ is also the point which corresponds to $P$, considered as belonging to the second range, then $P''$ must coincide with $P$.

We have $$t'' = \frac{\alpha t' + \beta}{\gamma t' + \delta} = \frac{\alpha(\alpha t + \beta) + \beta(\gamma t + \delta)}{\gamma(\alpha t + \beta) + \delta(\gamma t + \delta)} = \frac{(\alpha^2 + \beta\gamma)t + \beta(\alpha + \delta)}{\gamma(\alpha + \delta)t + (\beta\gamma + \delta^2)}.$$

Hence, if $t'' = t$, $$\gamma(\alpha + \delta)t^2 - (\alpha^2 - \delta^2)t - \beta(\alpha + \delta) = 0.$$

This equation is satisfied identically by $\alpha + \delta = 0$, and if this condition is not satisfied, we have the quadratic equation

$$\gamma t^2 - (\alpha - \delta)t - \beta = 0.$$

But this equation is just the condition that $t' = t$, and its roots determine the two double-points of the homography.

Hence *either* (1) *every pair of points are connected in pairs so that to P on the first range corresponds P′ on the second, and to P′ on the first range corresponds P on the second, or* (2) *this relation only holds in the case of the double-points.*

Case (1) is of great importance, and we shall return to it presently.

In general, we obtain an unending chain of points. To $P_1$ on $l$ corresponds $P_2$ on $l'$, to $P_2$ on $l$ corresponds $P_3$ on $l'$, and so on. If it happens that one point $P_{n+1}$ of the chain coincides with $P_1$, the homography is *periodic* with period $n$. The case $n = 2$ is that which we have just considered. The conditions that the homography should be of period 2 are either $\alpha + \delta = 0$, or (in order that the remaining quadratic should be an identity) $\alpha = \delta$, $\beta = 0$, $\gamma = 0$. The latter conditions give $t' = t$, and the two ranges are identical. In the former case the two ranges are said to be in *involution*. An involution is therefore a periodic homography of period 2, in which the elements are connected definitely in pairs, and the defining relation is

$$t' = \frac{\alpha t + \beta}{\gamma t - \alpha},$$

or $$\gamma t t' - \alpha(t + t') - \beta = 0,$$

*i.e.* it is a *lineo-linear symmetrical* equation in $t$ and $t'$.

**Examples.**

1. Prove that the condition that the homography $t' = (\alpha t + \beta)/(\gamma t + \delta)$ should be of period 3 is $\alpha^2 + \delta^2 + \alpha\delta + \beta\gamma = 0$.

2. Prove that the condition that the homography should be of period 4, but not of period 2, is $\alpha^2 + \delta^2 + 2\beta\gamma = 0$.

**16. Double points of an involution.** As in the case of the general homography, an involution possesses two double points $D_1$, $D_2$, whose parameters are the roots of the quadratic

$$\gamma t^2 - 2\alpha t - \beta = 0.$$

These may be real or imaginary. The condition for equal roots is $\alpha^2 + \beta\gamma = 0$. But this is excluded, as it makes the determinant $\begin{vmatrix} \alpha & \beta \\ \gamma & -\alpha \end{vmatrix}$ vanish, and the relation $t' = (\alpha t + \beta)/(\gamma t - \alpha)$ would reduce to $t' = \alpha/\gamma$. When the double points are real the involution is said to be *hyperbolic*, when they are imaginary it is called *elliptic*.

These relations have all been explained with reference to ranges of points on a straight line, but they may equally well be applied to pencils of lines through a point, or points on any curve, or tangents to any curve, or indeed to any one-dimensional figure whose elements are determined by a single parameter.

As an example of an involution, which will at the same time fix the names elliptic and hyperbolic, consider the pairs of conjugate diameters of a central conic.

Taking the *ellipse* $x^2/a^2 + y^2/b^2 = 1$, the equations of a pair of conjugate diameters are

$$y = \mu x, \quad y = \mu' x,$$

where $\mu$, $\mu'$ are connected by the symmetrical lineo-linear relation

$$\mu\mu' = -b^2/a^2.$$

Hence the pairs of conjugate diameters form an involution. The double lines are formed by putting $\mu' = \mu$, and we get the imaginary values

$$\mu = \pm ib/a.$$

Hence the involution is *elliptic*.

For the *hyperbola* $x^2/a^2 - y^2/b^2 = 1$, we have the conjugate diameters

$$y = \mu x, \quad y = \mu' x,$$

where $\mu$, $\mu'$ are connected by the relation

$$\mu\mu' = b^2/a^2.$$

The double lines in this case are real, given by $\mu = \pm b/a$, and are in fact the asymptotes. The involution is therefore *hyperbolic*.

**17.** *Every pair of points of an involution are harmonic conjugates with regard to the double points.*

Let the parameters be chosen so that the parameters of the double points are $\pm\sqrt{k}$; then the equation of the involution is

$$tt' = k.$$

The cross-ratio $\quad (t, t'; \sqrt{k}, -\sqrt{k}) = \dfrac{t-\sqrt{k}}{t+\sqrt{k}} \Big/ \dfrac{t'-\sqrt{k}}{t'+\sqrt{k}} = -1.$

If the parameter $t$ is the distance of the point $P$ from a fixed point $O$ on the fixed line, the equation $tt' = k$ means geometrically that the points $P$, $P'$ are so related that

$$OP \,.\, OP' = k.$$

This relation is sometimes taken as the defining relation for an involution of points. It is not convenient as a general method of defining an involution, however, as it would not apply, for example, to a pencil of lines.

The point $O$ is called the *centre* of the involution. It is the point which corresponds to the point at infinity on the line. In an involution-pencil there are two double lines, and the bisectors of the angles between these lines are the central lines of the involution. They are, in general, the only pair of lines of the involution which are at right angles.

## 18. Examples of involutions.

1. *Pairs of points on a line which are conjugate with regard to a circle or conic form an involution,* the double points being the points of intersection of the straight line with the conic.

2. *Pairs of straight lines at right angles to one another through a fixed point $O$ form an involution of lines,* for the correspondence between such pairs is (1, 1) and symmetrical. In this case the double rays are the two lines with respect to which each pair are harmonic conjugates, *i.e.* they are the lines $OI$ and $OJ$ joining $O$ to the circular points. Since the double rays are imaginary the involution is elliptic.

3. *Any straight line is cut by a system of coaxal circles in an involution.* Take any point $P$ on the line $l$; then there is just one circle of the system which passes through $P$, and this cuts $l$ again in $P'$. The same circle again is determined by $P'$; hence to $P'$ corresponds $P$. There is thus a (1, 1) involutory correspondence between the points $P$ and the points $P'$. The double points are the points of contact of the circles of the system which touch $l$.

4. **Desargues' Theorem.** *A straight line is cut in involution by a pencil of conics passing through four fixed points.*

This is the general theorem of which the last example, relating to coaxal circles, is a particular case, and the proof is exactly similar.

5. By the principle of duality, or independently by applying the same method of proof, we have also the theorem: *The pairs of tangents from a fixed point to a range of conics touching four fixed lines form an involution pencil.*

As a particular case, *pairs of tangents from a fixed point $O$ to the conics of a confocal system form an involution*; the double lines are the tangents to the two conics of the confocal system which pass through $O$.

**19. An involution is completely determined by two pairs of elements.** Consider two pairs of lines through the origin, and let their equations be

$$S \equiv ax^2 + 2hxy + by^2 = 0,$$
$$S' \equiv a'x^2 + 2h'xy + b'y^2 = 0.$$

Then there is a unique pair of lines

$$F \equiv Ax^2 + 2Hxy + By^2 = 0,$$

with respect to which the pairs $S$ and $S'$ are harmonic conjugates. For the conditions that $S$ and $S'$ should be apolar to $F$ are

$$bA - 2hH + aB = 0,$$
$$b'A - 2h'H + a'B = 0.$$

These determine uniquely the ratios

$$A : -2H : B = ha' - h'a : ab' - a'b : bh' - b'h.$$

Hence the double elements of the involution of which $S$ and $S'$ are two pairs are given by

$$F \equiv (ha' - h'a)x^2 - (ab' - a'b)xy + (bh' - b'h)y^2 = 0,$$

and the involution is completely determined.

**20. The cross-ratio of two pairs of elements.** Let the two pairs of elements be given by the quadratic equations

$$at^2 + 2ht + b = 0,$$
$$a't^2 + 2h't + b' = 0,$$

and let the two pairs be $t_1$, $t_2$ and $t_1'$, $t_2'$, so that

$$t_1 + t_2 = -2h/a, \qquad t_1 t_2 = b/a,$$
$$t_1' + t_2' = -2h'/a', \quad t_1' t_2' = b'/a'.$$

The cross-ratio

$$\lambda = (t_1 t_2, t_1' t_2') = (t_1 - t_1')(t_2 - t_2')/(t_1 - t_2')(t_2 - t_1')$$
$$= (t_1 t_2 + t_1' t_2' - t_1 t_2' - t_1' t_2)/(t_1 t_2 + t_1' t_2' - t_1 t_1' - t_2' t_2).$$

Let $\quad aa'(t_1 t_2' + t_1' t_2) = \mu_1, \quad aa'(t_1 t_1' + t_2' t_2) = \mu_2;$

then $\quad \mu_1 + \mu_2 = aa'(t_1 + t_2)(t_1' + t_2') = 4hh',$

and $\quad \mu_1 \mu_2 = a^2 a'^2 \{t_1 t_2 (t_1'^2 + t_2'^2) + t_1' t_2' (t_1^2 + t_2^2)\}$

$$= 4(abh'^2 + a'b'h^2 - aba'b').$$

Therefore $\mu_1$, $\mu_2$ are the roots of the equation

$$\mu^2 - 4hh'\mu + 4\{h^2 h'^2 - (ab - h^2)(a'b' - h'^2)\} = 0,$$

or, writing for shortness,

$$ab - h^2 = C, \quad a'b' - h'^2 = C', \quad ab' + a'b - 2hh' = 2K,$$

the roots $\quad \mu_1, \mu_2 = 2hh' \pm 2\sqrt{(CC')}.$

We have then $\quad \lambda = (ab' + a'b - \mu_1)/(ab' + a'b - \mu_2)$

$$= (K \pm \sqrt{CC'})/(K \mp \sqrt{CC'}),$$

or the two values of the cross-ratio, which correspond to the two different orders of the pairs, are the roots of the equation

$$\lambda^2 - 2\lambda(K^2 + CC')/(K^2 - CC') + 1 = 0.$$

**21. Elements common to two involutions.** Let the double elements of two involutions be determined by the quadratic equations

$$At^2 + 2Ht + B = 0,$$
$$A't^2 + 2H't + B' = 0;$$

and suppose the pair of elements

$$at^2 + 2ht + b = 0$$

to be common to both involutions.

Then $\quad aB - 2hH + bA = 0,$

$$aB' - 2hH' + bA' = 0;$$

hence $\quad a : -2h : b = HA' - H'A : AB' - A'B : BH' - B'H.$

Therefore the pair of common elements is uniquely determined.

The two common elements will be real if $h^2-ab>0$, *i.e.* if

$$(AB'-A'B)^2-4(HA'-H'A)(BH'-B'H)>0.$$

This may be written

$$(AB'-2HH'+BA')^2>4(AB-H^2)(A'B'-H'^2).$$

Now, if the two pairs of double elements are both real, with this condition (*i.e.* $K^2>CC'$), the cross-ratio of the two pairs of double elements is positive (§ 20), *i.e.* the double elements do not separate each other.

If either of the pairs is imaginary, say $AB-H^2>0$, then we have

$$4h^2AB-(aB+bA)^2>0;$$

therefore $$4AB(h^2-ab)>(aB-bA)^2;$$

but $AB>H^2>0$, therefore $h^2>ab$, and the two common elements are real.

Hence *two involutions on the same range have a unique pair of elements in common, which are real if either* (1) *one or both of the involutions are elliptic, or* (2) *if the two involutions are both hyperbolic, provided their double elements do not separate one another.*

**Examples.**

1. Pairs of conjugate diameters of a conic form an involution which is elliptic in the case of the ellipse and hyperbolic in the case of the hyperbola, and the double lines are the tangents to the conic from the centre, *i.e.* the asymptotes.

Pairs of rectangular lines through the centre also form an elliptic involution, and there are two real lines which are common to this involution and the involution of conjugate diameters. These are the rectangular conjugate diameters, or the principal axes of the conic.

2. Two rectangular hyperbolas determine a pencil of conics which cut the line at infinity in an involution of points of which the double points are the circular points. Hence every conic of the pencil is a rectangular hyperbola.

3. A pencil of conics cuts the line at infinity in an involution of points whose double points are the points in which two conics of the system touch the line at infinity. These two conics are therefore parabolas. Hence a pencil of conics contains two parabolas, and the directions of the axes of these parabolas are harmonic conjugates with regard to the asymptotes of each conic of the system.

4. If a pencil of conics contains a circle, the involution on the line at infinity has the circular points as a pair. The double points $D_1$, $D_2$ are therefore harmonic conjugates with regard to the circular points. Hence the axes of the two parabolas of the system are at right angles. Further, if $C$ is the centre of any conic of the system, $CD_1$ and $CD_2$ are conjugate diameters and are at right angles, and are therefore the principal axes of the conic. Hence the principal axes of all conics of the system are in two fixed directions at right angles.

5. Consider a circle and a conic. These determine a pencil of conics, and one conic of the system is the pair of common chords. If these meet the line at infinity in $L_1$, $L_2$, and $I, J$ are the circular points, we have

$$(L_1D_1, IJ)=(L_2D_1, JI).$$

Hence the two chords are equally inclined to an axis of the conic.

6. Consider two confocal conics through a point $P$. These determine a range of conics. The tangents from $P$ to the range of conics form an involution of which the double lines are the tangents at $P$ to the two given conics. One conic of the range is the degenerate point-pair consisting of the circular points $I, J$. Hence the tangents at $P$ are harmonic conjugates with regard to the lines $PI$, $PJ$, and are therefore at right angles, *i.e.* two confocal conics cut orthogonally.

Another degenerate conic of the range is the point-pair $F_1$, $F_2$, the foci. The two tangents at $P$ are therefore harmonic conjugates with regard to $PF_1$, $PF_2$, *i.e.* they bisect the angle $F_1PF_2$, *i.e.* each tangent at $P$ is equally inclined to the two focal lines.

7. Consider a conic with foci $F_1$, $F_2$, and take a point $P$ not on the conic. The tangents $PT, PT'$ belong to the involution determined by $PI, PJ$ and $PF_1, PF_2$. The double lines of this involution are conjugate with regard to $PI$ and $PJ$ and are therefore at right angles, and they bisect the angles $TPT'$ and $F_1PF_2$; hence the angles $TPT'$ and $F_1PF_2$ have the same bisectors.

**22. Correspondence between two conics.** The points of two conics may be in (1, 1) correspondence. This will happen when the parameters $t$, $t'$ of the points on the two conics are connected by the usual lineo-linear relation

$$att' + bt + ct' + d = 0.$$

In particular we can have a correspondence between the points of the same conic.

*If points $P$, $P'$ on a conic are connected by a (1, 1) correspondence, the envelope of the line $PP'$ is a conic having double contact with the given conic at the two double points of the homography.*

Let the parameters of the double points $X, Z$ be $\infty$, $0$; then the equation of the correspondence reduces to the form

$$bt + ct' = 0. \qquad \text{(1)}$$

Take as triangle of reference the chord $XZ$ and the tangents at $X$ and $Z$, which intersect in $Y$; then the parametric equations of the conic are

$$\left.\begin{aligned} x &= t^2, \\ y &= 2t, \\ z &= 1. \end{aligned}\right\} \qquad \text{(2)}$$

The equation of the chord joining the points $t$, $t'$ is

$$x - \tfrac{1}{2}(t+t')y + tt'z = 0. \qquad \text{(3)}$$

Substituting for $t'$ in terms of $t$, the line-coordinates of this line become

$$\left.\begin{aligned} l &= c, \\ m &= \tfrac{1}{2}(b-c)t, \\ n &= -bt^2. \end{aligned}\right\} \qquad \text{(4)}$$

Provided $b \neq c$, these are the parametric line-equations of a conic touching the given conic at $X$ and $Z$.

**23. Involution on a conic.** An involution is a symmetrical (1, 1) correspondence, and is represented by the symmetrical lineo-linear equation

$$att' + b(t+t') + d = 0 \qquad \text{(5)}$$

Comparing this with the equation (3) of the chord joining the points $t, t'$, we see that (5) is the condition that the chord should pass through the fixed point $(d, -2b, a)$. Hence, *an involution on a conic is determined by its intersections with a pencil of lines, and the double points of the involution are the points of contact of the tangent lines of the pencil.*

The proviso $b \neq c$ in the theorem of last paragraph is just the condition that the homography should not be an involution. If $b=c$, we have $m=0$. and the envelope degenerates to the point $Y$, *i.e.* the lines joining pairs of corresponding points of the involution are concurrent in $Y$.

**Examples.**

1. $A$ and $B$ are two fixed points on a conic, and $l$ is a fixed line cutting the conic in $U, V$. $PA$ cuts $l$ in $R$ and $BR$ cuts the conic in $Q$. Find the envelope of $PQ$.

From the construction there is a (1, 1) correspondence between $P$ and $Q$; and this correspondence is not symmetrical, for the construction which carries $P$ into $Q$ would carry $Q$ into another point $P'$ such that $QA$ and $BP'$ intersect on $l$. Also when $R$ coincides with $U$ or $V$, $P$ and $Q$ will coincide; therefore $U, V$ are the double points of the homography. Hence the envelope of $PQ$ is a conic having double contact with the given conic at $U$ and $V$.

2. $S$ is a fixed conic, and $A, B$ two fixed points not on the conic. $R$ is a variable point on the conic, and $RA, RB$ cut the conic in $P, Q$. Prove that the envelope of $PQ$ is a conic having double contact with $S$ at its points of intersection with $AB$.

3. $A$ and $B$ are two fixed points on a conic. A variable circle through $A, B$ cuts the conic again in $P, Q$. Prove that $PQ$ passes through a fixed point.

Starting with the point $P$, the point $Q$ is determined by drawing the circle $PAB$ cutting the conic again in $Q$, and the same construction applied to $Q$ gives $P$. Hence $P, Q$ are connected by a symmetrical (1, 1) correspondence, and therefore $PQ$ passes through a fixed point $O$. One circle through $A, B$ consists of $AB$ and the line at infinity, which cuts the conic in $H, K$. Hence the line at infinity is one position of the line $PQ$; therefore $O$ is a point at infinity, and all the lines $PQ$ are parallel.

## EXAMPLES XVIII.

1. $ABCD$ are four points on a conic, and $abcd$ are the tangents at these points. Show that the quadrangle $ABCD$ and the quadrilateral $abcd$ have the same harmonic triangle.

2. Show that the cross-ratio of the tangents to four conics of a pencil $S+\lambda S'=0$ at each of the four base-points is equal to the cross-ratio of the parameters $(\lambda_1\lambda_2\lambda_3\lambda_4)$.

3. $P$... and $P'$... are two homographic ranges on different lines, and $l$ is any other line. $PP'$ cuts $l$ in $Q$, and $M$ is the harmonic conjugate of $Q$ with regard to $P, P'$. Prove that the locus of $M$ is a conic passing through the points of intersection of $l$ with the two given lines.

4. Deduce from Ex. 3 that if $P$..., $P'$... are *similar* ranges (*i.e.* if the ratio of corresponding segments $PQ : P'Q'$ is constant) the locus of the mid-point of $PP'$ is a straight line.

5. Prove that if
$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} = 0.$$
the three lines $x/l_1 + y/m_1 + z/n_1 = 0$, etc., touch a conic inscribed in the triangle of reference. Prove further that if the same condition is satisfied, the three lines form a triangle inscribed in a conic in which the triangle of reference is inscribed, and they also form a triangle self-conjugate with respect to a conic with respect to which the triangle of reference is self-conjugate. (Math. Tripos I., 1910.)

6. Rays making angles $\alpha$, $\beta$, $\gamma$ with a zero line are in involution when paired with rays whose angles are $\alpha'$, $\beta'$, $\gamma'$. Show that they are also in involution when paired with rays whose angles are $\alpha'$, $\alpha+\alpha'-\gamma'$, $\alpha+\alpha'-\beta'$. (Math. Tripos II., 1913.)

7. Two homographic ranges of points on a line are such that no (real) point on the line is self-corresponding. Prove that two positions of a point $P$ may be found in any plane through the line such that all pairs of corresponding points of the ranges subtend the same angle at $P$. (Pembroke, 1910.)

8. One vertex $F$ of a variable triangle $MFN$ is fixed, and the angle $MFN$ is constant, while the other vertices $M$ and $N$ move respectively on fixed straight lines $TH$ and $TK$. Prove that the envelope of the side $MN$ is a conic of which $F$ is a focus, and which touches the given lines $TH$ and $TK$. Give a construction for the other focus. (Trinity, 1911.)

9. Prove that if the three sides $QR$, $RP$, $PQ$ of a movable triangle $PQR$ pass through the fixed points $D$, $E$, $F$ respectively, and $P$ lies on a fixed conic through $E$ and $F$, and $Q$ lies on a fixed conic through $F$ and $D$, then $R$ lies on a fixed conic through $D$ and $E$. (Trinity, 1900.)

10. The sides of a square $ABCD$ are trisected, the points being named in order $AHKBLMCNPDQR$. Prove that, if points $X$, $Y$ move on $BC$ and $AD$ respectively so that the pencils $A(LNDX)$ and $B(QDCY)$ are homographic, the ranges traced by $X$ and $Y$ are in perspective. Hence construct (i) a pair of corresponding rays of the pencils (other than $AD$, $BC$) which are parallel to one another, (ii) the tangents at $A$ and $B$ to the locus of intersection of $AX$ and $BY$; and sketch roughly this locus. (Math. Tripos II., 1912.)

11. A variable triangle $PQR$ is inscribed in a conic, and the sides $QR$, $RP$ pass through fixed points $A$, $B$. Show that if $A$ and $B$ are conjugate with regard to the conic, $PQ$ passes through a fixed point $C$. Show that the triangles $PQR$ and $ABC$ are in perspective and that the locus of the centre of perspective is the conic itself. (Math. Tripos II., 1914.)

12. Through a fixed point $O$ a line is drawn cutting the sides of a given triangle in $X$, $Y$, $Z$, and $P$ is the harmonic conjugate of $X$ with regard to $Y$, $Z$. Show that the locus of $P$ is a conic through $O$, $A$, $B$, $C$.

13. If in Ex. 12 $OA$ cuts $BC$ in $L$, and $L'$ is the harmonic conjugate of $L$ with regard to $B$, $C$, show that the tangents at $A$ and $O$ are $AL'$ and $OL'$.

14. A variable triangle $PQR$ is such that two of its vertices $Q$, $R$ lie on fixed straight lines $OY$, $OZ$, the sides $PQ$, $PR$ pass through fixed points $B$, $C$, and the third side $QR$ touches a fixed conic which touches $OY$, $OZ$. Show that the locus of $P$ is a conic.

15. If a polygon is inscribed in a conic, and all its sides but one pass through fixed points, prove that the envelope of that side is a conic having double contact with the given conic.

16. If $P_1P_2P_3 \dots$, $Q_1Q_2Q_3 \dots$ are corresponding points of a homography on a conic $S$, prove that the double points are the points of intersection of the conic with the Pascal line of the hexagon $P_1Q_2P_3Q_1P_2Q_3$.

17. Hence find a construction for a conic touching three given lines and having double contact with a given conic.

18. Find a construction for a polygon inscribed in a given conic and having each of its sides passing through a given point.

19. A variable conic cuts a fixed conic in two fixed points and passes through two other fixed points. Show that the line joining the variable points of intersection of the two conics passes through a fixed point.

# CHAPTER XIX.

## SYSTEMS OF POINTS ON A CONIC.

**1.** In this chapter we shall study the geometry of one dimension, where the figures, instead of being curves in two dimensions, consist of systems of points lying on a line or curve. The study of this narrow domain is useful in two ways: first, as an introduction to the invariant theory of curves in two dimensions, and second, as an application to the theory of algebraic equations, whose roots can always be represented by points lying on a line or curve.

Intrinsically there is no difference between the geometry on a straight line and that on a conic or any other curve which can be represented by rational algebraic freedom-equations. But in the case of a conic we shall derive much external assistance from the geometry of the conic itself.

**2.** We shall take therefore a fixed conic, which we shall call the base-conic. Choosing any two tangents and the chord of contact as triangle of reference, we can write the equations as follows:

Point-equation $S \equiv y^2 - 4zx = 0$, Line-equation $\Sigma \equiv m^2 - nl = 0$,

Parametric equations $\left.\begin{aligned} x &= t^2, \\ y &= 2t, \\ z &= 1, \end{aligned}\right\}$ $\left.\begin{aligned} l &= 1, \\ m &= -t, \\ n &= t^2, \end{aligned}\right\}$

the line $t$ being the tangent at the point $t$.

To avoid infinite values of the parameter it is sometimes convenient to use the homogeneous parametric equations

$$x = t^2, \quad y = 2tt', \quad z = t'^2.$$

## DYADS OF POINTS ON A CONIC.

**3.** A pair of points on the base-conic can be represented by the quadratic equation

$$f \equiv a_0 t^2 + 2a_1 t + a_2 = 0.$$

The form of the equation shows at once that the two points are the intersections of the line

$$a_0 x + a_1 y + a_2 z = 0$$

with the base-conic.

**4. Apolar pairs of points on a conic.** The cross-ratio of two pairs of points on a conic is equal to the cross-ratio of the pencil formed by joining the two pairs of points to any other point on the conic, and is equal to the cross-ratio of their parameters (Chap. XVIII. § 6). Two pairs of points,

$$f \equiv a_0t^2 + 2a_1t + a_2 = 0,$$
$$g \equiv b_0t^2 + 2b_1t + b_2 = 0,$$

are said to be *apolar* when their cross-ratio has the value $-1$. Analytically the condition is represented by the equation (see Chap. II. § 21)

$$a_0b_2 - 2a_1b_1 + a_2b_0 = 0.$$

To interpret this geometrically, we see that two lines

$$l_1x + m_1y + n_1z = 0,$$
$$l_2x + m_2y + n_2z = 0$$

are conjugate with regard to the conic, whose line-equation is $m^2 - nl = 0$, if

$$2m_1m_2 = n_1l_2 + n_2l_1.$$

Substituting $a_0$, $a_1$, $a_2$ for $l_1$, $m_1$, $n_1$ and $b_0$, $b_1$, $b_2$ for $l_2$, $m_2$, $n_2$, we get the above condition. Hence *two dyads on a conic are apolar if their chords are conjugate or apolar with regard to the conic.*

**5. Involution on a conic.** An involution of points on a conic consists of a symmetrical (1, 1) correspondence in which the points are connected in pairs. Let

$$f \equiv a_0t^2 + 2a_1t + a_2 = 0,$$
$$g \equiv b_0t^2 + 2b_1t + b_2 = 0$$

represent two of the pairs of points, $P$, $P'$ and $Q$, $Q'$; and let the chords $PP'$ and $QQ'$ intersect in $O$. Then there is a unique dyad $\phi \equiv T_1, T_2$, which is apolar to both $P$, $P'$ and $Q$, $Q'$, viz. the chord $T_1T_2$ is the polar of $O$ with regard to the base-conic. Now every chord conjugate to $T_1T_2$ passes through $O$; hence, the equations of the chords $PP'$ and $QQ'$ being

$$a_0x + a_1y + a_2z = 0,$$
$$b_0x + b_1y + b_2z = 0,$$

the equation of any other chord through $O$ is

$$(a_0x + a_1y + a_2z) + \lambda(b_0x + b_1y + b_2z) = 0.$$

The involution is then determined by the intersections of these chords with the conic, and is therefore represented by the linear system of dyads

$$f + \lambda g = 0.$$

**Examples.**

1. Show that the double points of the involution determined by the two dyads $f \equiv a_0t^2 + 2a_1tt' + a_2t'^2$ and $g \equiv b_0t^2 + 2b_1tt' + b_2t'^2$ is the Jacobian of $f$ and $g$, viz.

$$\begin{vmatrix} \dfrac{\partial f}{\partial t} & \dfrac{\partial f}{\partial t'} \\ \dfrac{\partial g}{\partial t} & \dfrac{\partial g}{\partial t'} \end{vmatrix} = 0.$$

**2.** If the Jacobian of two dyads consists of two coincident points, show that the two dyads have a point in common.

3. Prove that the three pairs of points whose parameters are $u, u'$; $v, v'$; $w, w'$ are in involution if

$$(u+u')(vv'-ww')+(v+v')(ww'-uu')+(w+w')(uu'-vv')=0.$$

**6. Polars of a point with regard to a system of points on a line.** The equation

$$f \equiv a_0 t^n + {}_nC_1 a_1 t^{n-1} t' + {}_nC_2 a_2 t^{n-2} t'^2 + \ldots + a_n t'^n = 0$$

represents a group of $n$ points on the line. If $(u, u')$ is any point on the line, the *first polar of* $(u, u')$ *with respect to* $f$ is defined to be

$$u \frac{\partial f}{\partial t} + u' \frac{\partial f}{\partial t'} = 0.$$

This represents a group of $n-1$ points on the line.

Similarly the first polar of $(u, u')$ with respect to the first polar, *i.e.*

$$\left(u \frac{\partial}{\partial t} + u' \frac{\partial}{\partial t'}\right)^2 f = 0,$$

or

$$u^2 \frac{\partial^2 f}{\partial t^2} + 2uu' \frac{\partial^2 f}{\partial t \, \partial t'} + u'^2 \frac{\partial^2 f}{\partial t'^2} = 0$$

is called the *second polar* of $(u, u')$ with respect to $f$; and similarly higher polars can be defined.

**7. Polar of a point with respect to a given dyad.** The first polar of $u$ with respect to the dyad

$$a_0 t^2 + 2a_1 t + a_2 = 0,$$

is

$$u(a_0 t + a_1) + (a_1 t + a_2) = 0,$$

*i.e.*

$$a_0 tu + a_1(t+u) + a_2 = 0.$$

But this expresses that the dyads

$$a_0 \lambda^2 + 2a_1 \lambda + a_2 = 0$$

and

$$\lambda^2 - (t+u)\lambda + tu \equiv (\lambda - t)(\lambda - u) = 0$$

should be apolar, and the latter consists of $u$ and its polar $t$. Hence *any point and its polar with regard to a given dyad form a harmonic set with the given dyad.*

This result shows that the polar-point can be found by a projective geometrical construction which is quite independent of the parametric representation. A figure which is obtained from a given figure by a projective or non-metrical geometrical construction, independent of any special parametric representation, is called a *covariant* of the given figure. Analytically this means that the equation which defines the new figure is unaltered if the parametric representation is altered, *i.e.* if the equations are transformed by a linear transformation of the form

$$t' = (\alpha t + \beta)/(\gamma t + \delta).$$

Similarly a projective relation between two figures which is expressed by an equation (containing only coefficients), which is unaltered by any such transformation, is called an *invariant* relation. Thus the expression $a_0 b_2 - 2a_1 b_1 + a_2 b_0$ is a *joint-invariant* of the two dyads in § 4.

## TRIADS OF POINTS ON A CONIC.

**8. Polar-dyad of a point with respect to a given triad.** Let

$$f \equiv a_0 t^3 + 3a_1 t^2 t' + 3a_2 t t'^2 + a_3 t'^3 = 0$$

represent a triad. The polar-dyad of the point $P \equiv (u, u')$ is

$$u\frac{\partial f}{\partial t} + u'\frac{\partial f}{\partial t'} = 0,$$

*i.e.* $$(a_0 u + a_1 u')t^2 + 2(a_1 u + a_2 u')tt' + (a_2 u + a_3 u')t'^2 = 0.$$

The chord joining this pair of points is called the *polar-axis* of the point $P$.

To exhibit the polar dyad as a covariant of the point $P$ and the given triad, we shall establish the following construction:

*The polar axis of a given point $P$ with respect to the triad $ABC$ is the polar line of $P$ with respect to the triangle $ABC$* (Chap. XII. § 12, Ex. 2).

Let $A$, $B$, $C$ be the given triad, and $P$ the given point. Join $PA$, $PB$, $PC$ cutting $BC$, $CA$, $AB$ in $X$, $Y$, $Z$. Join $YZ$, $ZX$, $XY$ cutting $BC$, $CA$, $AB$ in $X'$, $Y'$, $Z'$. Then $X'$, $Y'$, $Z'$ are collinear in a line which is the polar of the point $P$ with respect to the triangle $ABC$.

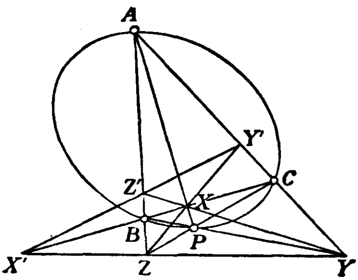


FIG. 114.

Also, if $ABC$ is chosen as triangle of reference, and the coordinates of $P$ are $(x', y', z')$, the equation of $X'Y'Z'$ is

$$\frac{x}{x'} + \frac{y}{y'} + \frac{z}{z'} = 0.$$

Now let the parameters of the points $A$, $B$, $C$ and $P$ be $t_1$, $t_2$, $t_3$ and $u$. Then $ABC$ being the triangle of reference, the equations of the conic can be written

$$x = (t - t_2)(t - t_3),$$
$$y = (t - t_3)(t - t_1),$$
$$z = (t - t_1)(t - t_2).$$

The equation of the polar axis is then

$$x(u - t_1) + y(u - t_2) + z(u - t_3) = 0,$$

and this cuts the conic in two points whose parameters are the roots of the equation in $t$,

$$(t - t_2)(t - t_3)(u - t_1) + (t - t_3)(t - t_1)(u - t_2) + (t - t_1)(t - t_2)(u - t_3) = 0.$$

But this is evidently the polar equation derived from the equation

$$f \equiv (t - t_1)(t - t_2)(t - t_3) = 0.$$

Hence the polar axis of any triad is derived by the above geometrical construction.

**Examples.**

1. Show that the equation of the polar axis of the point $u$ with regard to the triad $a_0t^3+3a_1t^2+3a_2t+a_3=0$ is $(a_0u+a_1)x+(a_1u+a_2)y+(a_2u+a_3)z=0$.

2. If $Q$ is the polar of $P$ with respect to a given dyad $f$, show that the dyad $PQ$ is apolar to $f$.

3. Show that the triads for which a given line $lx+my+nz=0$ is the polar axis of a given point $u$ form a linear system.

4. Prove that the second polar of $P$ is the fourth point in which the polar conic of $P$ with respect to the triangle $ABC$ (Chap. XII. § 12, Ex. 2) cuts the base-conic.

**9. Hessian of a triad.** The question arises: can the point $(u, u')$ be chosen so that its polar-dyad may consist of two coincident points? The condition that the two points be coincident is

$$(a_0u+a_1u')(a_2u+a_3u')-(a_1u+a_2u')^2=0,$$

or, in determinant notation,

$$\begin{vmatrix} a_0u+a_1u' & a_1u+a_2u' \\ a_1u+a_2u' & a_2u+a_3u' \end{vmatrix}=0,$$

*i.e.*

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial u^2} & \dfrac{\partial^2 f}{\partial u\,\partial u'} \\ \dfrac{\partial^2 f}{\partial u\,\partial u'} & \dfrac{\partial^2 f}{\partial u'^2} \end{vmatrix}=0.$$

This equation represents a pair of points. Hence *there are two points whose first polars with respect to a given triad consist of coincident points.* These two points are called the *Hessian points* or the *Hessian dyad* of the given triad.

**10. Reduction of the equation of a triad to the canonical form.** The investigation of the properties of a triad can be simplified by referring its equation to the Hessian points as base-points. The equation of the Hessian points thus reduces to $tt'=0$. This requires that

$$a_0a_2-a_1^2=0,$$
$$a_1a_3-a_2^2=0.$$

Hence, either $\qquad a_1=0 \quad \text{and} \quad a_2=0$ ......................................(1)

or

$$\frac{a_0}{a_1}=\frac{a_1}{a_2}=\frac{a_2}{a_3}.$$

The second alternative would make $f$ a perfect cube, which is not in general the case. Hence, adopting the first alternative, $f$ reduces to

$$f\equiv a_0t^3+a_3t'^3.$$

As in the general system of homogeneous coordinates, we can replace $t$ and $t'$ by any multiples of them; thus if we write $a_0^{\frac{1}{3}}t=\xi$, $a_3^{\frac{1}{3}}t'=\eta$, the triad reduces further to $\qquad f\equiv\xi^3+\eta^3=0,$

and its Hessian is $\qquad H\equiv\xi\eta=0.$

We can now prove that *each of the Hessian points of a triad taken double is the first polar of the other.*

For, taking $f \equiv t^3 + t'^3$, its Hessian is $tt' = 0$. The polar of the Hessian point (1, 0) is $t^2 = 0$, which represents the other Hessian point (0, 1) taken double.

**11.** Let
$$f \equiv t^3 + t'^3 = 0$$
be the canonical equation of a triad of points $A$, $B$, $C$ on the base-conic
$$x = t^2, \quad y = 2tt', \quad z = t'^2.$$
Then we know (Chap. XIII. § 7, Ex. 2) that the tangents at $A$, $B$, $C$ meet the opposite sides of the triangle $ABC$ in three collinear points $L$, $M$, $N$. If $t' = 1$ and $t = t_1, t_2, t_3$ are the values of the parameters for $A$, $B$, $C$ respectively, the equation of $BC$ is
$$x - \tfrac{1}{2}(t_2 + t_3)y + t_2t_3z = 0,$$
and that of the tangent at $A$ is
$$x - t_1y + t_1^2z = 0.$$
But
$$t_2 + t_3 = -t_1 \quad \text{and} \quad t_2t_3 = -t_1^3t_2t_3 = t_1^2;$$
therefore the equation of $BC$ becomes
$$x + \tfrac{1}{2}t_1y + t_1^2z = 0.$$
Hence $L$, the intersection of $BC$ with the tangent at $A$, is
$$(-t_1^2, 0, 1) \equiv (1, 0, t_1),$$
and the points $L$, $M$, $N$ lie on the line $y = 0$. The line $LMN$ cuts the base-conic in two points whose parameters are given by $tt' = 0$. This represents the Hessian of the given triad $f$, and the chord $LMN$ through the Hessian points is called the *Hessian axis*. Hence *the tangents at three points $A$, $B$, $C$ on the base-conic cut the opposite sides of the triangle $ABC$ in three collinear points $L$, $M$, $N$, and the line $LMN$ cuts the conic in a pair of points forming the Hessian of $A$, $B$, $C$.*

From this construction it is evident that the Hessian is a covariant of the given triad.

**Ex.** Show that the polar axes of points of the conic all pass through the pole of the Hessian axis with regard to the conic.

**12.** It is obvious from geometry that if two of the points $A$, $B$, $C$ coincide, the Hessian points will coincide with the two coincident points.

Conversely, if the Hessian points coincide at $T$, two of the three points $A$, $B$, $C$ will coincide at $T$. Take the two points $L$, $M$ on the tangent at $T$, then the point $A$ on the conic which corresponds to $L$ is either $T$ or the point of contact of the other tangent from $L$; similarly for $B$. Suppose neither $A$ nor $B$ coincides with $T$, and let the lines $LA$, $MB$ meet in $O$. Then $LB$, $MA$, $OT$ are concurrent in a point within the conic. But this point should be $C$, a point on the conic, which is impossible. Hence one at least of the points $A$, $B$ must coincide with $T$. Similarly one at least of the points $A$, $C$, and one at least of the points $B$, $C$, must coincide with $T$, and therefore two of the points $A$, $B$, $C$ coincide with $T$.

This leads to the necessary and sufficient condition that the cubic

$$f \equiv a_0t^3 + 3a_1t^2 + 3a_2t + a_3 = 0$$

should have two roots equal, viz. that the Hessian

$$H \equiv (a_0a_2 - a_1^2)t^2 + (a_0a_3 - a_1a_2)t + (a_1a_3 - a_2^2) = 0$$

should be a perfect square, and the equal roots are the roots of the equation $H = 0$. The condition for this is

$$\Delta \equiv (a_0a_3 - a_1a_2)^2 - 4(a_0a_2 - a_1^2)(a_1a_3 - a_2^2) = 0.$$

The expression $\Delta$ is called the *discriminant* of the cubic, and is an invariant.

If the roots of the cubic are all equal the Hessian vanishes identically. The conditions for this are

$$a_0 : a_1 : a_2 = a_1 : a_2 : a_3.$$

When the points $A$, $B$, $C$ are all real the line $H$ is entirely outside the conic; hence we have the condition for real roots of the cubic, viz. $\Delta < 0$.

**Examples.**

1. Prove that there are two points which form with $A, B, C$ an equianharmonic tetrad (Chap. XII. § 16), and that they coincide with the Hessian points.

2. If, using homogeneous parameters, the Hessian points $H_1, H_2$ of three points $P_1$, $P_2$, $P_3$ are represented by $t=0$, $t'=0$, show that the three points $P_1$, $P_2$, $P_3$ can be represented by

$$t+t'=0, \quad t+\omega t'=0, \quad t+\omega^2t'=0,$$

where $\omega$ is an imaginary cube root of unity.

3. Hence show that the range $P_1P_2P_3H_1$ is equianharmonic.

4. Prove that the three ranges

$$P_1P_2P_3H_1H_2,$$
$$P_2P_3P_1H_1H_2,$$
$$P_3P_1P_2H_1H_2$$

are homographic. (The three points $P_1$, $P_2$, $P_3$ are said to be *cyclically projective*, and $H_1$, $H_2$ are the double points.)

**13. Apolarity of two triads.** Taking the two cubic equations

$$f \equiv a_0t^3 + 3a_1t^2 + 3a_2t + a_3 = 0,$$
$$g \equiv b_0t^3 + 3b_1t^2 + 3b_2t + b_3 = 0,$$

the relation

$$a_0b_3 - 3a_1b_2 + 3a_2b_1 - a_3b_0 = 0,$$

which is linear and symmetrical in the coefficients of $f$ and $g$, is exactly analogous to the harmonic or apolar relation between two quadratics, and when this relation is satisfied the two triads are said to be *apolar*.

The geometrical meaning of this relation can be obtained as follows. Let $u_1$, $u_2$, $u_3$ be the parameters of the three points of the second triad. Then the polar of $u_1$ with respect to the first triad is represented by

$$(a_0u_1 + a_1)t^2 + 2(a_1u_1 + a_2)t + (a_2u_1 + a_3) = 0,$$

and the pair $u_2$, $u_3$ are represented by the equation

$$t^2 - (u_2 + u_3)t + u_2u_3 = 0.$$

These two pairs are harmonic or apolar if

$$(a_0u_1+a_1)u_2u_3+(a_1u_1+a_2)(u_2+u_3)+(a_2u_1+a_3)=0,$$

*i.e.* if $$a_0u_1u_2u_3+a_1(u_2u_3+u_3u_1+u_1u_2)+a_2(u_1+u_2+u_3)+a_3=0.$$

But $$u_1u_2u_3=-b_3/b_0,\quad \Sigma u_2u_3=3b_2/b_0,\quad \Sigma u_1=-3b_1/b_0.$$

Hence we get $$a_0b_3-3a_1b_2+3a_2b_1-a_3b_0=0.$$

Therefore *two triads ABC, A′B′C′ are apolar if the polar dyad of one point A with respect to A′B′C′ is harmonic with respect to BC.* (For another geometrical meaning for the apolarity of two triads see § 19, Ex. 5.)

**Examples.**

1. Show that the two triads $t_1$, $t_2$, $t_3$ and $u_1$, $u_2$, $u_3$ are apolar if

$$(t_1-u_1)(t_2-u_2)(t_3-u_3)+(t_1-u_2)(t_2-u_3)(t_3-u_1)+(t_1-u_3)(t_2-u_1)(t_3-u_2)=0.$$

2. Prove that any triad is apolar to itself.

3. Prove that the polar axis of $A$ with respect to the triad $ABC$ passes through $A$ and is conjugate to $BC$ with regard to the conic.

4. Prove that the polar axis of each of the Hessian points is the tangent to the base-conic at the other Hessian point.

5. Prove that any point $t$ forms with the Hessian points of $ABC$ a triad apolar to $ABC$.

**14.** From a given triad $ABC$ we can obtain another triad $A'B'C'$, such that each of the three points is the harmonic conjugate of one of the given points with regard to the other two.

The line $AA'$ is then the polar axis of $A$ with regard to $ABC$, and the three lines $AA'$, $BB'$, $CC'$ are concurrent in $O$, the pole of the Hessian axis with regard to the base-conic. The two triads $ABC$, $A'B'C'$ therefore form an involution.

$A'B'C'$ is a second covariant of the given triad. It is generally called simply the *Cubic covariant.*

**Examples.**

1. Prove that the parameter of the point $A'$, the harmonic conjugate of $A$ with respect to $B$, $C$, is $(a_2t_1+a_3)/(a_0t_1^2+a_1t_1)$.

2. Prove that the cubic covariant of $a_0t^3+3a_1t^2+3a_2t+a_3=0$ is

$$(a_0^2a_3-3a_0a_1a_2+2a_1^3)t^3+3(a_0a_1a_3+a_1^2a_2-2a_0a_2^2)t^2$$
$$-3(a_0a_2a_3+a_1a_2^2-2a_1^2a_3)t-(a_0a_3^2-3a_1a_2a_3+2a_2^3)=0.$$

3. Show that the cubic covariant of $t^3+t'^3=0$ is $t^3-t'^3=0$.

4. If $A'B'C'$ is the cubic covariant of $ABC$, prove for both $A$ and $A'$ that its second polar with regard to $ABC$ coincides with $A$.

5. Show that the cubic covariant of the triad $ABC$ consists of the polars of $A$, $B$, $C$ with regard to the Hessian.

6. Prove that the cubic covariant of a triad is the Jacobian of the triad and its Hessian.

7. Show that a triad and its cubic covariant have the same Hessian.

8. Hence show that the cubic covariant of a given triad can be constructed by drawing tangents to the base-conic from the points $L$, $M$, $N$ on the Hessian axis.

9. Show that the cubic covariant of a given triad $ABC$ can be constructed thus: draw the tangents at $A$, $B$, $C$ forming the triangle $PQR$, then $AP$, $BQ$, $CR$ cut the conic in the points $A'$, $B'$, $C'$ of the cubic covariant.

10. If $A'B'C'$ is the cubic covariant of $ABC$, show that $ABC$ is the cubic covariant of $A'B'C'$.

11. Show that if a triad is apolar to its cubic covariant, two of the points of the triad must coincide. Verify algebraically by showing that the discriminant $\Delta = 0$.

12. Show that the discriminant of the cubic covariant is $\Delta^3$.

13. If $f + \lambda g$ is a linear system, or involution, of triads on a conic, $\lambda$ being a variable parameter, prove that the poles of their Hessian axes describe a conic.

(By choosing the parameters of three points on the conic suitably one of the cubics can be reduced to the simple form $g \equiv t^3 + 1$. The coefficients in the equation of the Hessian axis will be found to involve $\lambda^2$.)

14. If $f$ and $g$ are apolar triads, prove that every pair of triads of the linear system $f + \lambda g$ are apolar.

15. Show that there is one triad which is apolar to each of three given triads.

16. If $f_1, f_2, f_3$ are three given triads, show that each triad of the linear system $\lambda_1 f_1 + \lambda_2 f_2 + \lambda_3 f_3 = 0$ is apolar to one and the same triad.

17. If $f_1$ and $f_2$ are two given triads, and $g_1$ and $g_2$ are two triads each apolar to both $f_1$ and $f_2$, show that each triad of the pencil $f_1 + \lambda f_2$ is apolar to each triad of the pencil $g_1 + \mu g_2$.

## TETRAD OF POINTS ON A CONIC.

**15.** Consider next the quartic equation

$$f \equiv a_0 t^4 + 4a_1 t^3 + 6a_2 t^2 + 4a_3 t + a_4 = 0,$$

which represents a tetrad of points on the base-conic $S \equiv y^2 - 4zx = 0$.

Since two conics intersect in four points, the quartic equation admits of a somewhat different treatment from the cubic, for we can consider the tetrad as defined by the intersection of a variable conic with the base-conic. In this connection we shall anticipate an important definition and theorem relating to a pair of conics (see Chap. XX. § 16 and Chap. XIV. § 19).

Two conics, a conic-locus

$$F \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

and a conic-envelope

$$\Phi' \equiv A'l^2 + B'm^2 + C'n^2 + 2F'mn + 2G'nl + 2H'lm = 0$$

are said to be *apolar* when their coefficients are connected by the equation

$$aA' + bB' + cC' + 2fF' + 2gG' + 2hH' = 0.$$

The geometrical properties which correspond to this relation are:

(1) In $F$ can be inscribed a triangle which is self-conjugate with regard to $\Phi'$;

(2) About $\Phi'$ can be circumscribed a triangle which is self-conjugate with regard to $F$.

**16.** Consider now a pencil of conics passing through the given tetrad. Let one conic of the pencil be

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy=0.$$

The parameters of the points in which this conic cuts the base-conic

$$x=t^2, \quad y=2t, \quad z=1$$

are given by the equation

$$at^4+4ht^3+(4b+2g)t^2+4ft+c=0.$$

Identifying this equation with $f=0$, we have

$$a=a_0, \quad h=a_1, \quad 2b+g=3a_2, \quad f=a_3, \quad c=a_4.$$

$b$ and $g$ are not fully determined, but we may take

$$b=a_2+\lambda, \quad \text{and then} \quad g=a_2-2\lambda,$$

where $\lambda$ has any value.

Then we have the pencil of conics

$$a_0x^2+a_2y^2+a_4z^2+2a_3yz+2a_2zx+2a_1xy+\lambda(y^2-4zx)=0.$$

The conic

$$F\equiv a_0x^2+a_2y^2+a_4z^2+2a_3yz+2a_2zx+2a_1xy=0$$

has a particular significance. It is the unique conic-locus through the four given points apolar to the base-conic $m^2-nl=0$; for the condition that a conic of the pencil should be apolar to the base-conic considered as an envelope is $(a_2+\lambda)-(a_2-2\lambda)=0$, *i.e.* $\lambda=0$.

The pencil of conics through the given tetrad is therefore represented by

$$F+\lambda S=0.$$

In a precisely similar way, we find that the range of conics touching the four tangents to the base-conic $\Sigma\equiv m^2-nl=0$, whose parameters are the roots of the tetrad, is $\Phi+\mu\Sigma=0,$

where $\Phi\equiv a_4l^2+4a_2m^2+a_0n^2-4a_1mn+2a_2nl-4a_3lm=0$

represents the unique conic-envelope of the range which is apolar to the base-conic $S$.

**17.** The pencil $F+\lambda S$ contains three degenerate conics consisting of pairs of straight lines, the pairs of common chords of $F$ and $S$. The values of $\lambda$ for these are found from the equation

$$\begin{vmatrix} a_0 & a_1 & a_2-2\lambda \\ a_1 & a_2+\lambda & a_3 \\ a_2-2\lambda & a_3 & a_4 \end{vmatrix}=0,$$

*i.e.* $$4\lambda^3-I\lambda-J=0, \quad \text{.........(1)}$$

where $$I\equiv a_0a_4-4a_1a_3+3a_2^2,$$
$$J\equiv a_0a_2a_4-a_0a_3^2+2a_1a_2a_3-a_1^2a_4-a_2^3.$$

These are two invariants of the quartic equation. The equation (1) is called the *Reducing Cubic* of the given quartic equation, because, as is shown in the Theory of Equations, the quartic equation can be solved algebraically when we know the roots of this cubic equation. One process of solution, in fact, consists in breaking up the quartic into quadratic factors, and this corresponds exactly to the geometrical process of separating the tetrad of points into two pairs.

**18.** It is obvious geometrically that if two points of the tetrad coincide, two of the line-pairs coincide, and the reducing cubic has two roots equal, and, conversely, if the reducing cubic has two roots equal, two of the line-pairs $(AB, CD)$, $(AC, BD)$, $(AD, BC)$ must be identical, and therefore two of the points $A$, $B$, $C$, $D$ must coincide. The necessary and sufficient condition that the quartic should have two roots equal is therefore that the reducing cubic should have two equal roots. The analytical condition that an equation of any degree should have two roots equal is expressed by the vanishing of a certain function of the coefficients called the discriminant $\Delta$, and this function is an invariant, but we have just seen that the discriminant of the given quartic is equal to or a numerical multiple of the discriminant of the reducing cubic, which can be expressed in terms of the invariants $I$ and $J$. Hence, we do not get a new invariant of the quartic independent of $I$ and $J$. In fact, the discriminant of the reducing cubic, or the condition that the equation (1), § 17, should have equal roots is obtained by eliminating $\lambda$ between this equation and its first derivative $12\lambda^2 - I = 0$. The result is easily found to be $I^3 - 27J^2 = 0$. We can therefore take as the discriminant of the quartic

$$\Delta \equiv I^3 - 27J^2.$$

**19. Geometrical meaning of the vanishing of the two invariants I and J.** If $J=0$, one root of the reducing cubic is zero, and the conic $F$ itself breaks up into two straight lines, and $F$ has the property that a triangle self-conjugate with regard to $F$ can be circumscribed about $S$. Now a triangle, in order to be self-conjugate with regard to a pair of straight lines considered as a degenerate conic, must have one vertex at the intersection of the two lines, and the two sides through that point harmonic conjugates with regard to the two lines. Hence the line-pair $F$, which consists of one pair of lines joining the four given points, are harmonic conjugates with regard to the tangents from their intersection to the base-conic, *i.e.* they are conjugate lines with regard to the base-conic. $J=0$ *is therefore the condition that the tetrad should be harmonic.*

To find the geometrical meaning of the condition $I=0$, let the parameters of the four points of the tetrad be $0, \infty, 1, u$. Then

$$f \equiv 12t(t-1)(t-u) \equiv 12t^3 - 12t^2(u+1) + 12tu,$$

so that $\quad a_0 = 0, \quad a_1 = 3, \quad a_2 = -2(u+1), \quad a_3 = 3u, \quad a_4 = 0.$

Then $\quad I \equiv a_0a_4 - 4a_1a_3 + 3a_2^2 = -36u + 12(u+1)^2 = 12(u^2 - u + 1).$

Hence, if $I=0$ we have $u = -\omega$ or $-\omega^2$, and the cross-ratio

$$(0\infty, 1u) = 1/u = -\omega^2 \text{ or } -\omega.$$

Hence $I=0$ *is the condition that the tetrad should be equianharmonic.*

**Examples.**

1. Show that if the tetrad $ABCD$ is equianharmonic the Hessian dyad of $ABC$ consists of $D$ and another point.

2. If $f$ and $g$ are apolar triads, show that the envelope of the Hessian axes of the linear system $f+\lambda g$ (which is a conic $\Psi$, see § 14, Ex. 13) cuts the base-conic in an equianharmonic tetrad.

3. Show also that the locus of the poles of the Hessian axes is a conic $F$ which touches the tangents to the base-conic at its points of intersection with the conic $\Psi$.

4. Show further that, $f$ and $g$ being still apolar triads, the conics $F$ and $\Psi$ are both apolar to the base-conic considered in each case either as a locus or as an envelope.

5. Show that the Jacobian of two triads is a tetrad, and that if the two triads are apolar their Jacobian is equianharmonic.

**20. The sextic covariant of a tetrad.** If the four points $A$, $B$, $C$, $D$ are grouped in pairs, *e.g.* $AB$, $CD$, there is one pair of points $X$, $X'$ which separate harmonically both $AB$ and $CD$. By grouping the four points in pairs in all possible ways, $AB$, $CD$; $AC$, $BD$; $AD$, $BC$, we obtain three pairs of points $XX'$, $YY'$, $ZZ'$. We shall see later that these form a covariant of the given tetrad; this covariant is called the *sextic covariant.*

**21. Reduction of the equation of a tetrad to the canonical form.** Take $XX'$, a pair of points of the sextic covariant of the tetrad $ABCD$, as the base-points, so that their equation is $tt'=0$. Then, since $XX'$ harmonically separate $AB$, this pair of points must be represented by an equation of the form $at^2+bt'^2=0$. Similarly the other pair $CD$ must be represented by an equation of the form $a't^2+b't'^2=0$. Hence the equation of the tetrad is

$$(at^2+bt'^2)(a't^2+b't'^2)=0,$$

which is of the form $$a_0t^4+6a_2t^2t'^2+a_4t'^4=0.$$

This may be further simplified by putting $a_0^{\frac{1}{4}}t=\xi$, $a_4^{\frac{1}{4}}t'=\eta$; then we have

$$f\equiv \xi^4+6\kappa\xi^2\eta^2+\eta^4=0.$$

**Examples.**

1. Prove that the sextic covariant of the tetrad $f\equiv x^4+6\kappa x^2y^2+y^4=0$ consists of the three dyads $xy=0$, $x^2-y^2=0$, $x^2+y^2=0$.

2. Show that each of the three pairs of points of the sextic covariant harmonically separates the other two pairs.

**22. The Hessian of a tetrad.** The first polar or polar-triad of the point $(u, u')$ with regard to the tetrad $f$ is

$$u\frac{\partial f}{\partial t}+u'\frac{\partial f}{\partial t'}=0. \qquad \text{.......}(1)$$

Let us find the condition that two of the points of the polar-triad may be coincident. We know that the equation $f(t)=0$ has two roots equal if $\dfrac{\partial f}{\partial t}$ vanishes also. Hence the condition that (1), as an equation in $t$, should have two roots equal is that (1) should vanish simultaneously with the equation

$$u\frac{\partial^2 f}{\partial t^2}+u'\frac{\partial^2 f}{\partial t\,\partial t'}=0. \quad\text{.....................................(2)}$$

Similarly, considering (1) as an equation in $t'$, we have

$$u\frac{\partial^2 f}{\partial t\,\partial t'}+u'\frac{\partial^2 f}{\partial t'^2}=0. \quad\text{.....................................(3)}$$

Eliminating $u$, $u'$ between (2) and (3), we have

$$\begin{vmatrix} \dfrac{\partial^2 f}{\partial t^2} & \dfrac{\partial^2 f}{\partial t\,\partial t'} \\ \dfrac{\partial^2 f}{\partial t\,\partial t'} & \dfrac{\partial^2 f}{\partial t'^2} \end{vmatrix}=0,$$

which represents the Hessian of $f$. Writing it out in full, we have

$$H\equiv\begin{vmatrix} a_0t^2+2a_1tt'+a_2t'^2 & a_1t^2+2a_2tt'+a_3t'^2 \\ a_1t^2+2a_2tt'+a_3t'^2 & a_2t^2+2a_3tt'+a_4t'^2 \end{vmatrix}=0.$$

*The Hessian therefore consists of the four points whose first polars contain two coincident points.*

Let us find the common tangents of the conic $F$ (§ 16) and the base-conic $S$. These can be best found by using line-coordinates. Since the tangent to $S\equiv y^2-4zx=0$ at $t$ is $x-ty+t^2z=0$, we can take as parametric line-equations of $S$, or, as we shall call it when it is considered as an envelope, $\Sigma$,

$$l=1,$$
$$m=-t,$$
$$n=t^2,$$

the parameter of the tangent being the same as that of the point of contact.

The line-equation of $F$ can be written in the form

$$\Phi\equiv\begin{vmatrix} a_0 & a_1 & a_2 & l \\ a_1 & a_2 & a_3 & m \\ a_2 & a_3 & a_4 & n \\ l & m & n & 0 \end{vmatrix}=0.$$

Then the parameters of the four common tangents of $\Sigma$ and $\Phi$, or of their points of contact, are the roots of the equation

$$\begin{vmatrix} a_0 & a_1 & a_2 & 1 \\ a_1 & a_2 & a_3 & -t \\ a_2 & a_3 & a_4 & t^2 \\ 1 & -t & t^2 & 0 \end{vmatrix}=0.$$

Multiply the second column of this determinant by $t$ and add to the third, and multiply the first column by $t$ and add to the second, and we get

$$\begin{vmatrix} a_0t+a_1 & a_1t+a_2 & 1 \\ a_1t+a_2 & a_2t+a_3 & -t \\ a_2t+a_3 & a_3t+a_4 & t^2 \end{vmatrix}=0.$$

Treat the rows in the same way, and we get finally

$$\begin{vmatrix} a_0t^2+2a_1t+a_2 & a_1t^2+2a_2t+a_3 \\ a_1t^2+2a_2t+a_3 & a_2t^2+2a_3t+a_4 \end{vmatrix}=0,$$

which is identical with the Hessian $H$. Hence *the Hessian of a tetrad consists of the points of contact of the common tangents to the base-conic and the conic-locus through the given tetrad which is apolar to the base conic-envelope.*

**23.** If we go through the same process, taking any conic of the pencil $F+\lambda S=0$, the equation for the parameters of the points of contact of the four common tangents to this conic and $S$ is found to be

$$\begin{vmatrix} a_0t^2+2a_1t+a_2+\lambda & a_1t^2+2a_2t+a_3-\lambda t \\ a_1t^2+2a_2t+a_3-\lambda t & a_2t^2+2a_3t+a_4+\lambda t^2 \end{vmatrix}=0,$$

which reduces to $$H+\lambda f=0.$$

This then represents a linear system of tetrads on the base-conic. When $\lambda$ is a root of the reducing cubic the conic $F+\lambda S$ breaks up into two straight lines; the common tangents of this degenerate conic and the base-conic are then simply the tangents to the base-conic from the point of intersection of the line-pair. The four points of contact then coincide in pairs, and for such values of $\lambda$ the expression $H+\lambda f$ must be a perfect square. We have

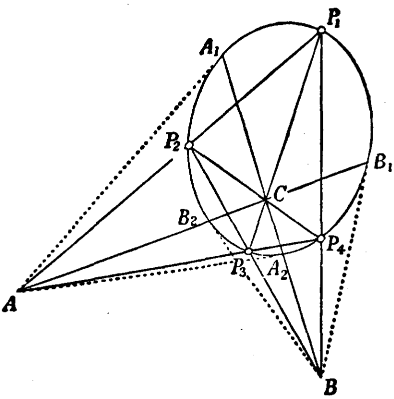



FIG. 115.

then three pairs of points, the points of contact of tangents to the base-conic from the points of intersection of the pairs of joins of the given tetrad. The parameters of these six points, which form the *Sextic Covariant* of the tetrad, are found by eliminating $\lambda$ between the equation $H+\lambda f=0$ and the reducing cubic, or we may express the covariant $T$ thus, noting that each factor is a perfect square,

$$T^2 \equiv 4(H+\lambda_1 f)(H+\lambda_2 f)(H+\lambda_3 f)$$
$$\equiv 4H^3 - IHf^2 + Jf^3 = 0,$$

where $\lambda_1$, $\lambda_2$, $\lambda_3$ are the roots of the reducing cubic.

In Fig. 115, $P_1P_2P_3P_4$ is the given tetrad and $A$, $B$, $C$ the points of intersection of the three pairs of chords. $A_1A_2$, $B_1B_2$, $C_1C_2$ (the latter imaginary), the points of contact of tangents from $A$, $B$, $C$ to the conic, form the sextic covariant, which may also be said to consist of the double points of the involutions determined by the four given points taken in different combinations.

The triangle $ABC$ is self-conjugate with regard to the conic; hence the three pairs of points of the sextic covariant, $A_1A_2$, $B_1B_2$, $C_1C_2$ are mutually harmonic.

Since the triangle $ABC$ is also self-conjugate with regard to every conic of the pencil $F+\lambda S$, it follows that every tetrad of the pencil $H+\lambda f$ has the same sextic covariant. The Hessians of these tetrads are, of course, different, but they form the same pencil of tetrads.

**24. Apolarity of two tetrads.** Two quartics

$$f \equiv a_0t^4+4a_1t^3+6a_2t^2+4a_3t+a_4=0,$$
$$g \equiv b_0t^4+4b_1t^3+6b_2t^2+4b_3t+b_4=0$$

are said to be apolar when their coefficients are connected by the lineo-linear relation

$$a_0b_4-4a_1b_3+6a_2b_2-4a_3b_1+a_4b_0=0. \quad \text{......................(1)}$$

This relation may be interpreted geometrically in various ways.

(1) Consider the pencil of conics

$$F+\lambda S=0,$$

determined by the four points of $f$, and the range of conics

$$\Psi'+\mu\Sigma=0,$$

determined by the four tangents of $g$; $F$ (or $\Phi$) being the conic-locus (or envelope) through $f$, which is apolar to the base-conic $\Sigma$, and $\Psi'$ (or $G'$) the conic-envelope (or locus) touching the tangents at $g$, which is apolar to the base-conic $S$. The condition that one of each should be apolar is

$$a_0b_4+(a_2+\lambda)(4b_2+\mu)+a_4b_0-4a_3b_1+(a_2-2\lambda)(2b_2-\mu)-4a_1b_3=0,$$

*i.e.*
$$a_0b_4-4a_1b_3+6a_2b_2-4a_3b_1+a_4b_0+3\lambda\mu=0.$$

This is satisfied if (1) is true and if either $\lambda$ or $\mu$ vanishes. Hence *if $f$ and $g$ are apolar every conic-locus of the pencil $F+\lambda S$ is apolar to the conic-envelope $\Psi'$, and every conic-envelope of the range $\Psi'+\mu\Sigma$ is apolar to the conic-locus $F$.* Moreover, the relation is symmetrical, so that *every conic-envelope of the range $\Phi'+\lambda\Sigma$ is apolar to the conic-locus $G$, and every conic-locus of the pencil $G+\mu S$ is apolar to the conic-envelope $\Phi'$.*

(2) Consider the locus of points $P\equiv(x', y', z')$ whose polars with respect to the two conics $F$ and $G$, which pass respectively through $f$ and $g$ and are apolar to the base-conic $\Sigma$, are conjugate with regard to the base-conic.

The polars of $P$ with respect to $F$ and $G$ are

$$x(a_0x'+a_1y'+a_2z')+y(a_1x'+a_2y'+a_3z')+z(a_2x'+a_3y'+a_4z')=0,$$
$$x(b_0x'+b_1y'+b_2z')+y(b_1x'+b_2y'+b_3z')+z(b_2x'+b_3y'+b_4z')=0.$$

These are conjugate with regard to the base-conic $\Sigma \equiv m^2 - nl$ if

$$2(a_1x' + a_2y' + a_3z')(b_1x' + b_2y' + b_3z') = (a_0x' + b_1y' + a_2z')(b_2x' + b_3y' + b_4z') + (a_2x' + a_3y' + a_4z')(b_0x' + b_1y' + b_2z').$$

Hence the locus of $P$ is a conic. This conic will be apolar to the base-conic $\Sigma$ if

$$2(a_1b_3 - 2a_2b_2 + a_3b_1) - (a_0b_4 + a_2b_2 + a_2b_2 + a_4b_0 - 2a_1b_3 - 2a_3b_1) = 0,$$

*i.e.* if

$$a_0b_4 - 4a_1b_3 + 6a_2b_2 - 4a_3b_1 + a_4b_0 = 0.$$

Hence, *if $f$ and $g$ are apolar, the conic-locus of points, whose polars with respect to the conics $F$ and $G$ are conjugate with respect to the base-conic, is apolar to the base-conic $\Sigma$.*

Reciprocally, *if $f$ and $g$ are apolar, the conic-envelope of lines, whose polars with respect to the conics $\Phi'$ and $\Psi'$ (the conics which touch the tangents to the base-conic at $f$ and $g$ respectively and are apolar to the base-conic $S$) are conjugate with respect to the base-conic, is apolar to the base-conic $S$.*

**Examples.**

1. If $\Phi'$ and $\Psi'$ are two conic-envelopes which are both apolar to the conic-locus $F$, prove that the points of intersection of $F$ and $\Phi'$ and the points of contact on $\Phi'$ of the common tangents to $\Psi'$ and $\Phi'$ form two apolar tetrads on $\Phi'$.

2. Prove that the first polar of the point $u$ with regard to the tetrad $f$ is the triad

$$u(a_0t^3 + 3a_1t^2 + 3a_2t + a_3) + (a_1t^3 + 3a_2t^2 + 3a_3t + a_4) = 0.$$

3. Prove that the second polar of $u$ with regard to $f$ is the point-pair

$$t^2(a_0u^2 + 2a_1u + a_2) + 2t(a_1u^2 + 2a_2u + a_3) + (a_2u^2 + 2a_3u + a_4) = 0.$$

4. Prove that the third polar of $u$ with regard to $f$ is the point

$$t(a_0u^3 + 3a_1u^2 + 3a_2u + a_3) + (a_1u^3 + 3a_2u^2 + 3a_3u + a_4) = 0.$$

5. Prove that any point and its first polar group with regard to a tetrad $f$ form a tetrad apolar to $f$.

6. Show that each polar, of any order, of a point $t_1$ of a tetrad $f$, with regard to $f$, contains the point $t_1$.

7. Prove that the second polar of one of the Hessian points of a tetrad $f$, with regard to $f$, consists of two coincident points, and that the third polar coincides with this coincident pair.

8. Prove that each Hessian point forms with its first polar group an equianharmonic tetrad.

9. Prove that each of the elements of the sextic covariant $T$ has its third polar coincident with one of the three points of the first polar, and forms with these three a harmonic tetrad.

10. Prove that each pair of elements of $T$ forms the double points of the involution determined by the other two pairs.

11. If the harmonic triangle $ABC$ is taken as triangle of reference, so that the parametric equations of the base-conic are $x = t^2 - 1$, $y = 2t$, $z = t^2 + 1$, show that the parameters of the tetrad can be taken to be $(t, -t, 1/t, -1/t)$.

## EXAMPLES XIX.

1. If a conic $F$ is apolar to the base-conic, both as a locus and as an envelope, show that $F$ cuts the base-conic in an equianharmonic tetrad.

2. If $f \equiv a_0t^3 + 3a_1t^2 + 3a_2t + a_3 = 0$ and $g \equiv t^3 + 1 = 0$ are two triads, show that the Hessian of the triad $f + \lambda g$ is $H_{11} + H_{12}\lambda + H_{22}\lambda^2$, where $H_{11}$ and $H_{22}$ are the Hessians of $f$ and $g$, and $H_{12} \equiv a_2t^2 + (a_0 + a_3)t + a_1$, a joint-covariant of $f$ and $g$.

3. Prove that the polar-dyads of either of the points $H_{12}$, with respect to the two triads $f$ and $g$, are mutually harmonic.

4. Prove that the Jacobians of all pairs of cubics of a linear system are the same.

5. Show that there are four values of $\lambda$ for which the Hessian of the cubic $f + \lambda g$ consists of two coincident points, and that these four double-point Hessians form the Jacobian of $f$ and $g$.

6. Show that the line-equation of the conic, which touches the tangents to the base-conic at the four points forming the Jacobian tetrad of the two triads $f \equiv a_0t^3 + 3a_1t^2 + 3a_2t + a_3$ and $g \equiv t^3 + 1$, and which touches the line joining one pair of points of one of the tetrads, say $t = -\omega$ and $t = -\omega^2$, is

$$a_2l^2 + \tfrac{4}{3}(a_0 - a_3)m^2 - a_1n^2 + 2a_2mn - \tfrac{1}{3}(a_0 - a_3)nl - 2a_1lm = 0\,;$$

and that this conic touches also all the sides of the two triangles formed by the two triads.

7. Show that the triangle formed by the three points of any triad of the linear system $f + \lambda g$ is circumscribed about a fixed conic.

8. Show that every triad of a linear system is a self-conjugate triangle with respect to a fixed conic which cuts the base-conic in the Jacobian of the linear system. In particular, for the linear system determined by the two triads $f \equiv a_0t^3 + 3a_1t^2 + 3a_2t + a_3$, $g \equiv t^3 + 1$, show that the fixed conic is

$$a_1x^2 + \tfrac{1}{12}(a_3 - a_0)y^2 - a_2z^2 - a_1yz + \tfrac{2}{3}(a_3 - a_0)zx + a_2xy = 0.$$

9. Show that the Jacobian of any two members of a linear system of quartics (or quantics of any degree) is the same.

10. Show that there are six values of $\lambda$ for which the tetrad $f + \lambda g$ has two points coincident, and that these six double-points form the Jacobian of the tetrads $f$ and $g$.

11. Show that the Jacobian of the two tetrads $t^4 + 6at^2 + 1 = 0$ and $t^4 + 6bt^2 + 1 = 0$ consists of the points in which the sides of their common harmonic triangle cuts the base-conic.

12. If $F$ and $G$ are the conic-loci through the tetrads $f$ and $g$ apolar to the base-conic envelope, prove that the locus of a point whose polars with respect to $F$ and $G$ are conjugate to the base-conic is a conic $U$, and that if $f$ and $g$ are apolar the conic-locus $U$ is apolar to the base-conic envelope.

13. Prove that the equation of the polar of the point $(x', y', z')$ with respect to the triangle formed by the triad $a_0t^3 + 3a_1t^2 + 3a_2t + a_3 = 0$ on the base-conic is

$$\begin{aligned} 8\{x(a_0x' + a_1y' + a_2z')^2 + y(a_0x' + a_1y' + a_2z')(a_1x' + a_2y' + a_3z') + z(a_1x' + a_2y' + a_3z')^2\} \\ + (y'^2 - 4z'x')\{2(a_0a_2 - a_1^2)x + (a_0a_3 - a_1a_2)y + 2(a_1a_3 - a_2^2)z\} \\ - (2yy' - 4z'x - 4zx')\{2(a_0a_2 - a_1^2)x' + (a_0a_3 - a_1a_2)y' + 2(a_1a_3 - a_2^2)z'\} = 0. \end{aligned}$$

14. Prove that the locus of a point such that its polars with respect to the triangles of the two triads $f$ and $g$ on the base-conic are conjugate with regard to the conic is a curve of the fourth degree which cuts the base-conic in the six points $f$ and $g$ and also in the two points $H_{12}$ of Ex. 2.

15. Show that the centre of the involution-range determined by four points whose abscissae are given by the equation $a_0x^4+4a_1x^3+6a_2x^2+4a_3x+a_4=0$ has for its abscissae $\frac{1}{2}(a_0a_3-a_1a_2+2a_1\lambda)/(a_1^2-a_0a_2+a_0\lambda)$, where $\lambda$ is a root of the reducing cubic $4\lambda^3-I\lambda-J=0$. (Trinity, 1911.)

# CHAPTER XX.

## INVARIANTS.

**1. Orthogonal or congruent transformations.** When a curve is referred first to one set of rectangular axes $Ox$, $Oy$ and then to another $O'x'$, $O'y'$, its equation in the first case is transformed into another equation in accordance with equations of the form (Chap. VIII. § 14).

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1, \\ y' &= l_2 x + m_2 y + n_2, \end{aligned}\right\} \qquad \text{(1)}$$

where

$$\left.\begin{aligned} l_1^2 + m_1^2 &= 1, \\ l_2^2 + m_2^2 &= 1, \\ l_1 l_2 + m_1 m_2 &= 0. \end{aligned}\right\} \qquad \text{(2)}$$

These equations represent a *transformation of rectangular axes* or an *orthogonal transformation.*

This can be considered also from another point of view. Instead of supposing that we are dealing with one and the same figure, referred to different axes, we may consider that we have two different figures referred to the same axes; and evidently in this case the two figures are equal in all respects or congruent. The equations (1), with the conditions (2), therefore represent a displacement of a figure from one position to another. From this point of view the transformation is called a *congruent transformation.*

**2. General linear transformation.** Leaving out the conditions (2), we have the general form of transformation between oblique axes. Still more generally, the equations

$$\left.\begin{aligned} x' &= l_1 x + m_1 y + n_1 z, \\ y' &= l_2 x + m_2 y + n_2 z, \\ z' &= l_3 x + m_3 y + n_3 z \end{aligned}\right\} \qquad \text{(3)}$$

represent the general transformation in homogeneous coordinates from one triangle of reference to another (Chap. XII. § **18**). This is called the *general linear transformation.*

We can consider this also from another point of view. If we suppose that we have always the same triangle of reference, the equations (3) represent a transformation by which a given figure is changed into another figure, just as one circle is changed into a different circle by the transformation of inversion.

**3. Projection.** Consider two separate planes $\Pi$ and $\Pi'$, and a point $O$ not on either of them. Then if $P$ is any point on the plane $\Pi$, a unique corresponding point $P'$ on $\Pi'$ is obtained as the point where $OP$ cuts the plane $\Pi'$. $P'$ is called the projection of $P$ on the plane $\Pi$, and $O$ is the centre of projection. Any figure in $\Pi$ is thus projected into a corresponding figure in $\Pi'$. In particular, *any straight line $l$ in $\Pi$ is projected into a straight line $l'$ in $\Pi'$*, for all points on the projection lie in the plane $Ol$, and also in the plane $\Pi'$, and therefore lie in the line of intersection of these two planes.

If $OP$ is parallel to the plane $\Pi'$, the point $P'$ which corresponds to $P$ will be at infinity, and if the plane $Oa$ is parallel to the plane $\Pi'$, all the points on the corresponding line $a'$ will be at infinity; thus to the particular straight line $a$ in $\Pi$ corresponds the line at infinity in $\Pi'$. In this way the line at infinity may be projected into any ordinary line, and from the point of view of projection there is no distinction between the line at infinity and any ordinary line.

A system of concurrent lines through a point $P$ is projected into a system of concurrent lines through the point $P'$, the projection of $P$. Thus concurrency and collinearity are unaltered by projection.

A curve in $\Pi$ which is cut by a straight line $l$ in $n$ points is projected into a curve in $\Pi'$, which is cut by the corresponding straight line $l'$ in $n$ corresponding points. Hence the degree of a curve is unaltered by projection. Similarly, since a tangent to the curve in $\Pi$ is projected into a tangent to the corresponding curve in $\Pi'$, the class of a curve will also be unaltered. Thus a conic is projected into a conic. An ellipse may be projected into another ellipse, or into a hyperbola or a parabola, according as the line which is projected into the line at infinity cuts the ellipse in imaginary points, real points or coincident points. Thus from the point of view of projection there is no distinction between the three types of conics. A degenerate conic, however, is projected into a similarly degenerate conic, *i.e.* two straight lines into two straight lines, and two coincident lines into two coincident lines.

Lengths and angles will generally be altered, *i.e.* metrical properties are not preserved by projection. The cross-ratio of four collinear points or of four concurrent lines is unaltered, however, because there is a (1, 1) correspondence between the points and also between the lines of the two planes $\Pi$ and $\Pi'$.

**4. Projective transformation in one plane.** We can now suppose the plane $\Pi'$ to be moved into coincidence with the plane $\Pi$, and then we have two figures in the same plane such that to any point $P$ in the one figure corresponds a unique point $P'$ in the other, and to a straight line $l$ through $P$ corresponds a unique straight line $l'$ through $P'$.

**5. Perspective.** There is a special type of projective transformation between two coincident planes $\Pi$ and $\Pi'$, which can be obtained by projecting one and the same figure in another plane $\Pi_1$ with different centres of projection. Let $S$ and $S'$ be the two centres of projection and $l$ the line of intersection of the plane $\Pi_1$ with $\Pi$ or $\Pi'$, and let $SS'$ cut $\Pi$ in $O$. Then starting with the point $P$ in $\Pi$, join $SP$ cutting $\Pi_1$ in $P_1$, join $S'P_1$ cutting $\Pi'$ in $P'$. We thus obtain a (1, 1) point-correspondence between $\Pi$ and $\Pi'$. Since $S$, $S'$, $P_1$, $P$, $P'$ all lie in the same plane, which cuts the plane $\Pi$ in $PP'$, it follows that $PP'$ passes through $O$ Hence *the line joining a pair of corresponding points $P$, $P'$ passes through a fixed point $O$.*

Let $a$ be any line in $\Pi$. Then the plane $Sa$ cuts $\Pi_1$ in a line $a_1$, and the plane $S'a_1$ cuts $\Pi'$ in the corresponding line $a'$. The planes $Sa$, $S'a'$, and $\Pi_1$ have the line $a_1$ in common, and this cuts the plane $\Pi$ in the point $A$ in which $a$ and $a'$ intersect. But this point lies on the line of intersection of the planes $\Pi_1$ and $\Pi$, *i.e.* $l$. Hence *the point of intersection of a pair of corresponding lines $a$, $a'$ lies on a fixed straight line $l$.*

This transformation is called the *perspective transformation.* $O$ is called the *centre* and $l$ the *axis of perspective.* It is completely specified, without going outside the plane $\Pi$ or $\Pi'$, when the centre and axis and a pair of corresponding points $A$, $A'$, or a pair of corresponding lines $a$, $a'$, are given.

**6. Equations of the perspective transformation.** To find the equations of the perspective transformation in the simplest form, take the centre of perspective $O$ as vertex of the triangle of reference, and the axis $l$ as the opposite side. Take any two points $X$, $Y$ on $l$ as the remaining vertices. Let $A \equiv (1, 0, a)$ be any point of the first figure lying on $OX$; then the corresponding point $A'$ lies also on $OX$, so that $A' \equiv (1, 0, a')$. Now let $P \equiv (x, y, z)$ and $P' \equiv (x', y', z')$ be any pair of corresponding points. The two sets of coordinates are then connected by two equations.

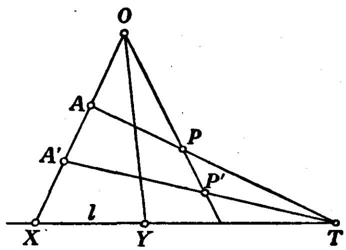


FIG. 116.

The first is found by the condition that $PP'$ passes through $O$. This gives

$$xy' = x'y. \qquad \text{.......(1)}$$

The second equation is found by the condition that $AP$ and $A'P'$ intersect on $l$. Taking $X$, $Y$, $Z$ as current coordinates, the equations of $AP$ and $A'P'$ are

$$\begin{vmatrix} X & Y & Z \\ 1 & 0 & a \\ x & y & z \end{vmatrix} = 0 \quad \text{and} \quad \begin{vmatrix} X & Y & Z \\ 1 & 0 & a' \\ x' & y' & z' \end{vmatrix} = 0.$$

Putting $Z = 0$, and equating the ratios of $X : Y$, we have

$$\frac{ay}{ax - z} = \frac{a'y'}{a'x' - z'}. \qquad \text{.......(2)}$$

By (1) this reduces to $\qquad ayz' = a'y'z.$

Hence the equations of transformation are

$$\left.\begin{aligned}\lambda x' &= x,\\ \lambda y' &= y,\\ \lambda z' &= kz,\end{aligned}\right\}$$

where $k = a'/a$. This ratio is equal to the cross-ratio $(OX, AA')$, and is called the cross-ratio of the perspective. When it has the value $-1$, the transformation is involutory, *i.e.* symmetrical, and is called *harmonic perspective*.

**Examples.**

1. If the line joining a pair of corresponding points $P, P'$ cuts the axis of perspective in $U$, prove that the cross-ratio $(OU, PP')$ is constant, $=k$.

2. If $u$ is the line joining $O$ to the point of intersection of a pair of corresponding lines $p, p'$, prove that the cross-ratio $(lu, pp')$ is constant, $=k$.

**7. Equations of the general projective transformation.** Let the figure $\Pi$ be transformed first by perspective into the figure $\Pi_1$, and then let $\Pi_1$ be moved rigidly in its plane into the figure $\Pi'$. We have then a transformation of the figure $\Pi$, through $\Pi_1$, into $\Pi'$. Let $\Pi$ be referred to the triangle $OXY$, $O$ being the centre of perspective of $\Pi$ and $\Pi_1$, and $X$, $Y$ any two points on the axis of perspective. Then $O$, $X$, $Y$ are the same points in $\Pi_1$. Let $O'$, $X'$, $Y'$ be the corresponding points in $\Pi'$, so that the triangles $OXY$ and $O'X'Y'$ are congruent. Let

$$P \equiv (x, y, z), \quad P_1 \equiv (x_1, y_1, z_1) \quad \text{and} \quad P' \equiv (x', y', z')$$

be three corresponding points of $\Pi$, $\Pi_1$ and $\Pi'$, all referred to the same triangle $OXY$. The coordinates of $P'$ referred to $O'X'Y'$ are the same as those of $P_1$ referred to $OXY$; hence $(x_1, y_1, z_1)$ are connected with $(x', y', z')$ by the equations of transformation of coordinates from the triangle $O'X'Y'$ to the triangle $OXY$, viz.

$$\begin{aligned}x' &= a_1x_1 + b_1y_1 + c_1z_1,\\ y' &= a_2x_1 + b_2y_1 + c_2z_1,\\ z' &= a_3x_1 + b_3y_1 + c_3z_1.\end{aligned}$$

Also the coordinates of $P$ and $P_1$ are connected by the equations

$$\lambda x_1 = x, \quad \lambda y_1 = y, \quad \lambda z_1 = kz.$$

Hence the coordinates of $P$ and $P'$ are connected by the equations

$$\begin{aligned}x' &= l_1x + m_1y + n_1z,\\ y' &= l_2x + m_2y + n_2z,\\ z' &= l_3x + m_3y + n_3z,\end{aligned}$$

where $l, m, n$ are written instead of $a/\lambda$, $b/\lambda$, $kc/\lambda$.

**8.** The general equations of the projective transformation are therefore of the same form as the equations of transformation of coordinates, or the congruent transformation. The only difference is that in the latter the coefficients must be connected by certain relations which express either that the two figures are congruent (when the triangles of reference are the same), or that the two triangles of reference are congruent (when the figures are the same), while in the general transformation there are no such limitations.

Now we have obtained a linear transformation by combining a perspective transformation with a congruent transformation. To see that this does give us the most general projective transformation, let us count up the constants at our disposal. The general linear transformation contains 9 constants, but since only the ratios of the coordinates are in question, only the ratios of the 9 constants are significant. Hence the general linear transformation depends upon 8 constants. The perspective transformation depends upon 5 constants, 2 for the coordinates of the centre, 2 for those of the axis, and 1 for the constant cross-ratio. And a congruent transformation, which is determined by moving a specified point into a given position and then rotating the whole figure through a certain angle, depends upon 3 constants, 2 for the translation and 1 for the rotation. Hence we have exactly the number of constants required, and therefore our equations represent the general projective transformation.

**9.** *Any conic can be projected into a circle with unit radius, and at the same time any point into the centre of the circle.*

Let $C$ be any conic and $O$ any point, and let $l$ be the polar of $O$. Then $l$ cuts the conic in two points $A$, $B$. If now the figure is projected so that $l$ becomes the line at infinity, $O$ is projected into the centre of the new conic; and if further the two points $A$, $B$ can be projected into the circular points the new conic will be a circle.

We shall suppose first that $O$ lies within the conic, so that $A$, $B$ are conjugate imaginary points. Choose a self-conjugate triangle of reference, so that $O$ and $l$ are a vertex and the opposite side. The equation of the conic is then of the form
$$ax^2+by^2-cz^2=0,$$
and $l$ is
$$z=0.$$
The conic being supposed to be real and the tangents from $(0, 0, 1)$ imaginary, we can assume that $a$, $b$, $c$ are all positive. In the projection take homogeneous rectangular cartesian coordinates, and let the origin $O'$ be at the centre of the circle. Then the line at infinity is $z'=0$. Further, the two lines $OA$, $OB$ are to be transformed into the absolute lines $O'I$, $O'J$, *i.e.*
$$ax^2+by^2=0 \quad \text{into} \quad x'^2+y'^2=0.$$
This can be done by putting
$$\sqrt{a}\,.\,x=x',$$
$$\sqrt{b}\,.\,y=y',$$
$$\sqrt{c}\,.\,z=z',$$
and the equation of the conic is transformed into $x'^2+y'^2=1$, which represents a circle of unit radius.

The equations of transformation are real since $a$, $b$, $c$ are all positive. But if $O$ is outside the conic, so that $A$, $B$ are real, then $a$ and $b$ are of opposite sign; $x'$ and $y'$ are then not both real. In this case we have still linear equations of transformation, but they involve imaginary numbers, and the transformation is imaginary. Real points and lines are in general transformed into imaginary points and lines. For the purpose of generalizing or simplifying theorems, and for the sake of continuity, we do not reject such imaginary transformations, but consider them as on the same footing, analytically, as real transformations.

**Examples.**

1. Show that any two conics can be projected into (i) two circles, (ii) two confocal conics, (iii) two parabolas with a common focus.

2. Show that two conics which have double contact can be projected into concentric circles.

3. A variable line through a fixed point $O$ cuts a given conic $S$ in $X$ and $Y$, and on this line is taken a point $P$ such that the cross-ratio $(OP, XY)$ is equal to a given constant. Find the locus of $P$.

4. From a variable point $P$ on a fixed line $l$ tangents $t, t'$ are drawn to a given conic, and a line $u$ is taken so that the cross-ratio $(tt', lu)$ is equal to a given constant. Find the envelope of $u$.

5. $A$ and $B$ are a pair of opposite vertices of the common circumscribed quadrilateral of two conics, and $P$ is one of their points of intersection. Prove that the tangents at $P$ are harmonic conjugates with regard to $PA$, $PB$.

**10. Metrical invariants of a conic.** We have seen in Chap. VIII. that when we transform from one system of rectangular coordinates to another, the expression

$$S \equiv ax^2 + by^2 + 2hxy + 2gx + 2fy + c \quad \text{.........(1)}$$

is transformed into another expression of the same form,

$$S' \equiv a'x'^2 + b'y'^2 + 2h'x'y' + 2g'x' + 2f'y' + c', \quad \text{.........(2)}$$

and that

$$\left.\begin{aligned} a' + b' &= a + b, \\ a'b' - h'^2 &= ab - h^2. \end{aligned}\right\} \quad \text{.........(3)}$$

$a + b$ and $ab - h^2$ are therefore invariants for the *orthogonal linear transformation*. Since we may also view this transformation as a transformation changing one figure into another, while keeping the axes fixed, and since from this point of view it is a *congruent* transformation, the conics $S = 0$ and $S' = 0$ only differ in position; they have the same eccentricity, the same lengths of axes, the same latus-rectum, and so on. These geometrical magnitudes are therefore invariants.

The actual equality of the expressions (3) is not of much significance geometrically, for if we only know that the conic $S' = 0$ is the conic $S = 0$ transformed by a congruent transformation, we cannot be sure that the actual values of the transformed coefficients $a'$, $b'$, etc., have been preserved, but only their ratios. If after the transformation the equation $S' = 0$ had been multiplied all through by $M$, then we should have

$$\left.\begin{aligned} a' + b' &= M(a + b). \\ a'b' - h'^2 &= M^2(ab - h^2). \end{aligned}\right\} \quad \text{.........(4)}$$

These are the significant equations rather than the equations (3).

To obtain an *absolute invariant*, which has a fixed value so long as the equation represents the same conic referred to any rectangular axes, whether the equation is multiplied or divided by any factor, we must form an expression of zero dimensions in the coefficients. Thus the ratio

$$(ab-h^2)/(a+b)^2$$

is an absolute invariant, and this has a definite geometrical significance; if $\varphi$ is the angle between the asymptotes it represents, in fact, the geometrical invariant $-\frac{1}{4}\tan^2\varphi$.

A conic is determined, without regard to position, when the lengths of its two axes are given. There are therefore just two independent geometrical invariants, and there ought to be two independent absolute invariants under the congruent transformation. Recalling the expressions for the lengths of the axes (Chap. X. § 12), we see that these involve in addition to $a+b$ and $ab-h^2$, also the value of the discriminant $\Delta$. $\Delta$ involves the coefficients of terms of the first degree and does not depend only on $a+b$ and $ab-h^2$. We ought therefore to have $\Delta$ also as an invariant, or, since it is of the third degree in the coefficients,

$$\Delta' = M^3\Delta.$$

Assuming this at present without proof, we see then that under the orthogonal transformation the expression $S$ has three invariants

$$a+b, \quad ab-h^2 \equiv C, \quad \text{and} \quad \Delta.$$

All measurements relating to the conic can be expressed in terms of these three invariants, *i.e.* every geometrical invariant can be expressed as a function of these, and, moreover, it will be a homogeneous expression of zero dimensions since it cannot be changed by multiplying all the coefficients by a common factor. The *absolute invariants* are therefore the ratios of the three invariants

$$I \equiv a+b, \quad J \equiv (ab-h^2)^{\frac{1}{2}}, \quad K \equiv \Delta^{\frac{1}{3}}.$$

**Examples.**

1. Prove that the area of the conic $=\pi(KJ^{-1})^3$.

2. Show that the product of the squares of the eccentricities $=4-(IJ^{-1})^2$.

3. Show that the tangent of the angle between the asymptotes $=2i(JI^{-1})$.

4. Show that the area of the triangle formed by the asymptotes and any tangent $=i(KJ^{-1})^3$.

5. Prove that the length of one of the equiconjugate diameters

$$=(-2K^3IJ^{-4})^{\frac{1}{2}}.$$

**11. Projective invariant of a conic.** We shall consider next the effect upon a conic of the general linear transformation, or the transformation to any other triangle of reference. Geometrically this is equivalent to a projection. Now any conic can be projected into any other conic, for we have shown (§ 9) that by a real transformation any conic can be projected into a circle with arbitrary radius and centre, and in such a way that any point in its interior is transformed into the centre of the circle, a second conic can then, by another real transformation, be projected into the same circle, and any point in its interior into the centre of the circle. Hence the only thing which is left unaltered by a *real* transformation is the distinction between the interior and the exterior of the conic, and the characteristic of being a proper conic. Both of these characters depend upon the value of the discriminant $\Delta$; hence we should expect that $\Delta$ is an invariant, and the only projective invariant of the conic.

12. We shall now prove that $\Delta$ is an invariant under the linear transformation,

$$x = l_1x' + m_1y' + n_1z',$$
$$y = l_2x' + m_2y' + n_2z',$$
$$z = l_3x' + m_3y' + n_3z'.$$

Let
$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy$$
$$\equiv x(ax + hy + gz) + y(hx + by + fz) + z(gx + fy + cz).$$

Then by the transformation $S$ is changed into

$$S' \equiv a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y'.$$

Write for shortness
$$al_1 + hl_2 + gl_3 = L_a,$$
$$hl_1 + bl_2 + fl_3 = L_b,$$
$$gl_1 + fl_2 + cl_3 = L_c,$$

and similar expressions for $M_a$, $N_a$, etc.

Then
$$ax + hy + gz = L_ax' + M_ay' + N_az', \text{ etc.,}$$

and we find
$$a' = l_1L_a + l_2L_b + l_3L_c,$$
$$f' = m_1N_a + m_2N_b + m_3N_c = n_1M_a + n_2M_b + n_3M_c$$
etc.

Then
$$\Delta' = \begin{vmatrix} a' & h' & g' \\ h' & b' & f' \\ g' & f' & c' \end{vmatrix} = \begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} L_a & L_b & L_c \\ M_a & M_b & M_c \\ N_a & N_b & N_c \end{vmatrix},$$

and the second determinant factor is equal to

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}.$$

Hence, writing the determinant or *modulus* of the transformation

$$\begin{vmatrix} l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \\ l_3 & m_3 & n_3 \end{vmatrix} \equiv D,$$

we have
$$\Delta' = D^2\Delta.$$

$\Delta$ is therefore a projective invariant of the conic. It must be left to geometrical intuition to satisfy us that it is the only projective invariant.

**13.** We can now supply the deferred proof that $\Delta$ is an invariant under the orthogonal transformation. The general transformation is in this case

$$x = lx' - my' + pz',$$
$$y = mx' + ly' + qz',$$
$$z = z',$$

where $$l^2 + m^2 = 1.$$

The determinant of the transformation is

$$D = \begin{vmatrix} l & -m & p \\ m & l & q \\ 0 & 0 & 1 \end{vmatrix} = l^2 + m^2 = 1.$$

Hence, with the orthogonal transformation, $\Delta' = \Delta$.

**14. Projective invariants of a system of two conics.** We shall pass on now to consider the projective invariants of two conics. Two conics have mutual relationships which are unaltered by projection. For example, if the two conics touch in one or in two points, or have contact of a higher order, the transformed conics will touch in the same way. If a triangle can be inscribed in one conic and circumscribed about the other, or a triangle self-conjugate with regard to one conic inscribed or circumscribed about the other, or a pair of common chords conjugate with regard to one of the conics, these relationships must be true also for the transformed conics. Such geometrical properties will be expressed by relations between invariants.

Consider the pencil of conics

$$\lambda S + S' = 0,$$

determined by the two conics $S$, $S'$ whose point- and line-equations are

$$S \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy = 0$$

and $$\Sigma \equiv Al^2 + Bm^2 + Cn^2 + 2Fmn + 2Gnl + 2Hlm = 0,$$

with similar expressions for $S'$ and $\Sigma'$.

Now, since the conics intersect in four points $A$, $B$, $C$, $D$, there are six common chords which, taken in pairs ($AB$, $CD$), ($AC$, $BD$), ($AD$, $BC$), form three degenerate conics of the pencil. Let us find the values of $\lambda$ corresponding to these three pairs of lines. The condition that the conic should break up into two straight lines is

$$\begin{vmatrix} \lambda a + a' & \lambda h + h' & \lambda g + g' \\ \lambda h + h' & \lambda b + b' & \lambda f + f' \\ \lambda g + g' & \lambda f + f' & \lambda c + c' \end{vmatrix} = 0.$$

When this determinant is expanded, we get a cubic equation in $\lambda$, which we shall write

$$\Delta\lambda^3 + \Theta\lambda^2 + \Theta'\lambda + \Delta' = 0.$$

$\Delta$ and $\Delta'$ are the discriminants of the two conics $S$ and $S'$. To evaluate $\Theta$ and $\Theta'$, we have

$$\Theta = \begin{vmatrix} a' & h & g \\ h' & b & f \\ g' & f & c \end{vmatrix} + \begin{vmatrix} a & h' & g \\ h & b' & f \\ g & f' & c \end{vmatrix} + \begin{vmatrix} a & h & g' \\ h & b & f' \\ g & f & c' \end{vmatrix}$$

$$= (a'A + h'H + g'G) + (h'H + b'B + f'F) + (g'G + f'F + c'C)$$

$$= a'A + b'B + c'C + 2f'F + 2g'G + 2h'H.$$

Similarly $\Theta' = aA' + bB' + cC' + 2fF' + 2gG' + 2hH'$.

Suppose now that the conics undergo a projective transformation, so that $S$, $S'$ are changed into $S_1$, $S_1'$; then $\lambda S + S'$ is changed into $\lambda S_1 + S_1'$. $\lambda$ being a constant is of course unchanged. If the conic $\lambda S + S'$ breaks up into two straight lines, the corresponding conic $\lambda S_1 + S_1'$ will also break up, since a pair of straight lines is projected into a pair of straight lines, *i.e.* $\lambda S_1 + S_1'$ will break up into factors for the same values of $\lambda$ as those for which $\lambda S + S'$ reduces. Hence, if $\Delta$, $\Delta'$, $\Theta$, $\Theta'$ are changed into $\Delta_1$, $\Delta_1'$, $\Theta_1$, $\Theta_1'$, the two equations

$$\Delta\lambda^3 + \Theta\lambda^2 + \Theta'\lambda + \Delta' = 0,$$

$$\Delta_1\lambda^3 + \Theta_1\lambda^2 + \Theta_1'\lambda + \Delta_1' = 0$$

are identical. Therefore

$$\frac{\Delta_1}{\Delta} = \frac{\Theta_1}{\Theta} = \frac{\Theta_1'}{\Theta'} = \frac{\Delta_1'}{\Delta'}.$$

But we have already seen that

$$\Delta_1 = D^2\Delta, \qquad \Delta_1' = D^2\Delta';$$

therefore $$\Theta_1 = D^2\Theta, \qquad \Theta_1' = D^2\Theta',$$

$D$ being the determinant or modulus of the transformation.

Hence $\Theta$ and $\Theta'$ are invariants in the same sense as $\Delta$ and $\Delta'$. They are *joint-invariants* of the two conics.

**15. Invariants for the reciprocal system.** In a similar way, if we take the range, or four-line system, of conics

$$\lambda\Sigma + \Sigma' = 0,$$

we get a cubic equation for $\lambda$ as the condition that this conic-envelope may break up into two points. Since $BC - F^2 = a\Delta$, etc., the equation is found to be

$$\Delta^2\lambda^3 + \Delta\Theta'\lambda^2 + \Delta'\Theta\lambda + \Delta'^2 = 0.$$

This shows that when we pass from the point-equations to the line-equations, the invariants corresponding to

$$\Delta, \quad \Theta, \quad \Theta', \quad \Delta',$$

become $$\Delta^2, \quad \Delta\Theta', \quad \Delta'\Theta, \quad \Delta'^2.$$

Hence, if there is a geometrical relation between two conics which is expressed by a relation between the invariants $\Delta$, $\Theta$, $\Theta'$, $\Delta'$, by reciprocation there is a geometrical relation which is expressed by the same function of the invariants $\Delta^2$, $\Delta\Theta'$, $\Delta'\Theta$, $\Delta'^2$.

**16. Apolarity.** The meanings of the invariants $\Delta$ and $\Delta'$ are already known, for $\Delta=0$ implies that the conic $S$ breaks up into two straight lines.

Let us investigate the meaning of the equation $\Theta=0$, or

$$Aa'+Bb'+Cc'+2Ff'+2Gg'+2Hh'=0.$$

This evidently implies some projective relation between the two conics. It connects the conic $S'$ as a locus with the conic $\Sigma$ as an envelope.

Let us take as triangle of reference $XYZ$ a triangle which is self-conjugate with regard to $\Sigma$. The equation of $\Sigma$ then reduces to

$$\Sigma \equiv Al^2+Bm^2+Cn^2=0,$$

and $$S' \equiv a'x^2+b'y^2+c'z^2+2f'yz+2g'zx+2h'xy=0.$$

Then $$\Theta \equiv Aa'+Bb'+Cc'=0.$$

Now we can choose the self-conjugate triangle with one vertex $X$ on the conic $S'$; then $a'=0$. A second vertex lies on the polar of $X$ with regard to $\Sigma$. This cuts $S'$ in two points. Choose one of these as the second vertex $Y$; then $b'=0$. The triangle of reference is now fixed, and $\Theta=Cc'$. Now $C$ cannot vanish, since then $\Sigma$ would reduce to two points and we would have $\Delta=0$. Hence, if $\Theta=0$, $c'=0$, and the third vertex $Z$ also lies on $S'$. Hence $\Theta=0$ *is the necessary and sufficient condition that the conic-locus $S'$ should have an inscribed triangle which is self-conjugate with regard to $\Sigma$.*

Again, let us take as triangle of reference a triangle self-conjugate with regard to $S'$, so that $f'=0$, $g'=0$, $h'=0$. Take the side $x=0$ as a tangent to $\Sigma$, so that $A=0$, and a second side $y=0$ as one of the tangents to $\Sigma$ which passes through the pole of $x=0$, so that $B=0$; then $\Theta=Cc'$. $c'\neq0$ unless $\Delta'=0$; therefore if $\Theta=0$, $C=0$, and the third side $z=0$ also touches $\Sigma$. Hence $\Theta=0$ *is also the necessary and sufficient condition that the conic envelope $\Sigma$ should have a circumscribed triangle which is self-conjugate with regard to $S'$.*

When a conic-locus and a conic-envelope are so related they are said to be *apolar.*

$\Theta=0$, which involves the coefficients of the point-equation of $S'$ and the line-equation of $\Sigma$, is therefore the condition that the conic-locus $S'$ should be apolar to the conic-envelope $\Sigma$, and similarly $\Theta'=0$ is the condition that the conic-locus $S$ should be apolar to the conic-envelope $\Sigma'$. Notice that this relation between two conics is not symmetrical when both conics are considered as loci, and it is always necessary to state which is a locus and which an envelope.

The expressions are also used: $\Sigma$ *is harmonically inscribed in $S'$, and $S'$ is harmonically circumscribed about $\Sigma$.*

It is clear from the preceding reasoning that when $\Sigma$ and $S'$ are apolar there is *an infinity* of triangles inscribed in the conic-locus $S'$ and self-conjugate with regard to $\Sigma$, and an infinity of triangles circumscribed about the conic-envelope $\Sigma$ and self-conjugate with regard to $S'$.

A line-pair locus is apolar to a conic-envelope $\Sigma$ when the two lines are conjugate with regard to $\Sigma$, and a point-pair envelope is apolar to a conic-locus $S$ when the two points are conjugate with regard to $S$. A point-pair and a line-pair are apolar when the line joining the two points is cut harmonically by the two lines.

**17. Reduction of the equations of two conics to the normal form.** The algebraic work of determining the relation between the invariants corresponding to a geometrical condition can be greatly simplified by reducing the equations of the conics to a normal form. When two conics intersect in four distinct points, the harmonic triangle of the complete quadrangle formed by these four points is self-conjugate for both conics, and when this triangle is taken as the triangle of reference, the equations of the conics become

$$S \equiv ax^2 + by^2 + cz^2 = 0,$$
$$S' \equiv a'x^2 + b'y^2 + c'z^2 = 0.$$

These equations may be still further simplified by taking multiples of the coordinates, or by the linear transformation

$$\sqrt{a'} \,.\, x = x', \quad \sqrt{b'} \,.\, y = y', \quad \sqrt{c'} \,.\, z = z'.$$

$S'$ then reduces to the form $x^2 + y^2 + z^2 = 0$. Of course if the original coordinates are real, and if the conic is real, these new coordinates are not all real, but this is of no consequence algebraically. We can therefore take as the normal form of the equations of two conics

$$S \equiv ax^2 + by^2 + cz^2 = 0,$$
$$S' \equiv x^2 + y^2 + z^2 = 0.$$

Then $\Delta = abc$, $\Theta = bc + ca + ab$, $\Theta' = a + b + c$, $\Delta' = 1$. $a$, $b$, $c$ are then the roots of the cubic equation

$$\Delta' t^3 - \Theta' t^2 + \Theta t - \Delta = 0.$$

Any relation between the invariants $\Delta$, $\Theta$, $\Theta'$, $\Delta'$ will then be a symmetric function of $a$, $b$, $c$.

**18. Harmonic conics of two conics.** Consider the two conics

$$S \equiv ax^2 + by^2 + cz^2 = 0,$$
$$S' \equiv x^2 + y^2 + z^2 = 0,$$

and let us investigate *the locus of a point $P$, such that the tangents from $P$ to the two conics form a harmonic pencil.* The condition is the same as that the tangents from $P$ to the conic $S'$ should be apolar to the conic-envelope $\Sigma$. Now the equation of the two tangents from $(x', y', z')$ to $S'$ is

$$(x^2 + y^2 + z^2)(x'^2 + y'^2 + z'^2) = (xx' + yy' + zz')^2,$$

*i.e.*

$$x^2(y'^2 + z'^2) + \dots - 2y'z'yz - \dots = 0.$$

The condition that this degenerate conic-locus should be apolar to

$$\Sigma \equiv bcl^2 + cam^2 + abn^2 = 0$$

is

$$bc(y'^2 + z'^2) + ca(z'^2 + x'^2) + ab(x'^2 + y'^2) = 0.$$

Hence the equation of the locus of $P$ is

$$F \equiv a(b + c)x^2 + b(c + a)y^2 + c(a + b)z^2 = 0.$$

The locus is therefore a conic; it is called the *harmonic conic-locus* of $S$ and $S'$.

In an exactly similar way, using line-coordinates instead of point-coordinates, it can be proved that *the envelope of a line which is cut harmonically by the two conics $S$ and $S'$ is a conic whose line-equation is*

$$\Phi \equiv (b+c)l^2+(c+a)m^2+(a+b)n^2=0.$$

This is called the *harmonic conic-envelope* of $S$ and $S'$.

It is at once obvious that the four conics, $S$, $S'$, F, $\Phi$ have a common self-conjugate triangle.

**Examples.**

1. If $S$ reduces to two straight lines intersecting in $O$, show that the F-conic reduces to the two tangents from $O$ to $S'$.

2. If $\Sigma$ and $\Sigma'$ reduce to pairs of points, show that the F-conic is a conic passing through these four points, and show how to construct the tangents to F at these points.

**19.** Consider the tangent to $S$ at one of its points of intersection $A$ with $S'$. Of the two pairs of points of intersection of this line with the two conics, one pair consists of coincident points at $A$ and one point of the other pair coincides with $A$; these therefore form a degenerate harmonic range, and therefore this line belongs to the $\Phi$ conic-envelope. Hence *the $\Phi$-conic touches the eight tangents to $S$ and $S'$ at their points of intersection.* Similarly it can be proved that *the F-conic passes through the eight points of contact of the common tangents of $S$ and $S'$.*

**Ex.** If two conics cut at right angles at each of their four points of intersection, show that in general these four points are concyclic.

Since the tangents at the four points are tangents to the $\Phi$-conic, and since they cut at right angles, the four points must lie on the orthoptic circle of the $\Phi$-conic. It may happen, however, that $\Phi$ reduces to a point-pair at infinity, harmonic with respect to the circular points. In this case the orthoptic circle of $\Phi$ is completely indeterminate, and the four points are not necessarily concyclic. (See Ex. X. 16.)

**20. Mutually apolar conics.** If the conic-locus $S$ is apolar to the conic envelope $\Sigma'$, by taking as triangle of reference a triangle $ABC$ inscribed in $S$ and self-conjugate with regard to $\Sigma'$, we can write their equations

$$S \equiv 2fyz+2gzx+2hxy=0,$$
$$S' \equiv a'x^2+b'y^2+c'z^2=0.$$

Suppose also that the conic-locus $S'$ is apolar to the conic-envelope $\Sigma$; then

$$\Theta = -a'f^2-b'g^2-c'h^2=0.$$

Hence the conic $S'$ passes through the four points $(\pm f, \pm g, \pm h)$. These are the vertices $P$, $Q$, $R$ of the triangle formed by the tangents to $S$ at $A$, $B$, $C$, and the point, $O$, of intersection of $AP$, $BQ$, $CR$. Hence the two conics have also the property that a triangle can be inscribed in one and circumscribed about the other, for the triangle $PQR$ is inscribed in $S'$ and circumscribed about $S$.

If we take the common self-conjugate triangle of the two conics as triangle of reference, we may take

$$S \equiv x^2 + y^2 + z^2 = 0,$$
$$S' \equiv a'x^2 + b'y^2 + c'z^2 = 0.$$

Then
$$\Theta \equiv a' + b' + c' = 0,$$
$$\Theta' \equiv b'c' + c'a' + a'b' = 0.$$

Hence $a'$, $b'$, $c'$ are the roots of an equation of the form $\lambda^3 - k = 0$; if we take $k = 1$, the roots are 1, $\omega$, $\omega^2$, and we can write

$$S' \equiv x^2 + \omega y + \omega^2 z = 0.$$

Hence (Chap. XVIII. § 7), the cross-ratio of the four points of intersection on each of the conics is $-\omega$ or $-\omega^2$, *i.e. the four points of intersection form an equianharmonic tetrad on each conic.*

The $\mathsf{F}$-conic of the two mutually apolar conics $S$ and $S'$ is

$$\mathsf{F} \equiv x^2 + \omega^2 y^2 + \omega z^2 = 0.$$

The line-equation of the $\Phi$-conic is

$$\Phi' \equiv l^2 + \omega m^2 + \omega^2 n^2 = 0,$$

and its point-equation is therefore

$$x^2 + \omega^2 y^2 + \omega z^2 = 0.$$

Hence *the* $\mathsf{F}$ *and* $\Phi$ *conics of two mutually apolar conics are identical.*

The three conics $S$, $S'$ and $\mathsf{F}$ are then all mutually apolar, and the four points of intersection of any two form an equianharmonic tetrad upon both. Also each conic is the harmonic conic of the other pair.

These conics appear to be imaginary, but that is only because of the particular choice of coordinates. If two real conics cut each other equianharmonically, they intersect in two real and two imaginary points, and the common self-conjugate triangle is imaginary (see Chap. XVIII. § 7, Ex. 4).

Further, since $\mathsf{F}$ passes through the points of contact of the common tangents to $S$ and $S'$, and $S'$ touches the tangents to $\mathsf{F}$ and $S$ at their points of intersection, the points of contact of the common tangents are also points of intersection of the conics. Hence the tangents at the 12 points of intersection coincide in pairs in 12 lines which are the common tangents. Only half of these tangents will be real, since any two of the conics can have only two real points of intersection.

Ex. Prove that if the $\mathsf{F}$ and $\Phi$-conics of two conics $S$ and $S'$ are identical, $S$ and $S'$ are mutually apolar.

**21. Reciprocal of a conic with respect to a given conic.**

*The envelope of the polar of a variable point $P$ on a given conic $S$ with regard to another fixed conic $S_0$ is a conic.*

Taking the harmonic triangle of the two conics as triangle of reference we can write

$$S \equiv ax^2 + by^2 + cz^2 = 0,$$
$$S_0 \equiv x^2 + y^2 + z^2 = 0.$$

The polar of $P \equiv (x', y', z')$ with regard to $S_0$ is

$$x'x + y'y + z'z = 0.$$

The line-coordinates of this line are $(l, m, n) \equiv (x', y', z')$.

Now, since $P$ lies on $S$, $ax'^2 + by'^2 + cz'^2 = 0$;

therefore $$al^2 + bm^2 + cn^2 = 0,$$

and this is the line-equation of a conic, $S'$, and its point-equation is

$$S' \equiv bcx^2 + cay^2 + abz^2 = 0.$$

Similarly it may be proved that the locus of the pole of a variable tangent to $S$ with regard to $S_0$ is a conic which is identical with $S'$, and also $S$ is the envelope of polars of points on $S'$ with regard to $S_0$.

The conics $S$ and $S'$ are said to be *reciprocal* with regard to $S_0$.

Ex. If $S_1$, $S_2$, $S_3$ are a set of mutually apolar conics, show that each is the reciprocal of the second with respect to the third.

**22.** If two conics are connected by some geometrical relationship which is unaltered by projection, this can generally be expressed by means of a relation connecting the invariants $\Delta$, $\Delta'$, $\Theta$, $\Theta'$. Any such relation between the invariants must satisfy the following conditions:

(1) It must be unaltered when the two conics are referred to any other homogeneous coordinates. Since only the ratios of $\Delta$, $\Delta'$, $\Theta$, $\Theta'$ are in general unaltered by this transformation, the equation must be homogeneous in $\Delta$, $\Delta'$, $\Theta$, $\Theta'$, each considered of the same dimensions.

(2) It must be unaltered when the equations of the conics are multiplied by any factor. Hence the equation in $\Delta$, $\Delta'$, $\Theta$, $\Theta'$ must be homogeneous in the coefficients of each of the conics, or the equation must be homogeneous when $\Delta$, $\Theta$, $\Theta'$, $\Delta'$ are considered of dimensions 3, 2, 1, 0 or 0, 1, 2, 3 respectively.

With the help of these principles the calculation of the invariant relations corresponding to a given geometrical relation can often be greatly simplified by reducing the equations of the conics to some simple type. The following example will illustrate this.

**23. Condition that two conics should have a triangle inscribed in the one and circumscribed about the other.** We have seen (Chap. XVIII. § 11) that if two conics $S$ and $S'$ are such that one triangle can be inscribed in $S'$ and circumscribed about $S$, an infinity of such triangles can be constructed.

Let the conics be transformed so that one such triangle is taken as triangle of reference; then the equations can be written

$$\Sigma \equiv 2Fmn + 2Gnl + 2Hlm = 0,$$

$$S' \equiv 2f'yz + 2g'zx + 2h'xy = 0,$$

and
$$\Delta S \equiv -F^2x^2 - G^2y^2 - H^2z^2 + 2GHyz + 2HFzx + 2FGxy = 0,$$

$$\Sigma' \equiv -f'^2l^2 - g'^2m^2 - h'^2n^2 + 2g'h'mn + 2h'f'nl + 2f'g'lm = 0.$$

Then
$$\Delta^2 = 2FGH, \quad \Delta' = 2f'g'h', \quad \Theta = 2(f'F + g'G + h'H),$$

$$\Delta\Theta' = f'^2F^2 + g'^2G^2 + h'^2H^2 + 2g'h'GH + 2h'f'HF + 2f'g'FG$$
$$= (f'F + g'G + h'H)^2.$$

Hence we get the relation $$\Theta^2 = 4\Delta\Theta',$$

This is homogeneous in $\Delta$, $\Theta$, $\Theta'$, $\Delta'$, and each side is of dimensions 4 and 2 respectively in the coefficients of the two conics. Hence this relation is unaltered by any linear transformation. It is therefore a necessary condition, and it is easily verified conversely that it is a sufficient condition, provided the conics do not degenerate.

**24. Absolute invariants of two conics.** The ratios

$$\Delta : \Theta : \Theta' : \Delta'$$

are absolutely unaltered by any linear transformation, provided the equations of the conics have not been multiplied or divided by any factor after the transformation. As we cannot be sure of this not having happened, we cannot, however, say that these ratios are absolute invariants. We must form expressions which are of zero dimensions, not only in $\Delta$, $\Theta$, $\Theta'$, $\Delta'$, but also in the coefficients of the two conics. Leaving out one of the four quantities at a time, we obtain the following as the simplest absolute invariants :

$$Q \equiv \frac{\Theta^2}{\Delta\Theta'}, \quad Q' \equiv \frac{\Theta'^2}{\Delta'\Theta}, \quad R \equiv \frac{\Theta^3}{\Delta^2\Delta'}, \quad R' \equiv \frac{\Theta'^3}{\Delta\Delta'^2}.$$

But these are not independent, for we have $R = Q^2Q'$ and $R' = QQ'^2$.

**25.** *Every invariant relation (except $\Delta = 0$ and $\Delta' = 0$) which is expressed in terms of $\Delta$, $\Theta$, $\Theta'$, $\Delta'$ can be expressed in terms of the absolute invariants $Q$ and $Q'$.*

Consider any homogeneous equation in $\Delta$, $\Theta$, $\Theta'$, $\Delta'$. We shall suppose it made rational and integral, so that it consists of a sum of terms of the form

$$k\Delta^{\delta}\Theta^{\theta}\Theta'^{\theta'}\Delta'^{\delta'}.$$

Then, since it is homogeneous in the coefficients of the two conics separately, each term is of the same dimensions, $r$, $r'$, say, in the coefficients. Therefore

$$3\delta + 2\theta + \theta' = r,$$

and

$$\theta + 2\theta' + 3\delta' = r'.$$

These lead to

$$\delta + \theta + \theta' + \delta' = \tfrac{1}{3}(r + r') = \text{const.},$$

and the equation is also homogeneous in $\Delta$, $\Theta$, $\Theta'$, $\Delta'$.

Now substitute for $\Delta$ and $\Delta'$ in terms of $Q$ and $Q'$, and the term becomes

$$kQ^{-\delta}Q'^{-\delta'}\Theta^{2\delta+\theta-\delta'}\Theta'^{2\delta'+\theta'-\delta}.$$

But

$$2\delta + \theta - \delta' = \tfrac{1}{3}(2r - r') = \text{const.},$$

$$2\delta' + \theta' - \delta = \tfrac{1}{3}(2r' - r) = \text{const.}$$

Therefore the factors involving $\Theta$ and $\Theta'$ can be cancelled out, and we are left with an equation in $Q$ and $Q'$ alone. This equation is of course not in general homogeneous in $Q$ and $Q'$.

For example, the invariant relation $\Theta^2 = 4\Delta\Theta'$ (§ 23) becomes $Q = 4$.

*Cor.* If two invariant relations in $\Delta$, $\Theta$, $\Theta'$, $\Delta'$ are given, the values of $Q$ and $Q'$ are determined, and no further relation between $Q$ and $Q'$ is possible.

**26. The common chords of two conics.** The cubic equation

$$\Delta\lambda^3+\Theta\lambda^2+\Theta'\lambda+\Delta'=0 \quad \text{.........(1)}$$

determines three values of $\lambda$ for which the conic

$$\lambda S+S'=0$$

breaks up into a pair of straight lines. Eliminating $\lambda$, we have the equation

$$\Delta S'^3-\Theta S'^2 S+\Theta' S' S^2-\Delta' S^3=0, \quad \text{.........(2)}$$

which is of the sixth degree, and represents the three pairs of common chords of the two conics.

Similarly, if we eliminate $\lambda$ between the equations

$$\Delta^2\lambda^3+\Delta\Theta'\lambda^2+\Delta'\Theta\lambda+\Delta'^2=0$$

and
$$\lambda\Sigma+\Sigma'=0,$$

we get the line-equation

$$\Delta^2\Sigma'^3-\Delta\Theta'\Sigma'^2\Sigma+\Delta'\Theta\Sigma'\Sigma^2-\Delta'^2\Sigma^3=0,$$

which is of the sixth degree, and represents the three pairs of points of intersection of the common tangents of the two conics.

**27. Condition for single contact.** If the conics touch, two pairs of common chords coincide, and the cubic equation in $\lambda$ will have two roots equal. The condition for this is that the Hessain quadratic (Chap. XIX. § 12) of the cubic equation (1), viz.

$$(3\Delta\Theta'-\Theta^2)\lambda^2+(9\Delta\Delta'-\Theta\Theta')\lambda+(3\Delta'\Theta-\Theta'^2)=0,$$

should have equal roots. Hence

$$(9\Delta\Delta'-\Theta\Theta')^2=4(\Theta'^2-3\Delta'\Theta)(\Theta^2-3\Delta\Theta'),$$

*i.e.*
$$\Theta^2\Theta'^2+18\Delta\Delta'\Theta\Theta'-27\Delta^2\Delta'^2-4\Delta\Theta'^3-4\Delta'\Theta^3=0,$$

which is the condition for *single contact*. The expression which is here equated to zero is called the *tact-invariant*. Expressed in terms of $Q$ and $Q'$, the condition is

$$Q^2Q'^2+18QQ'-4QQ'(Q+Q')-27=0.$$

**28. Osculating contact.** If the three roots of the cubic in $\lambda$ are all equal, three of the points of intersection coincide, and the conics osculate. Equation (2) then represents the common tangent and the common chord, each taken three times. The conditions for this are

$$\frac{3\Delta}{\Theta}=\frac{\Theta}{\Theta'}=\frac{\Theta'}{3\Delta'}.$$

These are the (two) conditions for *osculation* or *three-point contact*. Expressed in terms of $Q$ and $Q'$, these become

$$Q=3, \quad Q'=3.$$

There are two other sorts of contact of two conics, four-point contact and double contact. The latter requires two conditions, but the former requires three, one condition in addition to those required for three-point contact. By § 25, Cor. these three conditions cannot be expressed in terms of $\Delta$, $\Theta$, $\Theta'$, $\Delta'$ alone. We shall see in § 34 what are the conditions for four-point contact and double contact.

**29. Examples.**

1. *Find the condition that the four points of intersection of two conics $S$ and $S'$ should subtend a harmonic pencil at any point on $S'$.*

Transform the conics to the common self-conjugate triangle as triangle of reference, so that their equations become

$$S \equiv ax^2 + by^2 + cz^2 = 0,$$
$$S' \equiv x^2 + y^2 + z^2 = 0.$$

The coordinates of the four points of intersection are $(\pm p, \pm q, \pm r)$, where

$$p^2 : q^2 : r^2 = b - c : c - a : a - b.$$

The values of the cross-ratio of the pencil subtended by these points at any point on $S'$ are $-q^2/r^2$, etc. (Chap. XVIII. § 7). Hence, if the pencil is harmonic one of these six values must be $-1$. Hence

$$(q^2 - r^2)^2(r^2 - p^2)^2(p^2 - q^2)^2 = 0.$$

Now $q^2 - r^2 = b + c - 2a = \Theta' - 3a$; the equation therefore becomes

$$(\Theta' - 3a)(\Theta' - 3b)(\Theta' - 3c) = 0,$$

*i.e.* $$\Theta'^3 - 3\Theta'^2\Sigma a + 9\Theta'\Sigma bc - 27abc = 0,$$

or $$27\Delta\Delta'^2 - 9\Delta'\Theta\Theta' + 2\Theta'^3 = 0,$$

on making the equation homogeneous with the help of $\Delta'$.

If this is expressed in terms of $Q$ and $Q'$, it becomes

$$2QQ'^2 - 9QQ' + 27 = 0.$$

2. *Find the conditions that the four points of intersection of two conics should form a harmonic tetrad on each conic.*

In addition to the condition in Ex. 1, we have also

$$2Q^2Q' - 9QQ' + 27 = 0.$$

Hence $$Q = Q' \quad \text{and} \quad 2Q^3 - 9Q^2 + 27 = 0,$$

*i.e.* $$(2Q + 3)(Q - 3)^2 = 0.$$

Hence there are two alternative conditions,

$$Q = Q' = 3,$$
$$Q = Q' = -\tfrac{3}{2}.$$

We shall see in the next example that the first alternative must be rejected.

3. *Find the condition that the four points of intersection of two conics $S$ and $S$ should form an equianharmonic tetrad on $S'$.*

Proceeding as in Ex. 1, we must have one of the values $-q^2/r^2$, etc., $= -\omega$ or $-\omega^2$. Hence

$$(q^2 - \omega r)(q^2 - \omega^2 r^2) = 0.$$

Putting in the values of $q^2$ and $r^2$, this becomes

$$(\omega a + \omega^2 b + c)(\omega^2 a + \omega b + c) = 0,$$

*i.e.* $$\Sigma bc = \Sigma a^2 = (\Sigma a)^2 - 2\Sigma bc.$$

Hence $$\Theta'^2 = 3\Delta'\Theta,$$

*i.e.* $$Q' = 3.$$

Hence we see that if the four points of intersection form an equianharmonic tetrad on both conics, $Q = Q' = 3$.

The conditions $Q=Q'=3$ are satisfied also if the two conics osculate. A simpler form of the conditions, which is also really the necessary and sufficient condition, is given in the next example.

4. *Show that the necessary and sufficient conditions that the four points of intersection of two conics should form an equianharmonic tetrad on both conics are* $\Theta=0=\Theta'$. (Cf. § 20).

5. *Prove that if* $\Theta^2=3\Delta\Theta'$ and $\Theta'^2=3\Delta'\Theta$, *either the two conics osculate, or* $\Theta=\Theta'=0$.

*Cor.* $Q=Q'=3$ are the necessary and sufficient conditions that the conics should osculate, and $Q=Q'=-\frac{3}{2}$ are the necessary and sufficient conditions that they should cut each other harmonically.

6. *Prove that if* $\lambda$ *is the value of one of the cross-ratios of the pencil subtended by the points of intersection of two conics S and S' at any point on S,*

$$\left(\frac{\lambda^2-\lambda+1}{\Theta^2-3\Delta\Theta'}\right)^3=\left\{\frac{(\lambda+1)(2\lambda-1)(\lambda-2)}{27\Delta^2\Delta'-9\Delta\Theta\Theta'+2\Theta^3}\right\}^2.$$

**30. Points of intersection of two conics S, S'.** The most symmetrical way to represent a group of points obtained by the intersection of two curves is to form the line-equation of the whole group of points. We have had an instance of this already in representing the circular points, where it is simpler to deal with the line-equation than the coordinates of the separate points.

Consider any conic passing through the four points of intersection of $S$, $S'$. Its point-equation is

$$\lambda S+S'=0.$$

Forming its line-equation, the coefficient of $l^2$ is

$$(\lambda b+b')(\lambda c+c')-(\lambda f+f')^2=\lambda^2 A+\lambda(bc'+b'c-2ff')+A',$$

and the coefficient of $2mn$ is

$$(\lambda g+g')(\lambda h+h')-(\lambda a+a')(\lambda f+f')=\lambda^2 F+\lambda(gh'+g'h-af'-a'f)+F'.$$

Hence the line-equation reduces to

$$\lambda^2\Sigma+\lambda\Phi+\Sigma'=0,$$

where $\Phi\equiv(bc'+b'c-2ff')l^2+\ldots+2(gh'+g'h-af'-a'f)mn+\ldots,$

and $\Sigma$, $\Sigma'$ have the usual meanings.

Now, if we vary $\lambda$ this system of conics will always pass through the four fixed points, and will therefore have just these four points as envelope. The equation of the envelope is formed by expressing the condition for equal roots of $\lambda$, and we have

$$\Phi^2=4\,\Sigma\Sigma'$$

as the line-equation of the four common points.

Similarly it can be proved that

$$\mathsf{F}^2=4\Delta\Delta' SS'$$

is the point-equation of the four common tangents, where

$$\mathsf{F}\equiv(BC'+B'C-2FF')x^2+\ldots+2(GH'+G'H-AF'-A'F)yz+\ldots;$$

**31. Covariants.** Let us now consider the equation

$$\mathsf{F}^2 = 4\Delta\Delta' SS'$$

from the point of view of invariants. If the conics are transformed by the general linear transformation, the equation of the four common tangents after the transformation will be

$$\mathsf{F}_1^2 = 4\Delta_1\Delta_1' S_1 S_1',$$

where $S_1$, $S_1'$, $\Delta_1$, $\Delta_1'$, $\mathsf{F}_1$ are exactly the same functions of the new variables and the coefficients of $S_1$ and $S_1'$ that $S$, $S'$, $\Delta$, $\Delta'$, $\mathsf{F}$ are of the original variables and the coefficients of $S$ and $S'$. But since this new equation can be obtained by substituting for $x$, $y$, $z$ their values in terms of $x'$, $y'$, $z'$, these two equations are identical, *i.e.*

$$\mathsf{F}_1^2 - 4\Delta_1\Delta_1' S_1 S_1' \equiv k(\mathsf{F}^2 - 4\Delta\Delta' SS').$$

But $S_1 = S$, $S_1' = S'$, $\Delta_1 = D^2\Delta$, $\Delta_1' = D^2\Delta'$; therefore $k = D^4$, and

$$\mathsf{F}_1 \equiv D^2\mathsf{F}.$$

The function $\mathsf{F}$ behaves therefore exactly like an invariant, but it involves not only the coefficients but also the variables. It is called a *covariant* of the two conics. The function

$$\mathsf{F}^2 - 4\Delta\Delta' SS'$$

is also a covariant, and so also are $S$ and $S'$ themselves.

Analytically a covariant of one or more functions $S_1$, $S_2$, ... of the point-coordinates $x$, $y$, $z$ is an expression $C$ involving the point-coordinates and the coefficients of the given functions, such that when the functions are transformed by a linear substitution into the functions $S_1'$, $S_2'$, ... of the new variables $x'$, $y'$, $z'$, the same function $C'$ of the new variables and the coefficients of $S_1'$, $S_2'$, ... is identically equal to the original function $C$ multiplied by a factor which is independent of the coefficients.

Geometrically a covariant of one or more loci is a locus which is connected with the given loci by a law which is unaltered by projection.

Thus $\mathsf{F}^2 - 4\Delta\Delta' SS' = 0$ represents the four common tangents of the two conics $S$ and $S'$, and when it is transformed it represents again the four common tangents of the transformed conics.

**32.** We can now, either directly by the method of § 18, or by comparison with that result, identify the covariant $\mathsf{F}$ with the *harmonic conic locus* of the two conics $S$ and $S'$, *i.e.* the locus of points at which the two conics subtend a harmonic pencil. For if

$$S \equiv ax^2 + by^2 + cz^2 = 0, \qquad \Sigma \equiv bcl^2 + cam^2 + abn^2 = 0,$$

$$S' \equiv x^2 + y^2 + z^2 = 0, \qquad \Sigma' \equiv l^2 + m^2 + n^2 = 0,$$

$$\mathsf{F} \equiv (ca + ab)x^2 + (ab + bc)y^2 + (bc + ca)z^2 = 0,$$

which is the equation of the harmonic conic-locus.

**33. Contravariants.** In a similar way, or by reciprocation, we see that $\Phi$ *is the envelope of lines which are cut harmonically by the two conics $S$ and $S'$*, or the *harmonic conic-envelope* of the two conics.

The function $\Phi$ also behaves as an invariant and is exactly analogous to a covariant. Since, however, it involves the line-coordinates instead of the point-coordinates, it is called a *contravariant*.

The distinction between a covariant and a contravariant lies in the point of view and the analytical expression, for a contravariant, which represents an envelope, can always be expressed by means of its point-equation, and then we get a corresponding covariant.

Thus expressing the conics as in § 17, we find

$$\Phi \equiv (b+c)l^2+(c+a)m^2+(a+b)n^2=0,$$

and the point-equation of this conic is

$$(c+a)(a+b)x^2+(a+b)(b+c)y^2+(b+c)(c+a)z^2=0.$$

This may now be written

$$(a+b+c)(ax^2+by^2+cz^2)+(bc+ca+ab)(x^2+y^2+z^2)-\Sigma a(b+c)x^2=0,$$

*i.e.*
$$\Theta' S+\Theta S'-\mathsf{F}=0.$$

Now, since this equation is homogeneous in the coefficients of the two conics and involves only invariants and covariants, its form will be unaltered when by a linear substitution the equations of the conics are made general. Hence $\Phi$ as a locus belongs to the linear system determined by $S$, $S'$, and $\mathsf{F}$.

Similarly the line-equation of $\mathsf{F}$ can be found to be

$$\Delta'\Theta\Sigma+\Delta\Theta'\Sigma'-\Delta\Delta'\Phi=0.$$

**Examples.**

1. If $S$ and $S'$ touch at a point $P$, prove that $\mathsf{F}$ and $\Phi$ both touch $S$ and $S'$ at $P$. (Use the property in § 18.)

2. If $S$ and $S'$ touch at two separate points, $\mathsf{F}$ and $\Phi$ both touch $S$ and $S'$ at these points.

3. If $S$ and $S'$ have three-point contact at $P$, $\mathsf{F}$ and $\Phi$ also have three-point contact with both $S$ and $S'$ at $P$.
(Take $S\equiv z^2+2xy=0,\quad S'\equiv c(z^2+2xy)+2y(my+nz)=0.$)

4. If $S$ and $S'$ have four-point contact at $P$, $\mathsf{F}$ and $\Phi$ also have four-point contact with both $S$ and $S'$ at $P$.
(Take $S\equiv z^2+2xy,\quad S'\equiv c(z^2+2xy)+by^2.$)

**34. Conditions for double contact of two conics S and S′.** When $S$ and $S'$ have double contact at $P$ and $Q$, $\mathsf{F}$ also has double contact with $S$ at $P$ and $Q$; hence in this case $\mathsf{F}$ belongs to the pencil or linear system of conics

$$\lambda S+\mu S'=0.$$

Hence denoting the coefficients of $\mathsf{F}$ by $\mathsf{a}$, $\mathsf{b}$, $\mathsf{c}$, $\mathsf{f}$, $\mathsf{g}$, $\mathsf{h}$, where

$$\mathsf{a}\equiv BC'+B'C-2FF',\quad \mathsf{f}\equiv GH'+G'H-AF'-A'F, \text{ etc.,}$$

$$\begin{aligned} \mathsf{a}&=\lambda a+\mu a', & \mathsf{f}&=\lambda f+\mu f',\\ \mathsf{b}&=\lambda b+\mu b', & \mathsf{g}&=\lambda g+\mu g',\\ \mathsf{c}&=\lambda c+\mu c & \mathsf{h}&=\lambda h+\mu h'.\end{aligned}$$

Eliminating $\lambda$ and $\mu$, we get all the determinants of the third order vanishing in the following array:

$$\left\|\begin{matrix} a & b & c & f & g & h \\ a' & b' & c' & f' & g' & h' \\ \mathsf{a} & \mathsf{b} & \mathsf{c} & \mathsf{f} & \mathsf{g} & \mathsf{h} \end{matrix}\right\| = 0.$$

Although there are 20 determinants here, this is only equivalent to two conditions. These are the (two) necessary and sufficient conditions for double contact.

These conditions are also necessary for four-point contact, for four-point contact is double contact in which the two points of contact coincide. For four-point contact also, the conditions for three-point contact are necessary, *i.e.*

$$\frac{3\Delta}{\Theta} = \frac{\Theta}{\Theta'} = \frac{\Theta'}{3\Delta'}.$$

These, together with the above conditions, give the (three) conditions for four-point contact.

The determinants in the above array are of dimensions (3, 3) in the coefficients of the two conics, and it might be expected that the conditions for double contact could be expressed in terms of the fundamental invariants. But the only terms involving $\Delta$, $\Theta$, $\Theta'$, $\Delta'$ of the required dimensions are $\Delta\Delta'$ and $\Theta\Theta'$, and two relations of the form $l\Delta\Delta' + m\Theta\Theta' = 0$ could not exist unless $\Delta\Delta'$ and $\Theta\Theta'$ both vanish.

**35.** The conics $\mathsf{F}$ and $\Phi$ perform rôles with respect to other covariants and contravariants of the two conics analogous to those performed by the invariants $\Delta$, $\Theta$, $\Theta'$, $\Delta'$ with respect to other invariants. Thus, just as most invariant relations can be expressed in terms of these four fundamental invariants, so most covariants can be expressed in terms of $S$, $S'$, $\mathsf{F}$ and the fundamental invariants, and similarly for the related envelopes or contravariants.

**Examples.**

1. Find the reciprocal of $S$ with regard to $S'$.

Taking
$$S \equiv ax^2 + by^2 + cz^2 = 0,$$
$$S' \equiv x^2 + y^2 + z^2 = 0,$$
the reciprocal of $S$ with regard to $S'$ is
$$bcx^2 + cay^2 + abz^2 = 0.$$
This can be expressed in the torm
$$(bc + ca + ab)(x^2 + y^2 + z^2) - \Sigma a(b + c)x^2 = 0,$$
*i.e.*
$$\Theta S' - \mathsf{F} = 0.$$

Similarly the reciprocal of $S'$ with respect to $S$ is
$$\Theta' S - \mathsf{F} = 0.$$

**2.** Show that the line-equation of the reciprocal of $S$ with regard to $S'$ is
$$\Theta'\Sigma' - \Delta'\Phi = 0.$$

**3.** Prove that the locus of points whose polars with regard to $S$ and $S'$ are conjugate with regard to the $\Phi$-conic is the $\mathsf{F}$-conic.

**36. System of three conics.** A system of three conics $S_1$, $S_2$, $S_3$ has the invariants $\Delta$ of each singly, the joint-invariants $\Theta$ of the conics taken in pairs, and further invariants of all three taken together. If we consider the net of conics

$$\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 = 0,$$

the discriminant is

$$\begin{vmatrix} \lambda_1 a_1 + \lambda_2 a_2 + \lambda_3 a_3 & \cdot & \cdot \\ \lambda_1 h_1 + \lambda_2 h_2 + \lambda_3 h_3 & \cdot & \cdot \\ \lambda_1 g_1 + \lambda_2 g_2 + \lambda_3 g_3 & \cdot & \cdot \end{vmatrix}.$$

Expanding this, we get

$$\Delta_1 \lambda_1^3 + \ldots + \Theta_{112} \lambda_1^2 \lambda_2 + \Theta_{113} \lambda_1^2 \lambda_3 + \ldots + \Theta_{123} \lambda_1 \lambda_2 \lambda_3,$$

where $\Theta_{112}$, etc., stand for the mutual invariants, and there is one invariant of all three, viz.

$$\Theta_{123} \equiv \begin{vmatrix} a_1 & h_2 & g_3 \\ h_1 & b_2 & f_3 \\ g_1 & f_2 & c_3 \end{vmatrix} + \begin{vmatrix} a_1 & h_3 & g_2 \\ h_1 & b_3 & f_2 \\ g_1 & f_3 & c_2 \end{vmatrix} + \text{etc.}$$

We shall not enter further into these invariants, but there is one *covariant* which is of great importance. This consists of the determinant

$$\begin{vmatrix} \frac{\partial S_1}{\partial x} & \frac{\partial S_1}{\partial y} & \frac{\partial S_1}{\partial z} \\ \frac{\partial S_2}{\partial x} & \frac{\partial S_2}{\partial y} & \frac{\partial S_2}{\partial z} \\ \frac{\partial S_3}{\partial x} & \frac{\partial S_3}{\partial y} & \frac{\partial S_3}{\partial z} \end{vmatrix} = 0,$$

and is called the ***Jacobian*** of the three conics. Since the differential coefficients are all linear expressions, the Jacobian is a cubic curve.

**37.** *The Jacobian of three conic-loci is the locus of points whose polars with regard to the three conics are concurrent.* The polars of the point $(x', y', z')$ are

$$x \frac{\partial S_1}{\partial x'} + y \frac{\partial S_1}{\partial y'} + z \frac{\partial S_1}{\partial z'} = 0,$$

etc. Hence the condition for concurrency is represented by the vanishing of the above determinant.

If the three conics have a point in common, evidently this point satisfies the condition for concurrency of its polars; hence the Jacobian passes through this point.

If the three conics have two points $A$, $B$ in common, the polar of any point $P$ on $AB$ with respect to each of the conics passes through the harmonic conjugate of $P$ with respect to $A$ and $B$. Hence the Jacobian consists of the line $AB$ and a conic.

If the conics have three points $A$, $B$, $C$ in common, the Jacobian consists of the three lines $BC$, $CA$, $AB$.

If the conics have four points in common, the Jacobian vanishes identically; and, conversely, if the Jacobian vanishes identically, the conics belong to the same pencil, or are linearly connected, *i.e.* for a certain system of values of $\lambda_1, \lambda_2, \lambda_3$, $\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 \equiv 0.$

To prove this, suppose first that $S_3$ belongs to the linear system

$$\lambda S_1 + \mu S_2 = 0;$$

then $S_3 \equiv \lambda S_1 + \mu S_2$, $\frac{\partial S_3}{\partial x} \equiv \lambda \frac{\partial S_1}{\partial x} + \mu \frac{\partial S_2}{\partial x}$, etc., and the determinant vanishes identically. Conversely, if the determinant vanishes identically, it is possible to find multiples $\lambda_1, \lambda_2, \lambda_3$, such that

$$\lambda_1 \frac{\partial S_1}{\partial x} + \lambda_2 \frac{\partial S_2}{\partial x} + \lambda_3 \frac{\partial S_3}{\partial x} \equiv 0, \text{ etc.}$$

Multiply these three equations respectively by $x, y, z$ and add; then, since $x \frac{\partial S_1}{\partial x} + y \frac{\partial S_1}{\partial y} + z \frac{\partial S_1}{\partial z} \equiv 2S_1$, it follows that

$$\lambda_1 S_1 + \lambda_2 S_2 + \lambda_3 S_3 \equiv 0.$$

**38.** *The Jacobian of any three conics of the linear system*

$$\lambda S + \lambda' S' + \lambda'' S'' = 0$$

*is the same as the Jacobian of $S, S', S''$.*

The Jacobian of the three conics

$$\lambda S + \lambda' S' + \lambda'' S'' = 0,$$
$$\mu S + \mu' S' + \mu'' S'' = 0,$$
$$\nu S + \nu' S' + \nu'' S'' = 0$$

is

$$\begin{vmatrix} \lambda \frac{\partial S}{\partial x} + \lambda' \frac{\partial S'}{\partial x} + \lambda'' \frac{\partial S''}{\partial x} & \cdot & \cdot \\ \mu \frac{\partial S}{\partial x} + \mu' \frac{\partial S'}{\partial x} + \mu'' \frac{\partial S''}{\partial x} & \cdot & \cdot \\ \nu \frac{\partial S}{\partial x} + \nu' \frac{\partial S'}{\partial x} + \nu'' \frac{\partial S''}{\partial x} & \cdot & \cdot \end{vmatrix} = \begin{vmatrix} \lambda & \lambda' & \lambda'' \\ \mu & \mu' & \mu'' \\ \nu & \nu' & \nu'' \end{vmatrix} \begin{vmatrix} \frac{\partial S}{\partial x} & \frac{\partial S'}{\partial x} & \frac{\partial S''}{\partial x} \\ \frac{\partial S}{\partial y} & \frac{\partial S'}{\partial y} & \frac{\partial S''}{\partial y} \\ \frac{\partial S}{\partial z} & \frac{\partial S'}{\partial z} & \frac{\partial S''}{\partial z} \end{vmatrix},$$

and the last factor is the Jacobian of $S, S', S''$.

**39.** *If three conics have a common self-conjugate triangle, the Jacobian consists of the sides of this triangle.* Let the three conics be

$$S \equiv ax^2 + by^2 + cz^2 = 0,$$
$$S' \equiv a'x^2 + b'y^2 + c'z^2 = 0,$$
$$S'' \equiv a''x^2 + b''y^2 + c''z^2 = 0.$$

Then $\frac{1}{2} \frac{\partial S}{\partial x} = ax$, etc., and

$$J = \begin{vmatrix} ax & by & cz \\ a'x & b'y & c'z \\ a''x & b''y & c''z \end{vmatrix} = 0,$$

*i.e.* $$xyz = 0.$$

**40. The cubic covariant of two conics.** Now consider the Jacobian of the three conics $S$, $S'$ and F, or that of $S$, $S'$ and $\Phi$ (as a locus). Since $\Phi$ belongs to the same linear system as $S$, $S'$, F, these two Jacobians are the same. Since the four conics have a common self-conjugate triangle, the Jacobian consists of the three sides of that triangle. This is a covariant of $S$ and $S'$, since it is a locus definitely connected with the two conics. We may call it the *cubic covariant*.

There are special cases in which the cubic covariant is further specialized.

(1) If $S$ and $S'$ have simple contact, so that F also touches them at the point of contact, part of the covariant consists of the tangent at the point of contact.

Ex. Prove that the complete Jacobian in this case consists of the tangent at the point of contact taken twice, and a line through this point which cuts the four conics $S$, $S'$, F, $\Phi$ in points the tangents at which meet in a point on the common tangent. (Show that we can choose $S \equiv x^2 + 2yz$, $S' \equiv ax^2 + cz^2 + 2fyz$, and use these equations.)

(2) If $S$ and $S'$ have three-point contact, the cubic covariant consists of the tangent at the point of contact taken three times.

(Take $S \equiv x^2 + 2yz$, $S' \equiv a(x^2 + 2yz) + 2gzx$.)

(3) If $S$ and $S'$ have double contact, or four-point contact, the cubic covariant vanishes identically.

**41. The cubic contravariant of two conics.** Corresponding reciprocally to the cubic covariant $J$ of the two conics $S$, $S'$ there is a cubic contravariant $\Gamma$. Analytically this is determined by the same process as that by which we formed the Jacobian $J$, using line-coordinates instead of point-coordinates. It is then the line-Jacobian of $\Sigma$, $\Sigma'$ and $\Phi$, or that of $\Sigma$, $\Sigma'$ and F (as an envelope). Geometrically it is the envelope of lines whose poles with regard to $\Sigma$, $\Sigma'$ and $\Phi$ are collinear, and it can be easily seen that in the general case it consists of the three vertices of the common self-conjugate triangle of $\Sigma$ and $\Sigma'$. In the other special cases it can be proved that:

(1) If $\Sigma$ and $\Sigma'$ have simple contact at $P$, and $p$ is the tangent at $P$, $\Gamma$ consists of $P$ taken twice and a point on $p$ such that the points of contact of the tangents from this point to $\Sigma$, $\Sigma'$, F, $\Phi$ are collinear in a line through $P$.

(2) If $\Sigma$ and $\Sigma'$ have three-point contact, $\Gamma$ consists of the point of contact taken three times.

(3) If $\Sigma$ and $\Sigma'$ have double contact, or four-point contact, $\Gamma$ vanishes identically.

**42. Application of invariants and covariants to metrical problems.** The projective invariants and covariants can be used also in metrical problems by expressing the relations between given conics and the fixed degenerate conic or Absolute consisting of the line at infinity and the circular points. Thus we have the fixed conic *

$$S' \equiv z^2, \quad \Sigma' \equiv l^2 + m^2.$$

If we take another conic $S$, the F-conic of $S$ and $S'$ is the locus of points such that the tangents to $S$ and $S'$ are harmonic conjugates, but since the tangents to $\Sigma'$ are the absolute lines, this condition makes the tangents to $\Sigma$ at right angles. Hence *the* F*-conic is the orthoptic locus of the conic S.* The $\Phi$-conic is the envelope of lines which are cut by $S$ and $S'$ harmonically. But since the two points of intersection with $S'$ coincide, one of the points of intersection with $S$ must also coincide with this point. Hence the line must pass through one of the points of intersection of $S$ with the line at infinity. Hence *the $\Phi$-conic consists of the two points at infinity on the conic.*

The invariants $\Theta$ and $\Theta'$ are (see, however, footnote on this page).

$$\Theta = C, \quad \Theta' = a + b.$$

Hence *the condition that a conic-locus should be apolar to the absolute* (*or circular points*) *is that it should be a rectangular hyperbola*, and *the condition that a conic-envelope should be apolar to the absolute* (*or line at infinity*) *is that it should be a parabola.*

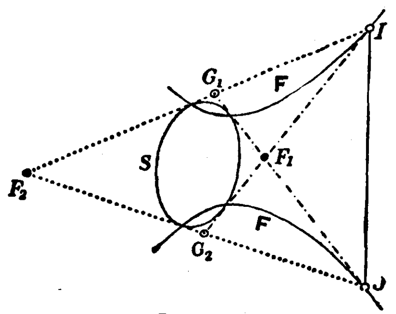


FIG. 117.

Since F passes through the 8 points of contact of the common tangents to $\Sigma$ and $\Sigma'$, and since these common tangents are the tangents to the conic from the circular points (absolute tangents), the 8 points of contact consist of four imaginary points on $S$, and the circular points $I, J$, each taken twice. Hence F passes through the circular points, and is therefore a circle (Fig. 117).

Moreover, any conic passing through the circular points must be regarded as having double contact with the absolute, since the common tangents coincide in pairs with the tangents at $I$ and $J$. Conversely,

* The point- and line-equations of the absolute should be written more strictly

$$S' \equiv z^2 + \epsilon(x^2 + y^2 + 2f'yz + 2g'zx) = 0,$$

$$\Sigma' \equiv \epsilon(l^2 + m^2) - \epsilon^2(f'^2l^2 + g'^2m^2 - n^2 + 2f'mn + 2g'nl - 2f'g'lm),$$

where $\epsilon \to 0$, or, putting $f' = 0 = g'$ and neglecting $\epsilon^2$,

$$S' \equiv z^2 + \epsilon(x^2 + y^2), \quad \Sigma' \equiv \epsilon(l^2 + m^2).$$

Then if $S = 0$ is any conic, $\Theta = C + \epsilon(A + B)$, $\Theta' = \epsilon(a + b)$, while $\Delta' = 0$.

Unless we proceed strictly in this way, we may draw wrong conclusions from invariant relations such as $\Theta^2 = 4\Delta\Theta'$. This relation means that a triangle can be circumscribed about $S$ and inscribed in $S'$, but such a triangle must collapse into the line at infinity, and the condition then implies that the conic $S$ touches the line at infinity, and is therefore a parabola. This is correctly indicated by substituting the above values for $\Theta$ and $\Theta'$ which give

$$C^2 + 2\epsilon C(A + B) = 4\Delta\epsilon(a + b),$$

reducing to $C = 0$ when $\epsilon \to 0$. If, on the other hand, we took $\Theta = C$ and $\Theta' = a + b$, we should get the relation $C^2 = 4\Delta(a + b)$ which is quite wrong.

the condition that $S$ and $S'$ should have double contact is the condition that $S$, $S'$ and $\mathsf{F}$ should belong to the same pencil, and therefore that $S$ should pass through the circular points. We find

$$\mathsf{F} \equiv Cx^2 + Cy^2 + (A+B)z^2 - 2Fyz - 2Gzx = 0,$$

and the conditions that $S$ and $S'$ should have double contact are

$$\left\| \begin{matrix} a & b & c & f & g & h \\ 0 & 0 & 1 & 0 & 0 & 0 \\ C & C & A+B & -F & -G & 0 \end{matrix} \right\| = 0.$$

It will be found that these conditions require either (i) $a=b$ and $h=0$ or (ii) $C=0$, $F=0$, $G=0$. The first alternative gives a circle, the second gives two parallel straight lines. It frequently happens that we have to consider a pair of parallel straight lines as a degenerate form of circle, since a pair of lines intersecting on the line at infinity is a degenerate hyperbola having double contact with the line at infinity. Thus the theory of reciprocity would require that four circles can be drawn to pass through three given points $A$, $B$, $C$, just as four circles can be drawn to touch three lines. These four "circles" are the ordinary circumscribed circle of the triangle $ABC$, and the three pairs of parallel lines consisting each of a side $BC$ and the line through the opposite vertex $A$ parallel to this side.

Considering the points of intersection of $\mathsf{F}$ and $S$, which are the points of contact of the tangents to $S$ from the circular points, we see that the *common chords of a conic and its orthoptic circle are the polars of the foci, i.e. the directrices, and the polars of the circular points, i.e. a pair of imaginary diameters.*

## EXAMPLES XX.

**1.** The conic whose equation in rectangular coordinates is

$$ax^2 + 2hxy + by^2 = 1$$

is such that $ab - h^2 = a + b - 1$, investigate the geometrical meaning implied by this relation.

2. If $2(a+b)^2 = 9(ab - h^2)$, prove that the circle whose diameter is the minor axis passes through the foci.

3. Determine the homogeneous relations which subsist among the invariants of pairs of the conics:

$$S_1 \equiv (\alpha x + \beta y)^2 - 2x = 0, \qquad S_2 \equiv \beta^2(x^2 + y^2) - 2x = 0,$$
$$S_3 \equiv 2xy + 2\beta\lambda x + 4\alpha\lambda y - 2\lambda/\beta = 0,$$

and verify the corresponding geometrical relations between the conics.

(Math. Tripos II., 1913.)

4. If $S$ and $S'$ have three-point contact at $O$ and cut again in $L$, and if $M$, $N$ are the points in which $\mathsf{F}$ cuts $S$ and $S'$, prove that $OM$, $ON$ are harmonic conjugates with regard to $OL$ and the tangent at $O$.

5. If the conic-locus $S$ is apolar to the conic-envelope $\Sigma'$, prove that the conic-envelope $\Sigma$ is apolar to $\mathsf{F}$, and the conic-locus $S'$ is apolar to $\Phi$. Prove also that in this case a triangle can be inscribed in $S$ and circumscribed about $\Phi$, and one circumscribed about $S'$ and inscribed in $\mathsf{F}$.

6. If the conic-locus $S$ is apolar to the conic-envelope $\Sigma'$, show that the reciprocal of $S$ with regard to $S'(\Sigma')$ is apolar to $S'$.

7. If a conic-locus $S$, which is apolar to a given conic-envelope $\Gamma$, is apolar to its reciprocal $\Sigma'$ with respect to $\Gamma(C)$, show that $S$ cuts $C$ in an equianharmonic tetrad, which is the same as that consisting of the four points of contact on $\Gamma$ of the common tangents of $\Sigma'$ and $\Gamma$.

8. Prove that the condition that $\mathsf{F}$ and $\Phi$ should be apolar is

$$\Theta\Theta'+3\Delta\Delta'=0.$$

9. Prove that the condition that a triangle can be inscribed in $\mathsf{F}$ and circumscribed about $\Phi$ is $\Theta^2\Theta'^2+10\Delta\Delta'\Theta\Theta'-3\Delta^2\Delta'^2-4\Delta'\Theta^3-4\Delta\Theta'^3=0$.

10. Show that all the conic-loci apolar to four given conic-envelopes form a pencil.

11. Prove that there is one conic-envelope which is apolar to four given conic-loci.

12. Prove that the Jacobian of three conics passes through the nine vertices of their common self-conjugate triangles taken in pairs.

13. The Jacobians of four conics taken in sets of three have six common points which are the degenerate members of the pencil of conic-envelopes apolar to the four conics.

14. If the conic-locus $S_1$ is apolar to the $\Phi$-conic of $S_2$ and $S_3$, then $S_2$ is apolar to the $\Phi$-conic of $S_3$ and $S_1$, and $S_3$ is apolar to the $\Phi$-conic of $S_1$ and $S_2$.

15. If the conic-envelope $\Sigma'$ is apolar to the orthoptic circle of $\Sigma$, show that $\Sigma$ is apolar to the orthoptic circle of $\Sigma'$, and that the two orthoptic circles cut at right angles.

16. If $S$ and $S'$ are circles, show that their $\Phi$-conic has its real foci at the centres of the circles.

17. If $S$ and $S'$ are homothetic, show that their $\Phi$-conic touches each of their asymptotes.

18. Prove that the discriminant of $\Phi$ is $(\Theta\Theta'-\Delta\Delta')^2$, and that of $\mathsf{F}$ is $\Delta\Delta'(\Theta\Theta'-\Delta\Delta')$, and hence that if $\mathsf{F}$ degenerates to a line-pair $\Phi$ degenerates to a point-pair, and conversely (provided $S$ and $S'$ do not degenerate).

19. Show that the $\Phi$-conic of a rectangular hyperbola and a circle is a parabola.

20. If the asymptotes of a rectangular hyperbola are parallel to the axes of another conic, show that the $\Phi$-conic of the two conics is a parabola.

21. If a circle and a rectangular hyperbola are such that their $\mathsf{F}$ and $\Phi$-conics both degenerate, show that one of their common chords is a diameter of the circle and another is a diameter of the hyperbola.

22. Show that a circle (locus) will be apolar to a rectangular hyperbola (envelope) if the circle passes through the centre of the hyperbola, and a circle (envelope) will be apolar to a rectangular hyperbola (locus) if the hyperbola passes through the centre of the circle.

23. Show that the $\mathsf{F}$-conic of two parabolas is a hyperbola whose asymptotes are parallel to the axes of the parabola.

24. If a circle (locus) is apolar to a concentric hyperbola (envelope), show that the circle is the orthoptic circle of the conjugate hyperbola.

25. If a circle (locus) is apolar to a parabola (envelope), show that the centre of the circle lies on the directrix of the parabola.

26. If a circle (locus) is apolar to a conic (envelope), show that the circle cuts orthogonally the orthoptic circle of the conic.

27. If a circle (envelope) of given radius is apolar to a given parabola (locus), show that the locus of its centre is an equal parabola.

28. If a parabola (locus) $S$ is apolar to a parabola (envelope) $\Sigma'$, show that the diameter of $S$ through the point at which the tangent is parallel to the axis of $\Sigma'$ touches $\Sigma'$.

29. If two parabolas are mutually apolar, show that they intersect in a point at which the tangent to each is parallel to the axis of the other. Find the F and $\Phi$ conics.

30. Verify for a circle $S$ and a parabola $S'$ that if $\Theta^2=4\Delta\Theta'$, the circle passes through the focus of the parabola.

31. $\Sigma$ is a given conic-envelope and $A, B, C, D$ are four points such that the lines $AB, CD$ and $AC, BD$ are conjugate with regard to $\Sigma$; prove that $AD, BC$ are also conjugate lines, and that every conic-locus through $A, B, C, D$ is apolar to $\Sigma$.

32. If two circles are mutually apolar, prove that their radii are equal and that they cut at an angle 120°.

Show further that the F and $\Phi$-conics of the two circles is a hyperbola whose foci are the centres of the two circles, latus rectum equal to the diameter, and eccentricity $=\sqrt{3}$. Also show that the tangents at the six real points of intersection of the three curves coincide in pairs and form two equal equilateral triangles.

33. If $\Sigma$ is apolar to $S'$, prove that the centre of perspective of any inscribed triangle of $S'$ and its polar triangle with regard to $\Sigma$ lies on $S'$; and that the axis of perspective of any circumscribed triangle of $\Sigma$ and its polar triangle with regard to $S'$ is a tangent of $\Sigma$.

34. If two of the vertices of a self-conjugate triangle with respect to $S$ lie on $S'$, prove that the locus of the third vertex is $\Theta S-\Delta S'=0$.

35. Two conics intersect in four points $A, B, C, D$. Show that if the tangents at $C$ are harmonic conjugates with regard to $CA, CB$, the tangents at $D$ are harmonic conjugates with regard to $DA, DB$; and prove that the condition that this relation should hold for some selection of the points of intersection is $\Theta\Theta'=\Delta\Delta'$. Verify that this relation is satisfied for two orthogonal circles.

36. If two circles are connected by the invariant relation $\Theta\Theta'=\Delta\Delta'$, show that either they are orthogonal or else $\frac{1}{2}d^2=r^2+r'^2$, where $r, r'$ are their radii and $d$ the distance between their centres.

37. Prove that if a triangle can be inscribed in the ellipse $x^2/a^2+y^2/b^2=1$ and circumscribed about the ellipse $x^2/a'^2+y^2/b'^2=1$, $\pm a'/a\pm b'/b=1$.

38. Prove that if a triangle can be inscribed in the hyperbola $x^2/a^2-y^2/b^2=1$ and circumscribed about the hyperbola $x^2/a'^2-y^2/b'^2=1$, then $a'/a-b'/b=-1$ or $+1$, according as the vertices of the triangle all lie on the same branch or on different branches.

39. Two confocal ellipses are such that a polygon can be inscribed in one and circumscribed about the other; prove that the perimeters of all such polygons are equal. (M. Chasles, 1845)

40. Two homothetic and concentric ellipses are such that a polygon can be inscribed in one and circumscribed about the other; prove that the areas of all such polygons are equal.

41. If two conics $S$, $S'$ are such that one quadrilateral can be inscribed in $S$ and circumscribed about $S'$, prove that an unlimited number of such quadrilaterals exist. Prove that their diagonals both pass through a fixed point, and that the joins of the points of contact of the opposite sides with $S'$ also pass through this point; and prove that the invariant relation connecting the two conics is $8\Delta'^2\Delta - 4\Delta'\Theta\Theta' + \Theta'^3 = 0$.

42. Deduce from the last example that if a circle through a pair of foci of a conic cuts the asymptotes in $P$, $P'$, $Q$, $Q'$, $PP'$ and $QQ'$ are tangents to the conic.

43. If a conic $S$ passes through the four foci of a conic $S'$ and cuts the tangents at the ends of any diameter in $P$, $P'$, $Q$, $Q'$, prove that $PP'$ and $QQ'$ are tangents to $S'$.

44. Determine the distance $d$ between the centres of two circles $S$, $S'$ of radii $r$, $r'$ if a triangle can be inscribed in $S$ and circumscribed about $S'$.

45. If a quadrilateral can be inscribed in the ellipse $x^2/a^2 + y^2/b^2 = 1$ and circumscribed about the ellipse $x^2/a'^2 + y^2/b'^2 = 1$, prove that $\pm a'^2/a^2 \pm b'^2/b^2 = 1$, and distinguish the different cases.

46. If two conics $S$ and $S'$ are such that quadrilaterals can be inscribed in $S$ and circumscribed about $S'$, prove (i) that the poles, with regard to $S'$, of one pair of common chords lie on $S$, and (ii) that the polars, with regard to $S$, of one pair of points of intersection of common tangents touch $S'$; and conversely, that if either of the relations (i) or (ii) is true, the conics admit inscribed and circumscribed quadrilaterals.

47. Prove that if two conics $S$, $S'$ intersect in four distinct points, and if $\Theta^3 - 4\Delta\Theta\Theta' + 8\Delta^2\Delta' = 0$, quadrilaterals can be circumscribed about $S$ and inscribed in $S'$.

48. If the centre of the circle $S$ lies on the circle $S'$, prove that quadrilaterals can be inscribed in $S'$ and circumscribed about $S$.

49. Points on the circle $x^2 + y^2 = r^2$ being represented by $r\cos\alpha$, $r\sin\alpha$, show that consecutive vertices of a quadrilateral inscribed in the circle

$$x^2 + y^2 = a^2 + b^2$$

and circumscribed about the ellipse $x^2/a^2 + y^2/b^2 = 1$ are connected by the relation $\tan\alpha_1 \tan\alpha_2 = -b^2/a^2$; and for the circle $x^2 + y^2 = a^2 - b^2$ and the same ellipse the relation is $\sin\alpha_1 \sin\alpha_2 = b^2/(a^2 - b^2)$.

50. If $A$, $B$, $C$ are three fixed points on a conic $S(\Sigma)$, prove that the pencil of conic-loci passing through $A$, $B$, $C$ and apolar to $\Sigma$ have as their fourth common point the pole of the Hessian axis of the triad $ABC$.

51. If $ABC$ and $DEF$ are two triangles such that $B$, $C$ are conjugate points with respect to the polar conic of $A$ with regard to the triangle $DEF$, prove that $C$, $A$ have the same property with regard to the polar conic of $B$, and similarly $A$, $B$ with regard to the polar conic of $C$.

# ANSWERS TO THE EXAMPLES.

## Chapter I. § 5.

1. (i) 2, 3·46 ; (ii) $-\sqrt{2}, \sqrt{2}$ ; (iii) 0, 4.
2. (i) $\sqrt{2}, \frac{1}{4}\pi$ ; (ii) 13, 112° 37′ ; (iii) $-\sqrt{7}$, 40° 54′ ; (iv) 0, any angle.
3. (i) 13, 10, 11·7 ; 34·7. (ii) 10·4, 18·8, 12·8 ; 42·0.
4. (i), (iv), (vi) isosceles ; (ii), (iii), (v) equilateral.

## Chapter I. § 9.

1. (i) $-22\frac{1}{2}$, (ii) 15, (iii) $8\frac{3}{4}$.
2. (*a*) (i) 48, (ii) $74\frac{1}{2}$ ; (*b*) (i) $-6\frac{1}{2}$, (ii) 46.

## Chapter I. § 12.

1. (i) 0, 7 ; (ii) −6, 4 ; (iii) 8, 11 ; (iv) 16, 15 ; (v) −14, 0.
2. 4 : 3.
3. (i) both bisected ; (ii) 1 : 1, 1 : 2 ; (iii) 1 : 1, 2 : 3.
4. −2 : 5, 1 : 2.

## EXAMPLES I.

1. 85, 53, 104 ; 242 ; 2244.
2. $31 + 24\frac{1}{2} + 26\frac{1}{2} = 82$.
4. $-\frac{1}{3}, \frac{3}{4}$.
5. 9, 8.
7. (i) 1 : 1 and 2 : 1, (ii) 2 : 3 and 2 : 1.
8. 5.
9. 11, 12. *PCAB* is a parallelogram.
15. 8, 2.
16. $k = -2$ or $\frac{1}{2}$. Points 3, 4 and −7, −1.

## Chapter II. § 4.

1. (i) $\frac{3}{4}$ ; −4, 3 ; $-\frac{3}{5}x + \frac{4}{5}y = \frac{12}{5}$. (ii) $-\frac{12}{5}$ ; $\frac{13}{4}, \frac{39}{5}$ ; $\frac{12}{13}x + \frac{5}{13}y = 3$.
(iii) $\frac{15}{8}$ ; $-\frac{34}{15}, \frac{17}{4}$ ; $-\frac{15}{17}x + \frac{8}{17}y = 2$. (iv) $-\frac{11}{60}$ ; $\frac{61}{11}, \frac{61}{60}$ ; $\frac{11}{61}x + \frac{60}{61}y = 1$.
(v) 1 ; 8, −8 ; $x \cos(-45°) + y \sin(-45°) = 4\sqrt{2}$.
2. (i) $3x + y + 1 = 0$, (ii) $x + 6y + 1 = 0$, (iii) $3x + y = 9$.
3. (i) $x = 2 + t,\ y = 3 + 2t$ ; (ii) $x = 1 - 3t,\ y = 4 + t$ ; (iii) $x = 1 + 3t,\ y = 3 - t$.
4. $4x - 3y = 11$ in each case.
5. (i) $x = 7 - 3t,\ y = t$ ; (ii) $x = 3 + 4t,\ y = -1 + 3t$ ; (iii) $x = 2 + 3t,\ y = 2 + 7t$ ;
(iv) $x = -\frac{1}{2} + 3t,\ y = 1 - t$.

## Chapter II. § 7.

1. (i) 45°, (ii) 8° 08′, (iii) 70° 34′.
2. (i) 5, (ii) 3, (iii) 8.
3. (i) 2, (ii) 2, (iii) 1.
6. (i) $y = 5x - 15$, (ii) $3y + x = 12$, (iii) $2x + 3y = 18$, (iv) $7x - 4y = \pm 14$, (v) $4x + 3y = \pm 40$

### Chapter II. § 12.

1. (i) $-\frac{17}{7}, -\frac{22}{7}$; (ii) $-\frac{1}{11}, -\frac{12}{11}$; (iii) $-\frac{11}{8}, \frac{17}{8}$; (iv) $8, -1$; (v) $-4, 5$; (vi) $-\frac{11}{6}, -\frac{3}{2}$.

2. (i) $29x+6y=0$, (ii) $4x-9y+19=0$, (iii) $23x+46y=78$, (iv) $46x-23y=31$.

4. (i) $3x+y-2=0$, $x-3y-8=0$; (ii) $8x+y+10=0$, $x-8y-15=0$; (iii) $11x+3y+3=0$, $3x-11y-11=0$.

### Chapter II. § 15.

1. (vi) $(2x-3)(5y+4)$, (vii) $(x+2)(2x-y+1)$, (viii) $(6x+y-2)(x-3y+4)$, (ix) $(2x-5y-3)(5x+y-7)$, (x) $(2x+3y)(3x-5y+8)$, (xi) $(2x-3)^2$, (xii) $(x+2y-3)(3x-y+1)$, (xiii) $(y-x\cot\frac{1}{2}\theta)(y-x\tan\frac{1}{2}\theta)$, (xiv) $(y-x\tan\frac{1}{2}\theta)(y+x\cot\frac{1}{2}\theta)$, (xv) imaginary, (xvi) irrational, (xvii) $(2x+3y-5)^2$, (xviii) irrational, (xix) irrational, (xx) imaginary.

2. (i) 0, 3; (ii) 4, $\frac{1}{2}$; (iii) $7\frac{1}{2}$; (iv) $-12$.

### Chapter II. § 18.

1. $3x^2-8xy-3y^2=0$.
2. $ax=by$.
3. $5x^2-11xy-5y^2=0$.
4. $45°$.
5. $(3x-2y)^2=0$. The locus consists of two lines through (2, 3).
6. (i) $45°$, (ii) $60°$, (iii) $\tan^{-1}2=63°\ 26'$, (iv) $60°$, (v) $45°$, (vi) $\tan^{-1}\frac{3}{5}=30°\ 58'$.
7. (iv) $1, -2$; (v) $2, -3$; (vi) $8/3, 5/3$.

### Chapter II. § 22.

1. (i) $2x^2-xy-2y^2=0$, (ii) $3x^2+2xy-3y^2=0$, (iii) $3(x-1)^2+(x-1)(y+2)-3(y+2)^2=0$, (iv) $5x^2-4xy-5y^2+34x-2y+52=0$.
2. $2y=x$, $5y+2x=0$, $y+4x=0$.
3. 22/7, 16, 19/4.
4. $ax+hy+\mu(hx+by)=0$.

### EXAMPLES II. A.

1. (i) $y=4$, (ii) $12x+5y+3=0$, (iii) 10.
3. $G\equiv(\frac{10}{3}, \frac{7}{3})$, $O\equiv(\frac{7}{2}, \frac{5}{2})$, $S\equiv(\frac{13}{4}, \frac{9}{4})$.
4. $7x-9y-310=0$, $x+2y+97=0$, $11x-3y-8=0$, concurrent in $(-11, -43)$.
6. (i) $\lambda\mu-\lambda+2\mu-3=0$, (ii) $11\lambda\mu-\lambda+13\mu-11=0$.
7. $2x+y=2\pm\sqrt{5}$.
11. (i) $t=2(u-1)$, (ii) $tu+2t-2u+1=0$.
12. $-220, 60$; $-12, -44$; $-45, -165$; $33, -9$.
13. 8, 6; $-31, -7$; $68, -34$; 29, 83.
15. $k=0$, $(3x+4y)(x-2y-5)=0$; $k=\frac{15}{2}$, $(7x+y+25)(3x-y-15)=0$; $k=-\frac{5}{3}$, $(x+3y-5)(x-7y-25)=0$.
16. $5x^2-24xy-5y^2-52x+260y=0$.
17. $9x^2+4y^2+13xy-75x-50y+150=0$.
20. 0, 1.
21. 1, $-2$.
23. (i) $(3, -1), (5, 5)$; (ii) $(2, -2), (0, 6)$; (iii) $(-2, 2), (4, 6)$; (iv) $(-b, a), (b, -a)$; (v) $\{\frac{1}{2}(x_1+x_2-y_1+y_2), \frac{1}{2}(x_1-x_2+y_1+y_2)\}$, $\{\frac{1}{2}(x_1+x_2+y_1-y_2), \frac{1}{2}(-x_1+x_2+y_1+y_2)\}$.

### EXAMPLES II. B.

7. $-\frac{11}{2}, -\frac{7}{3}, -\frac{9}{7}$.
8. Four lines $x\pm y\pm k\cot\alpha=0$.
12. $x^2+y^2=a^2$.

### Chapter III. § 3.

(i) $3x+4y=25$, $4x=3y$; (ii) $3x\pm2y+13=0$, $2x\mp3y=0$; (iii) $x\cos\alpha+y\sin\alpha=a$, $y=x\tan\alpha$.

### Chapter III. § 6.

Q. The line at infinity.

## Chapter III. § 7.

Q. The point at infinity in the direction perpendicular to the diameter.

## Chapter III. § 8.

1. $x+3y=4$, $2x+y=4$, $3x-y=4$; concurrent in $(\frac{8}{5}, \frac{4}{5})$. 2. 24, $-8$.

## Chapter III. § 13.

1. 3. 2. (i) $y=\sqrt{3}(x\pm 2)$, (ii) $y=x\pm\sqrt{6}$. 3. 3, 4; $-4$, 3.

## Chapter III. § 14.

Q. The circle degenerates to a straight line and the line at infinity.

1. (i) 3, $-1$; 7. (ii) $\frac{1}{2}, \frac{1}{2}$; $\frac{1}{2}\sqrt{2}$. (iii) $\frac{1}{2}, \frac{5}{8}$; $\frac{5}{8}$.
(iv) 0, $\frac{7}{6}$; $\frac{7}{6}$. (v) $a$, 0; $a$. (vi) $-\frac{2}{3}, \frac{2}{3}$; $\frac{1}{3}\sqrt{11}$.
2. (i) $x^2+y^2+6x-3y-1=0$, (ii) $x^2+y^2+x-2y-1=0$, (iii) $x^2+y^2+3x-5y-4=0$, (iv) $x^2+y^2-2x-4y=0$, (v) $x^2+y^2-ax-by=0$.

## Chapter III. § 16.

2. (i) $3x-4y+14=0$, $4x+3y+2=0$; (ii) $4x-9y=0$, $9x+4y=0$.
3. $x+y+2=0$. 5. (i) $-13$, $-2$; (ii) 5, 2.
6. $S=-39, -20, 72, 49$; $S'=76, -21, -17, 52$. 7. (i) 60°, (ii) 45°.
8. (i) $8x^2=15xy$, (ii) $(x-y)(17x+31y)=0$.
9. (i) $x^2+y^2-ax\pm ay+\frac{1}{4}a^2=0$,
(ii) $x^2+y^2-8x-6y+21=0$, $x^2+y^2-28x+14y+101=0$,
$x^2+y^2-16x-22y+149=0$, $x^2+y^2+4x-2y-11=0$.

## EXAMPLES III. A.

1. $x^2+y^2=6y$, $x^2+y^2+24y=0$. 2. (i) 0, 0; (ii) 2, $-1$; (iii) $-3$, $-5$. 4. $\frac{4}{3}$, $-\frac{4}{3}$.
6. The point of intersection $(3, -1)$ of the line of centres with the polar of the first point; also the point at infinity $(2, 1, 0)$. 7. The chord of contact.
8. $(x-1\pm 3)^2+(y+3)^2=9$, $(x-1\pm\sqrt{21})^2+(y-3)^2=9$.
9. $x^2+y^2-6x-6y+9=0$, $x^2+y^2-30x-30y+225=0$.
11. $ax+by=0$. 12. 1 and 5. 13. $4x^2+4y^2-x=18$.
14. $4x-3y+25=0$, $3x+4y=25$. 15. $24x^2-14xy-24y^2=0$.
18. Two circles passing through the two centres; if the angle is a right angle the locus consists simply of the circle whose diameter is the join of the two centres.
19. 3, 0; 7, 0; $3x+y=9$, $x-3y=3$; $3x-y=21$; $x+3y=7$.

## EXAMPLES III. B.

1. $p=a(\cos\alpha\pm 1)$, cardioid curve. 5. $x^2+y^2-2\alpha x-2\beta y+c^2=0$.
8. The locus is a curve of the fourth degree. If the centres and radii of the given circles are $(\alpha, 0)$, $r$ and $(-\alpha, 0)$, $r'$, the bisectors touch one of the circles

$$x^2+(y\pm\alpha)^2=\tfrac{1}{4}(r\mp r')^2.$$

11. If the given circle is $x^2+y^2=r^2$ and $A\equiv(a, 0)$ the locus is the circle

$$(r^2-a^2)(x^2+y^2)+2ar^2x=2r^4,$$

which becomes the tangent at $A$ when $a=r$.
12. A circle with centre half way between the centre and the given point.

## Chapter IV. § 6.

4. Outside, inside. 5. $-1<t<\frac{1}{3}$. 6. 2, 1.

### Chapter IV. § 9.

2. $y=\sqrt{3}x\pm 7$.
5. (i) $-10{\cdot}5$, 1; (ii) $-1{\cdot}5$, $-1$; (iii) $-4{\cdot}5$, 1.
6. $\frac{5}{21}$, $\frac{8}{21}$.
7. $2x+3y=5$.
8. $3x^2+4y^2-6x-12y=0$, a similar ellipse.

### Chapter IV. § 19.

1. Take as the eccentric angles of the two points $\phi\pm\alpha$.

### Chapter IV. § 20.

Q. All four foci coincide at the centre, and the four directrices coincide with the line at infinity.

### Chapter IV. § 22.

Q. No. See § 20, Q.

### Chapter IV. § 30.

Q. Of the four normals from any point two are real and coincide with the diameter through the point, the other two are the lines joining the point to the circular points.

### EXAMPLES IV. A.

1. $\pm\cos^{-1}e$, $\pi\pm\cos^{-1}e$.
2. $l/r=e\cos\theta+\cos\{\frac{1}{2}(\theta_1+\theta_2)-\theta\}\sec\frac{1}{2}(\theta_1-\theta_2)$.
6. $x^2/a^2+y^2/b^2=\sec^2\alpha$.
7. $x^2+y^2=(a+b)^2$.
9. Use § 16.
10. Ellipse whose foci are the centres of the two circles.
32. The tangents at $A$ and $A'$, and the two ellipses $(x^2-a^2e^2)/(1\pm e)+y^2/(1\mp e)=0$.
33. $a^2/l^2+b^2/m^2=c^4/n^2$.
37. $(x^2+y^2-a^2-b^2)^2\tan^2\alpha=4(b^2x^2+a^2y^2-a^2b^2)$.
47. ·017.
48. ·0564.
49. ·017, 281° 00′.

### EXAMPLES IV. B.

2. Use Ex. IV. A, 12.
6. $a^2x^2+b^2y^2=a^4+b^4$.
7. $x^2/a^2+2y^2/(a^2+b^2)=1$.
8. The ellipse $x^2/(b-b')^2+y^2/(a-a')^2=(ab'-a'b)^2/\{(a-a')^2-(b-b')^2\}^2$.
9. The ellipse $x^2/b^2+y^2/a^2=a^2b^2/(a^2-b^2)^2$.
10. If the fixed line is $lx+my=1$, and the ellipse $x^2/a^2+y^2/b^2=1$, the locus is
$$a^2lx+b^2my=\tfrac{1}{2}(a^2+b^2).$$
11. The given ellipse, and the ellipse $x^2/a^2+y^2/b^2=4$.
17. The circle is the auxiliary circle of the ellipse, and $O$ is a focus. If $O$ is on the circle the envelope reduces to two points.
18. $x/a-4y/b+2=0$.

### EXAMPLES V. A.

3-7. Use equation $xy=c^2$.
15. Let $CA$, $CB$ be the asymptotes, $AB$ a tangent. Draw the circle $ABC$, and let the exterior bisector of the angle $C$ cut the circle again in $O$. Then the circle with centre $O$ and radius $OA$ passes also through $B$ and cuts the interior bisector of the angle $C$ in the two foci.
16. Two conics whose foci are the centres of the fixed circles, their asymptotes being perpendicular to the common tangents.
17. If $\phi+\phi'=2\alpha$, the line passes through $(0, -b\cot\alpha)$.

### EXAMPLES V. B.

2. Use converse of theorem: if a quadrilateral is circumscribed about a circle the sums of its opposite sides are equal; or Ex. 1.

6. If $\tan\alpha = \pm b/a$ the normal is perpendicular to an asymptote and is at infinity There is no real normal for $\tan\alpha > b/a$.
13. The point $(-a, 0)$, the line $x = a$.
16. A hyperbola with the given points as foci.

### Chapter VI. § 7.

3. Lat. rect. $= a$, focus $\{(\frac{1}{4}a^2 - b)/a, 0\}$.
4. Lat. rect. $= 1/a$, vertex $\{-b/a, (ac - b^2)/a\}$.
5. (i) $(-2, 5)$, $\tan^{-1}4 = 75° 58'$ ; (ii) $(1, -3)$, $180° - \tan^{-1}\frac{1}{2} = 153° 26'$ ; (iii) $(-0{\cdot}83, 1{\cdot}79)$ ; $26° 34'$.
6. (i) $(\frac{7}{5}, 0)$, $\tan^{-1}\frac{4}{3} = 53° 08'$ ; (ii) $(-\frac{5}{3}, 0)$, $180° - \tan^{-1}\frac{3}{2} = 123° 41'$ ; (iii) $(\frac{13}{20}, 0)$, $55° 17'$

### Chapter VI. § 15.

Q. The diameter through the point is a normal at infinity.

### EXAMPLES VI. A.

2. $4x + 3y + 1 = 0$.
6. 1890 ft., $16\frac{3}{4}°$.
7. 95 ft. from shore end ; 66·4 ft. below vertex ; $35° 36'$, $55° 04'$.
8. $\tan^{-1}\frac{13}{21} = 31° 46'$ ; 52 ft.
9. $\tan^{-1}8 = 82° 52'$.
11. The chord joining the ends of two rectangular chords through a fixed point $M$ on the parabola cuts the normal at $M$ in a fixed point (the Frégier point of $M$. Cf. VIII. § 9).
17. $3x^2 - y^2 + 10ax + 3a^2 = 0$. The other branch corresponds to the angle 120°.
18. The directrix.
22. The point $(-2a, 0)$, and the directrix.
23. $-\frac{2}{3}(2a - h)$, 0.
29. If $\theta$ is the vectorial angle, referred to the focus, of one extremity of the chord, the vectorial angle of the point of intersection of two consecutive chords is $\theta + 45°$ ; and the equation of the envelope is $l/r = e(1 + \cos\theta)$.

### EXAMPLES VI. B.

8. If $\theta$ is the inclination of either tangent to the axis, $r \pm r' = 2a\cos\theta\,\mathrm{cosec}^2\theta$.
14. If $(x, y)$ is the given point, $\theta_1 + \theta_2 + \theta_3 = \tan^{-1}y/x$.

### Chapter VII. § 5.

3. $(l^2 + m^2)(cm^2 + n^2) = 0$. The line passes through one of the limiting points, or through one of the "circular points" (see XI. § 14).

### Chapter VII. § 10.

Q. The remainder of the Jacobian is the line at infinity.

### EXAMPLES VII. A.

1. (i) 1, 0 ; −1, 0. (ii) 2, 1 ; $\frac{1}{5}$, $\frac{8}{5}$.
3. −2, −1.
4. $x^2 + y^2 - 4x - 2y + 2 = 0$.
6. ±3, 0.
7. $(a^2 + b^2)(x^2 + y^2) + 2c(ax + by) = 0$.
12. $lxy - mx^2 = ny - mc$ ; in general a hyperbola, but a parabola when $l = 0$. When $m = 0$ we get really only the axis of $x$, but the equation gives also $x = n/l$. This line is the locus of harmonic conjugates of any point on the line $x = -n/l$ with respect to the degenerate circle which consists of the radical axis $x = 0$ and the line at infinity.
13. $2\alpha\alpha' + 2\beta\beta' = c - c' + 2(\alpha'^2 + \beta'^2)$.

### EXAMPLES VII. B.

1. $2x-2y=a+b$, $2(b+c)x-2by=(b+2c)(a+b)$, $2(a+c)x-2ay=(a+2c)(a+b)$.
Radical centre $a+b$, $\frac{1}{2}(a+b)$. Orthotomic circle
$$\{x-(a+b)\}^2+\{y-\tfrac{1}{2}(a+b)\}^2=\tfrac{1}{4}(a-b)^2.$$
2. $-g/(1+\lambda)$, $-f/(1+\lambda)$, where $\lambda$ is a root of the equation
$$(f^2-c')\lambda^2-(c+c')\lambda+(g^2-c)=0.$$
15. A straight line. 16. $a+b+c=0$.
20. $x^2+y^2+2ax\cot 2\phi-a^2=0$, $x^2+y^2-2ay\coth 2\psi+a^2=0$.
21. $x^2+y^2-x/\phi=0$, $x^2+y^2+y/\psi=0$.
24. Invert with regard to one of the points of intersection and apply Ex. 23.

### Chapter VIII. § 14.

1. $xy=0$. 2. $3x^2+xy=0$. 3. $8x^2+4y^2=1$.
4. Rotate through $\tan^{-1}\frac{3}{4}$; $15x^2-10y^2=1$. 5. Rotate through $\tan^{-1}2$; $3x^2+2y^2=1$.
6. Axes through $(-1, 1)$, rotated through $\tan^{-1}\frac{3}{4}$; $18x^2-7y^2+54=0$.
7. Axes through $(-\frac{3}{5}, -\frac{1}{5})$, rotated through $\tan^{-1}\frac{1}{3}$; $10(x^2-y^2)=1$.

### EXAMPLES VIII.

12. $x(a+b\cos\omega)+y(b+a\cos\omega)=ab\cos\omega$.
16. If parallels through the given point to the given straight lines meet the latter in $M$ and $N$, and $O$ is the point of intersection of the given straight lines, the hyperbola passes through $O$, and its asymptotes are the perpendiculars to $OM$ and $ON$ at their mid-points.
21. Sum of squares $=2(p^2+q^2+r^2+s^2)$; $\sin^{-1}2(ps-qr)/(p^2+q^2+r^2+s^2)$.

### EXAMPLES IX.

1. (i) $-1, 3$; (ii) $\frac{1}{5}, \frac{2}{5}$; (iii) $1, -\frac{1}{2}$.
2. (i) $x+y+1=0$, $x-y+2=0$; 1, 2. (ii) $2x+y+1=0$, $x-2y-1=0$; 1, 3.
(iii) $3x-2y+1=0$, $2x+3y-2=0$; $a^2=1$, $b^2=-1$.
3. (i) $3x+2=0$, $2y+3=0$; (ii) $5x+5y+4=0$, $10x-15y+12=0$.
(iii) $y-2\pm\sqrt{3}(x+y+1)=0$.
4. (i) Hyperbola. Centre $(-1, 2)$. Axes $x+2y=3$, $2x-y+4=0$.
(ii) Ellipse. Centre (1, 2). Axes 4 and 2, parallel to coord. axes.
(iii) Ellipse. Centre (1, 2). Axes $x-3y+5=0$, $3x+y=5$.
(iv) Ellipse. Centre (2, 1). Axes $x-2y=0$, $2x+y=5$.
(v) Parabola. Vertex $(-\frac{1}{5}, -\frac{3}{5})$. Axis $2x+y+1=0$.
(vi) Ellipse. Centre (1, 1). Axes $2x+y=3$, $x-2y+1=0$.
(vii) Rect. Hyp. Centre $(-1, 2)$. Axes $4x-3y+10=0$, $3x+4y=5$.
5. $(3-\sqrt{2})x^2+(3+\sqrt{2})y^2=12$. 6. $2\sqrt{5}$, $x+y=0$; $2\sqrt{\frac{5}{3}}$, $x-y=4$.
7. $x^2+2y^2=2$.
8. $2x^2-3y^2+4xy-5x-5y+2=0$. Hyperbola. Centre $(\frac{1}{2}, 0)$. Asymptotes parallel to $y=\frac{1}{3}(2\pm\sqrt{10})x$.
10. Hyperbola. Centre (3, 3). Axes $2x-(3\pm\sqrt{13})y+3(1\pm\sqrt{13})=0$.
11. Parabola. Lat. rect. $=7\sqrt{10}/100$.
12. Hyperbola. $e=2/\sqrt{3}$. Centre $(-\frac{1}{5}, \frac{3}{5})$. Axes $x+2y=1$, $2x-y+1=0$.
13. Ellipse. Centre $(-\frac{18}{25}, \frac{1}{25})$. Axes $3x+4y+2=0$, $4x-3y+3=0$. $e=\frac{1}{2}\sqrt{2}$.
14. Focus $(-\frac{2}{5}, \frac{2}{5})$. Vertex $(-\frac{32}{25}, \frac{13}{25})$. 15. $36x-36y+77=0$. $(-\frac{73}{72}, -\frac{71}{72})$.
16. $7x^2+7y^2+2xy+10x-10y+7=0$. 17. $1, -1, \frac{1}{2}$.

## EXAMPLES X.

1. $7y^2 - 24xy + 20x = 0$.  3. $2fg\lambda = af^2 + bg^2$.
4. $(ab - h^2)(x^2 + y^2) = (ax + hy)^2 + (hx + by)^2$.
6. $(2ab - \lambda)(x^2 + y^2) - 2bx - 2ay = 0$, tangent circles.
9. (i) a similar conic, (ii) a rect. hyperbola.
16. Orthogonal conic $ax^2 + 2hxy - by^2 = (b - a)/(b + a)$.

### Chapter XI. § 3.

1. Point $(2, -1)$.  2. Point $(-\frac{1}{3}, -\frac{2}{3})$.  3. Point $(-\frac{3}{5}, \frac{2}{5})$.
4. Parabola.  5. Circle.  6. Circle.

### Chapter XI. § 4.

1. Circle, $l^2 + m^2 = 1$.  2. Circle, centre $O$.  3. $25l^2 + 16m^2 = 1$, ellipse.
4. $144l^2 - 25m^2 = 1$, hyperbola.  5. $5(l^2 + m^2) = 1$, circle.
6. $|lm| - 2|l| - 2|m| + 2 = 0$, with $|l|$ and $|m|$ each $>2$, four arcs of circles.
7. $lm + l + m = 0$, parabola.  8. $l = m^2$, parabola.

### Chapter XI. § 13.

1. (i) $10l^2 + 3m^2 + 2n^2 + 10mn - 2nl + 14lm = 0$, (ii) $m^2 + 4mn - 2nl + 2lm = 0$, (iii) $44l^2 + 23m^2 + 23n^2 + 10mn - 4nl + 32lm = 0$.
2. (i) $x^2 + y^2 - 4x + 6y - 3 = 0$, (ii) $p(2x + 1)^2 + 2q(2x + 1)(2y + 1) + r(2y + 1)^2 = 0$, (iii) $z^2 = 0$.
3. $-3, 3$.  4. $0, -\frac{3}{2}$; $3, 0$.  5. $2, -1$.  6. $20x - 28y - 45 = 0$.

### Chapter XI. § 17.

3. $(1, -1)$, $2x - 3y + 2 = 0$; $(-1, 2)$, $2x - 3y + 1 = 0$.
4. $(-1, 3)$, $2x - y + 1 = 0$; $(3, 1)$, $2x - y - 1 = 0$; $e = \sqrt{5}$.

### Chapter XI. § 20.

2. (i) 2, 1; 1, −1. (ii) 3, −2; 2, 1. (iii) focus $(-2, 1)$, axis of parabola $y = 3x + 7$.
3. (i) $1, -2$; $2, -\frac{3}{2}$.  (ii) $-2, 3$; $-\frac{1}{5}, \frac{42}{5}$.
(iii) focus $(0, 3)$, axis of parabola $y = 2x + 3$. (iv) $\frac{1}{4}(4 \pm \sqrt{14})$, $-\frac{1}{4}(4 \pm \sqrt{14})$.

## EXAMPLES XI.

1. Directrix $3x + 2y + 4 = 0$. Second focus $(\frac{14}{13}, \frac{8}{13})$, directrix $3x + 2y = 8$. $e = 1/\sqrt{3}$.
2. $16(x^2 + y^2) - 18xy - 8(x + y) + 1 = 0$, $2xy = 1$.
3. Ellipse. Centre $(-\frac{3}{5}, \frac{1}{5})$. Axes $2x - y + 2 = 0$, $x + 2y + 1 = 0$.
Foci $-\frac{1}{5}(3 \pm \sqrt{3})$, $\frac{2}{5}(2 \pm \sqrt{3})$.
4. Centre $(\frac{3}{2}, \frac{1}{2})$. Foci $\frac{3}{2} \pm \sqrt{2}$, $\frac{1}{2} \pm \frac{1}{2}\sqrt{2}$.
13. Parabola with focus at the centre of the circle and vertex at mid-point of $OA$; part of the envelope is also the point at infinity on $OA$.
14. When $\lambda = \mu = 1$ the two conics are homothetic and concentric; if $\lambda = \mu = \frac{1}{2}$ the second conic reduces to the asymptotes of the first.
15. $FG = CH$ and $BC - F^2 = CA - G^2$. Centre $(1/F, 1/G, 1/H)$.
20. $Al^2 + Bm^2 + 2Hlm = 0$ represents the points at infinity on the tangents from the origin. The point $2Gl + 2Fm + Cn = 0$ is the point of intersection of the two tangents which are parallel to the first pair.
21. $Gl + Fm + Cn = 0$ is the centre, and $bl^2 - 2hlm + am^2 = 0$ the two points at infinity on the conic.

## EXAMPLES XII.

3. $\Delta=0,\ A+B+C+2F+2G+2H=0$.

4. Euler line in trilinears $\Sigma\alpha(b^2-c^2)\cos A=0$, in areals $\Sigma x\cos A\sin(B-C)=0$.

6. $x/(m-n)+y/(n-l)+z/(l-m)=0$.

7. (i) $\dfrac{\beta}{\gamma}=\dfrac{\sin\mu_1/\sin(B+\mu_1)}{\sin\nu_1/\sin(C+\nu_1)}$;

(ii) $\dfrac{\sin\lambda_3\sin\mu_1\sin\nu_2}{\sin\lambda_2\sin\mu_3\sin\nu_1}=\dfrac{\sin(A+\lambda_3)\sin(B+\mu_1)\sin(C+\nu_2)}{\sin(A+\lambda_2)\sin(B+\mu_3)\sin(C+\nu_1)}$;

(ix) $\cot\lambda=(k-\cos 2A)/\sin 2A$, etc., where $k$ may have any value; in particular, if $k=1$, $\lambda=90°-A$, etc.; if $k=0$, $\lambda=180°-2A$, etc.

(x) $\tan\lambda=k\tan A$, etc., where $k$ may have any value; in particular, if $k=1$, $\lambda=A$, etc.; if $k\to\infty$. $\lambda=\mu=\nu=90°$.

15. The line $\Sigma x\sin(B-C)=0$. $\theta=0$, the Lemoine point; $\theta=\pm60°$, the isodynamic points; $\theta=90°$, the circumcentre.

16. $\sin(\beta-\gamma)$, $\sin(\gamma-\alpha)$, $\sin(\alpha-\beta)$.

17. Parallel to line $\alpha\cos A+\beta\cos B+\gamma\cos C=0$, the polar of the orthocentre.

## Chapter XIII. § 17.

1. (i) $(l+m+n)^2=0$, (ii) $(l+m-2n)(l-2m+n)=0$,
(iii) $l^2+m^2+n^2-mn-nl-lm\equiv(l+\omega m+\omega^2 n)(l+\omega^2 m+\omega n)=0$,
(iv) $(l+m+n)^2=0$ for all values of $s$.

2. (i) $z^2=0$, (ii) $\{(bc'-b'c)x+(ca'-c'a)y+(ab'-a'b)z\}^2=0$,
(iii) $(ax+by+cz)^2-(a^2+b^2+c^2)(x^2+y^2+z^2)=0$, *i.e.*
$\{(b^2+c^2)x-aby-acz\}^2+(a^2+b^2+c^2)(cy-bz)^2=0$.

## EXAMPLES XIII.

5. The polar of $O$.

11. 8, 2.

15. Let the equations of the conics referred to the triangle $ABC$ be
$$S_1\equiv f_1yz+g_1zx+h_1xy=0,\quad S_2\equiv f_2yz+g_2zx+h_2xy=0.$$
Then, if $T_1\equiv(x_1, y_1, z_1)$ and $T_2\equiv(x_2, y_2, z_2)$,
$$x_1/f_1:y_1/g_1:z_1/h_1=x_2/f_2:y_2/g_2:z_2/h_2.$$

19. Cf. Chap. XVI, § 10.

21. $b(x^2+y^2)+2gx=0$.

22. $x=0,\ y=0,\ x+2z=0,\ -x+12y+4z=0$.

24. The points of intersection of the polar of the Fregier point corresponding to the given point.

## EXAMPLES XIV.

1. If the conic touches $AB$ at $B$ and $AC$ at $C$, the centre-locus is the line joining $A$ to the mid-point of $BC$.

8. $mn\cos A+nl\cos B+lm\cos C=0$.

14. A straight line.

24. $c(a\alpha-b\beta-c\gamma)\pm a(a\alpha+b\beta-c\gamma)=0$.

26. $\Sigma x^2\cot A=0$, $\Sigma l^2\tan A=0$.

27. $(x+y+z)\{s^2x+(s-c)^2y+(s-b)^2z\}=\Sigma a^2yz$, $-mn\cot\tfrac12A+nl\tan\tfrac12B+lm\tan\tfrac12C=0$.

33. $\Sigma a\alpha\,.\,\Sigma a\beta\gamma=0$, *i.e.* the line at infinity and the circumcircle.

34. $lmn\Sigma a^2=\Sigma l(c^2m^2+b^2n^2)\cos A$.

35. $\Sigma\beta^2\gamma(\cos C-\cos A\cos B)=\Sigma\beta\gamma^2(\cos B-\cos C\cos A)$.

38. Express squares of distances of centre from $A$, $B$, $C$.

40. Verify with the help of the identities $2bc\cos A=b^2+c^2-a^2$.

41. Substitute $\xi\xi'=a^2,\ldots\ \eta\zeta'+\eta'\zeta=-2bc\cos A$.

44. The Lemoine circle in areals is $\Sigma x \,.\, \Sigma b^2c^2(b^2+c^2)x = (\Sigma a^2)^2 \,.\, \Sigma a^2yz$.

45. The Cosine circle in trilinears is $\Sigma a\alpha \,.\, \Sigma \alpha b^2c^2 \cos A = \frac{1}{4}(\Sigma a^2)^2 \,.\, \Sigma a\beta\gamma$.

46. If the equations of $Y'Z$, $Z'X$, $X'Y$ in areals are $-px+y+z=0$, $x-qy+z=0$, $x+y-rz=0$, $K \equiv \{1/(p+1),\ 1/(q+1),\ 1/(r+1)\}$. The equation of the conic is $\Sigma x \,.\, \Sigma px - \Sigma(q+1)(r+1)yz = 0$.

47. If $K \equiv (f, g, h)$ in areals the equation of the centre-locus is $\Sigma(g-h)x=0$. If $K$ is the centroid, $f=g=h$, and the straight line becomes indeterminate. In this case all the conics have the centroid as centre.

49. Express that the polar of the centre of either circle with regard to the other is the common chord.

## EXAMPLES XV.

13. A conic. If $\lambda$, $\mu$ are fixed and $d$ is variable, the conics are confocal.

14. In trilinears the inscribed conic is

$$(\theta_1 - \cos A)mn + (\theta_2 - \cos B)nl + (\theta_3 - \cos C)lm = 0,$$

where $\theta_1, \theta_2, \theta_3 = \pm 1$, and the circumscribed conic is

$$(\theta_1 + \theta_2\theta_3)yz + (\theta_2 + \theta_3\theta_1)zx + (\theta_3 + \theta_1\theta_2)xy = 0.$$

$\theta_1, \theta_2, \theta_3$ must be either all + (ellipse), or one + and two − (hyperbola). The points of concurrence of the normals are $(k_1, k_2, k_3)$, $(-k_0, k_3, k_2)$, $(k_3, -k_0, k_1)$, $(k_2, k_1, -k_0)$, where $k_0 \equiv 1 + \cos A + \cos B + \cos C$, $k_1 \equiv 1 + \cos A - \cos B - \cos C$, etc.

15. The tangent to the parabola which makes the given angle with the focal radius to its point of contact.

16. The equation of the locus is $(x^2+y^2-a^2)\tan^2\alpha + 2yae \tan\alpha - b^2 = 0$

17. The equation of the locus is

$$(x+2\lambda)^2 + (x+2\lambda')^2 - 2(x+2\lambda)(x+2\lambda')\cos 2\alpha = (x^2+y^2)\sin^2 2\alpha,$$

which may be written $\{y^2 - 4\lambda(x+\lambda)\}^2 = \{(x+2\lambda)\cos 2\alpha - (x+2\lambda')\}^2$.

25. If the outer parabola has freedom-equations $x=at^2$, $y=2at$, any tangent to the inner cuts the outer in points whose parameters have a constant difference. The distance between corresponding tangents measured parallel to the axis is constant, and therefore the perpendicular distance is proportional to the sine of the inclination of the tangent to the axis.

27. If $S \equiv x^2/a^2 + y^2/b^2 - 1 = 0$, $S' \equiv Ax^2 + 2Hxy + By^2 - 1 = 0$, the condition is

$$a^2A + b^2B = 1.$$

This occurs, for example, when $S'$ is the orthoptic circle of $S$.

30. An ellipse with semi-axes $a-kb$, $b-ka$. If $k=1$, it is a circle with radius $a-b$; $k=-1$, a circle with radius $a+b$; $k=2ab/(a^2+b^2)$ gives the locus of the Frégier point (see Ex. 29).

31. $(am^2+bl^2)(x^2+y^2) - 2blx - 2amy + a + b - l^2 - m^2 = 0$; $ax^2+by^2 = (a-b)^2/(a+b)^2$.

## EXAMPLES XVI.

2. A straight line through the point of contact.

5. A circle. It meets the parabola again in the points $(a, -2a)$ and $(9a, -6a)$ on the line $x+2y+3a=0$

6. Circles $x^2+y^2 = \pm 4(x\cos\alpha + y\sin\alpha)$. Chords of contact $x\cos\alpha + y\sin\alpha = \pm 1$.

9.

| | |
|---|---|
| $\lambda > 1$, hyperbola, | $\lambda = -1$, parabola, |
| $\lambda = 1$, two parallel imag. lines, | $-2 < \lambda < -1$, hyperbola, |
| $0 < \lambda < 1$, virtual conic, | $\lambda = -2$, two real lines, |
| $\lambda = 0$, two imag. lines, | $\lambda < -2$, hyperbola. |
| $-1 < \lambda < 0$, ellipse, | |

**10.** (i) $\lambda < -8$, hyperbola,
$\lambda = -8$, two real lines,
$-3 > \lambda > -8$, hyperbola,
$\lambda = -3$, two real lines,
$0 > \lambda > -3$, hyperbola,

(i) (cont.) $\lambda = 0$, two real lines,
$1 > \lambda > 0$, hyperbola,
$\lambda = 1$, parabola,
$\lambda > 1$, ellipse,
$\lambda \to \infty$, parabola.

(ii) $\lambda < 0$, hyperbola,
$\lambda = 0$, two real lines,
$0 < \lambda < 1$, hyperbola,
$\lambda = 1$, two parallel imag. lines,

(ii) (cont.) $1 < \lambda < 17$, virtual conic,
$\lambda = 17$, two imag. lines,
$\lambda > 17$, ellipse,
$\lambda \to \infty$, parabola.

(iii) $\lambda < -2$, hyperbola,
$\lambda = -2$, parabola,
$-2 < \lambda < -\frac{1}{2}$, ellipse,
$\lambda = -\frac{1}{2}$, two imag. lines,
$-\frac{1}{2} < \lambda < -\frac{2}{13}$, virtual conic,

(iii) (cont.) $\lambda = -\frac{2}{13}$, two imag. lines,
$-\frac{2}{13} < \lambda < 0$, ellipse,
$\lambda = 0$, parabola,
$\lambda > 0$, hyperbola,
$\lambda \to \infty$, two real lines.

## EXAMPLES XVII.

**1.** (i) $2y^2 + 11z^2 + 5yz - 6zx - 3xy = 0$, (ii) $x^2 - 42y^2 - 2z^2 - 25yz + 48zx + 11xy = 0$, (iii) $254x^2 + 45y^2 + 88z^2 - 153yz - 289zx + 170xy = 0$.

**2.** (i) $7l^2 - 4m^2 - n^2 + 4mn + 2nl + 8lm = 0$, (ii) $l^2 + 8m^2 + n^2 - 4mn - 26nl + 8lm = 0$, (iii) $9l^2 - 7m^2 - 20n^2 - 68mn - 34lm = 0$.

**3.** (i) $x = 2t^2 + t + 3$, $y = -t^2 + 2t - 3$, $z = -3t + 1$. (ii) $x = 3t^2 - 4t + 1$, $y = -t^2 + 2t + 1$, $z = 4t - 6$. (iii) $x = 6t^2 - 2t - 1$, $y = -4t + 2$, $z = -2t^2 + 2t - 1$.

**4.** (i) $\frac{1}{2}, \frac{1}{2}$; (ii) $0, \frac{1}{2}$; (iii) $\frac{3}{2}, \frac{1}{2}$; (iv) 1, 2. **5.** (i) 1, 3, 1; (ii) 0, −5, 3; (iii) −2, 1, 1

**6.** (i) Hyperbola; $x - y = 0$, $x + 3y = 2$; (ii) Hyperbola; $2y = 1$, $4x - 2y + 1 = 0$; (iii) Ellipse; (iv) Hyperbola, $x - y + 1 = 0$, $x = 1$.

**7.** (i) Hyperbola; $x^2 + y^2 + z^2 - 3yz + 7zx - 3xy = 0$.
(ii) Hyperbola; $x = 0$, $11x - 6y - 10z = 0$. (iii) Parabola.

**10.** (i) $0, -\frac{1}{3}$; (ii) $4\frac{3}{5}, 3\frac{1}{5}$; (iii) 1, −2 and −3, 1.

**12.** Directrix of parabola $4a(x - b) + c^2 = 0$. **13.** $l^2 - m^2 + n^2 + 4nl = 0$.

**14.** $a, b, c$. **15.** $ax + cy + b^2 = 0$, $4b^2c^2(a^2 + c^2)^{-\frac{3}{2}}$.

## EXAMPLES XVIII.

**8.** Draw $PF'$ making $\angle F'PT = FPH$, and $QF'$ making $\angle F'QT = FQK$.

**10.** A hyperbola with asymptotes parallel to $AD$ and the line $AX$, where, if $X$ and $Y$ are corresponding points on $BC$ and $AD$, $AX \parallel BY$.

## Chapter XX. § 9.

**3.** Let the polar of $O$ cut the conic in $I, J$. The locus of $P$ is a conic touching the given conic at $I$ and $J$.

**4.** Let $l$ cut the conic in $I, J$. The envelope of $u$ is a conic touching the given conic at $I$ and $J$.

## EXAMPLES XX.

**1.** One of the semi-axes $= 1$.

**3.** $\Theta_{32} = 0$, *i.e.* locus $S_3$ is apolar to envelope $\Sigma_2$; $3\Delta_1/\Theta_{12} = \Theta_{12}/\Theta_{21} = \Theta_{21}/3\Delta_2$; therefore $S_1$ and $S_2$ osculate.

**29.** If $S \equiv y^2 - 4azx$, $S' \equiv x^2 - 4byz$, then $F \equiv 2abz^2 + xy$, $\Phi \equiv n^2 + 8ablm$. $F$ and $\Phi$ coincide.

**39.** Use Graves' Theorem. **44.** $d^2 = r^2 \pm 2rr'$.

# INDEX

*(The numbers refer to the pages.)*


Abscissa, 2.
Absolute, the, 185, 290.
Absolute invariants, 271, 280.
Absolute points and lines, 185.
— position, 1.
Altitudes of triangle, concurrent, 17.
— of triangle of reference, 159.
Angle between conjugate diameters, 42.
— — two straight lines (rect. axes), 12, (oblique axes), 111; line-pair of the quadratic equation, 20.
—, logarithmic expression for, 186.
— of intersection of two circles, 32.
Apolar conics, 255, 275; mutually, 277.
— pairs of points or lines, 22; on a conic, 247.
— points or lines w.r.t. a conic, 187 (*see also* Conjugate points and lines).
— tetrads on a conic, 261-262.
— triads on a conic, 253.
Apolarity, 187, 275.
Apollonius, Treatise on conics, 53.
Area of ellipse, 44, 128.
— of hyperbolic sector, 65.
— of parabolic segment or sector, 77.
— of triangle (rect. axes) 4, (oblique axes) 101.
— — inscribed in ellipse, 45.
—, sign of, 4.
Areal coordinates, 156; plotting points, 164; as coordinates in three dimensions, 164.
Asymptotes, defined, 60; tangents at infinity, 61, 113; of circle or ellipse imaginary, 61; general cartesian equation, 125; directions of, 114; relation to line at infinity, 181; from freedom-equations of conic, 218, 221.
—, equation of hyperbola referred to, 65, 103, 171.
Auxiliary circle, of ellipse, 38; of hyperbola, 63; as pedal locus, 49; degenerate form for parabola, 74.
— —, minor, 39.
Axes, coordinate, rectangular, 1; oblique, 101; limiting case of trilinears, 163.
Axes, of ellipse, 37; hyperbola, 60; relation to asymptotes, 113.
—, equations of, for general conic, 115; parabola, 141.
—, homothetic, 100.
—, lengths of, 116, 127.
—, major and minor, 37.
—, transverse and conjugate, 60.
—, equation of conic referred to, 126.
Axis of perspective, 267.
—, radical, 84.

Barycentric coordinates, 157.
Bisectors of angles of line-pair, 17, 23.
— — of triangle, 159.
Boscovich, R. J., 40.
Brianchon's theorem, 236.
Brocard points, 166, 191.
— circle, 194.
Bundle of circles, 90.

Canonical equation of circle, 32; coaxal circles, 86; ellipse, 36; four straight lines, 150; hyperbola, 60; parabola, 70; straight line, 11; tetrad, 258; triad, 251; two conics, 173, 276.
Carnot's theorem, 169.
Cartesian coordinates, 2; general, 101; as special case of proj. coord., 156, 163.
Casey's theorem on tangent circles, 95.
Cayley, on Pascal's theorem, 235.
Centre of circle, 31; ellipse, 37; hyperbola, 60; parabola, 73.
— of conic, relation to line at infinity, 113; cartesian coord., 114, 123; homogeneous coord., 180; from parametric equations of conic, 221.
—, equation of conic referred to, 126.
— of curvature, ellipse, 51; parabola, 77.
— of gravity, 6, 157.
— of inversion, 91.
— of involution, 240.
— of perspective, 267.
—, homothetic, 100, 204.
Centre-locus of pencil of conics, 211; range of conics, 213.

Centroid, 6; of triangle of reference, 159.
Chasles, Michel, 68, 293.
Chord of contact, 27.
— of curvature, 56.
Chords, common, of circle and ellipse, 50; of two conics, 281; of conic and orthoptic circle, 194.
—, parallel, 43; locus of mid-points, 40, 62, 72; normals at ends of, 76.
— subtending right angle at point on conic, 104.
—, supplemental, 42.
— joining two points on conic, 43, 75, 220; polar coord., 54.
Circle, 25-35; in homogeneous coord., 181-184.
—, extended meaning, 291.
— of curvature, 51.
—, inscribed and escribed, 183, 184.
—, cosine, 193.
—, Tucker, 193.
—, Lemoine, 193.
—, nine-points, 184.
— of inversion, 91.
—, auxiliary, 38.
—, osculating, 51.
Circular points at infinity, 181-194, 113; as degenerate conic, 185; line-equation of, 138, 182.
Circumcentre of triangle, 159.
Circumcircle, 181.
Circumscribed conic, 170.
— parallelogram, 43, 64.
Class of a curve, 136.
Classification of conics, w.r.t. eccentricity, 127; w.r.t. points at infinity, 114.
— of pencils of circles, 86.
Coaxal circles, 85-88.
Collinearity, 5, 148.
Common chords of circle and ellipse, 50.
— — of two conics, 281.
— elements of two involutions, 241.
— tangents of two conics, 283; circles, 94.
Complete quadrangle, coordinates of vertices, 150; inscribed in conic, 172.
Complete quadrilateral, equations of sides, 150; circumscribed about conic, 173; mid-points of diagonals collinear, 167, 213.
Complex numbers, applied to orthogonal conics, 68, 196, 197; to inversion, 95-97; to coaxal circles, 100.
Composition of transformations, 107.
Concurrency, 16, 148, 164.
Cone, 36, 53.
Confocal conics, 189, 195-203.
Conformal transformation, 93.
Congruent transformation, 265.
Conic, defined, 36; history of, 53; analytical definition, 113; referred to oblique axes, 102; in homogeneous coord., 168; determined by five points, 122; generated by homographic pencils, 233.
Conjugate axis of hyperbola, 60.
— diameters, 40, 62, 103; relation to asymptotes, 124.
— hyperbolas, 63; ellipses, 127.
— imaginary lines and points, 20.
— points and lines w.r.t. conic, 28, 37, 122, 137, 187.
— systems of coaxal circles, 88.
— triangles, 29; in perspective, 173.
Conjugates, harmonic, 22.
— isogonal and isotomic, 159.
Constraint equation, 10.
Contact of conics, 174, 281, 285-286.
— of circle with ellipse, 51.
Contravariants, 284.
Coordinates, 1-8; areal, 156; barycentric, 157; cartesian, 2, 101; curvilinear, 3, 202; elliptic, 202; homogeneous, 145-167; homogeneous cartesian, 15, 156, 163; line or tangential, 130-135, 147, 160; metrical, 157; oblique, 2, 101-105; parabolic, 203; polar, 2; projective, 145; rectangular, 2; rectilinear, 3; superabundant, 157; trilinear, 157.
— network, 1.
Correspondence, 223, 227-246.
— between two conics, 243.
Corresponding points on ellipse and auxiliary circle, 38, 39.
— — on conics of same species, 200.
— — on homothetic figures, 203.
Cosine circle of triangle, 193.
Covariant, 249, 284.
—, cubic, of triad, 254; of two conics, 289.
—, sextic, of tetrad, 258.
Cross-ratio, 22, 152-154.
— in a (1, 1) correspondence, 224, 229.
— in a homography, 230, 237.
— in an involution, 241.
— of four points on a conic, 55, 231-233, 283.
— of base-points of a pencil of conics, 231-233.
— unaltered by projection, 266.
Cubic covariant of a triad, 254.
— — of two conics, 289.
Cubic equation, 250.
—, reducing, 256.
Curvature, 51.
Curvilinear coordinates, 3, 202.
Cyclic projectivity, 253.
— quadrangle, 24A.

Dandelin's theorem, 36A.
Degenerate conics, 175-176.

Degree of a curve, 135.
Desargues' theorem on involutions, 240; for perspective triangles, 167.
— conception of parallel lines, 145.
Descartes, René, 2.
Determinants, 123.
Diameter of conic, 113; general equation, 124; conjugate to line at infinity, 181.
— of ellipse, 37; hyperbola, 60, 62; parabola, 72. (*See also* Conjugate diameters.)
Direction, 14.
Director circle, 40, 49. (*See also* Orthoptic circle.)
Directrix, 36B, 47, 66, 140, 194.
— of parabola, 73, 125.
Discriminant of cubic, 253.
— of quartic, 257.
— of quadratic equation in $x$, $y$, 19.
Distance between two points (cartesians and polars), 3; (oblique coord.), 101; (areals), 193.
— of point from line, (cartesians), 13; (homogeneous coord.), 162.
Double contact, circle and ellipse, 51; two conics, 174; conditions for, 285.
Double points of homography, 237.
— — of involution, 239, 248.
Duality, principle of, 148.
Duplication of the cube, 53.
Dyad of points on a conic, 247.

Eccentric angle, 41.
Eccentricity, 47, 66, 73; two values, 127; for similar conics, 206.
Ellipse, 36-59; etymology, 54.
—, Fagnano's, 128.
—, Steiner's, 59.
Elliptic compasses, 38.
— coordinates, 202.
— involution, 239.
Envelopes, 131.
— of normals, 52.
Equation of a locus, 9.
—, constraint, 10.
—, freedom, 10.
—, line, 134.
—, parametric, 10.
—, pedal, 52.
—, point, 134.
—, tangential, 30.
(*See also* Canonical.)
Equianharmonic tetrad or range, 153; condition for, 257; intersection of two conics, 233, 278, 282.
Equiconjugate diameters, 42.
Equilateral hyperbola, 61.
Escribed circles of triangle, 184.
Eudoxus, 53.
Euler line of triangle, 165, 191.
Evolute, 52, 76.
Excentres of triangle, 159.

Fagnano's ellipse, 59, 98, 128.
*F*-conic, 276. (*See* Harmonic conic.)
Fermat points, *see* Isogonic points.
Foci, 46, 66, 73; history of, 54; optical property, 54, 74; relation to circular points, 138, 188; coordinates of, 119, 139-142; from line-equation, 140; from freedom-equations, 221; Dandelin's construction, 36A.
— of conic inscribed in triangle, 189, 236.
—, imaginary, 48.
Four-line system of conics, 212.
Four-point contact, 175, 286.
Four-point system of conics, 209.
Freedom-equations, 216-226; indeterminateness of, 222.
— of circle, 41; ellipse, 41, 216; hyperbola, 63, 65, 66, 216; parabola, 75, 218; straight line, 10, 11, 148.
Frégier-point, 105; locus of, 208.

Gaskin, T., 40.
Gradient, 9.
— of tangent, 26.
Graves' theorem, 199.

Harmonically inscribed or circumscribed conics, 275.
Harmonic conic, 276, 284.
— — of ellipse and auxiliary circle, 57.
— — of circle and two diameters, 69.
Harmonic conjugates, 22.
— perspective, 268.
— properties of pencil of conics, 210; range, 213.
— property of pole and polar, 27.
— ranges and pencils, 22, 146, 153.
— tetrad, 257; intersection of two conics, 232, 282.
— triangle, 149.
Hessian of a tetrad, 258-260.
— of a triad, 251.
— axis of a triad, 252.
— point of triangle, *see* Isodynamic points.
Homogeneous cartesian coordinates, 15, 156.
— coordinates in general, 145-167.
— line-coordinates, 135, 147, 160.
— quadratic equation in $x$, $y$, 19.
Homographic pencils, conic generated by, 233.
— ranges, 230; on one line, 237.
Homography, 230, 237; on a conic, 243.
Homothetic axes, 100.
— centre, 100, 204.
— conics, 181, 204-205.
Hyperbola, 36A, 60-69, 103; etymology, 54.
—, rectangular or equilateral, 61, 65.
Hyperbolic functions, 63.
— involution, 239.

Imaginary foci, 48.
— lines and points, 20.
In-centre of triangle, 159.
Infinity, line at, 179-194; equation (cartesians), 16, 156; (areals and trilinears), 156, 157; projection, 266.
—, points at, 6, 14; on conic, 112, 290; line-equation of, 144.
Initial line, 2.
Inscribed circle, 183.
— conic, 171.
— parabola, 190, 236.
— and circumscribed polygons of two conics, 56, 236, 279, 293, 294.
— and circumscribed confocal conics of a triangle, 207.
—, harmonically, conics, 275.
Intercepts, 10.
Intersection of circle and ellipse, 50.
— of straight line and conic, 168.
— of two lines, 14, 148.
— of two conics, 281, 283.
—, lines through, 16, 21.
Invariants, 249, 265-294.
—, absolute, 271, 280.
—, metrical, 109, 270.
—, projective, 271-274.
— of cubic, 253.
— of quartic, 256-257.
— of two circles, under inversion, 94.
Inverse transformation, 105.
Inversion, 91-97.
Involution, 238-244.
— on a conic, 243, 248.
Isodynamic points, 166, 191.
Isogonal conjugates, 159.
— — of straight line, 177.
Isogonic points, 166, 191.
Isoptic locus, 56; of parabola, 178.
Isotomic conjugates, 159.
Isotropic lines, *see* Absolute lines.
Ivory's theorem, 202.

Jacobian of three circles, 89, 100.
— of three conics, 287-288.
— of two dyads, 248; triads, etc., 263.
Joachimsthal's section-formulae, 5, 101, 158.
— ratio-equation, 27, 30, 71, 122, 170.

Kelvin, 93.
Kepler, J., 54.
Kirkman, T. P., 235.

La Hire, P. de, 40.
Lamé, G., 203.
Latus rectum, 47, 53.
Lemoine point, 159, 165.
— circle, 193.
Limiting points, 87.
Linear systems of circles, 90.
— — of conics, 209, 287.
Linear systems of dyads, 248.
— — of tetrads, 260.
— — of triads, 251, 255.
— transformation, 223, 249, 265.
Line-coordinates, 130-135, 147, 160.
Line-equation, *see* Tangential.
Lineo-linear relation, 223-224, 228-229.
Line-pair, 18; condition for, 19.
Liouville, J., 93.
Locus, equation of, 9.

Maclaurin, C., 234.
Major axis, 37.
Maximum inscribed ellipse, 59, 181, 191, 205.
— — triangle, 46.
Mean points, 6.
— proportionals, 53.
Mechanical description of ellipse, 38, 47; hyperbola, 67, 68; parabola, 75.
Medians of triangle, 159.
— of quadrilateral, 8.
Menaechmus, 53.
Menelaus' theorem, 169.
Metrical coordinates, 157.
— geometry, 145, 155, 179.
— invariants of conic, 270.
Minimum circumscribed ellipse, 59, 180, 181, 191, 205.
— — triangle, 46.
Minor axis, 37.
— auxiliary circle, 39.
Modulus of transformation, 272.
Monge, G., 40.

Net of circles, 90; conics, 287.
Network, coordinate, 1.
Nine-points circle, 184; centre, 159, 165.
— conic, 211.
Normal to circle, 26, 52.
— to ellipse, 49.
— to parabola, 76.
Normal forms, *see* Canonical.

Oblique axes, 101-105.
One-to-one correspondence, 152, 223, 227.
Ordinate, 2.
Origin, 1.
Orthocentre, 159, 165.
Orthocentric quadrangle, 68, 187, 212.
Orthocycle, 40.
Orthogonal circles, 87-90; condition for (rect. coord.), 32, (oblique coord.) 110, (trilinears) 194; projective invariant, 293.
— conics, 129, 215, 277; confocal, 196, 197.
— transformation, 265.
Orthoptic locus (circle), 39, 62, 125; directrix of parabola, 75; of inscribed conic, 194; projective properties, 290-291.

Orthotomic circle of three circles, 88.
— — of a net of circles, 90.
Osculating circle, 51.
— contact, 51, 281.
Oval chuck, 38.

Pappus, 54.
Parabola, 36A, 70-83; oblique coord., 104; etymology, 53; line at infinity a tangent, 73, 112; conic-envelope apolar to absolute, 290.
—, condition for (cartesians) 114, (homogeneous coord.) 180, (freedom equations) 218.
Parabola, semi-cubical, 77.
Parallelism (rect. coord.) 12, (oblique coord.) 111, (homogeneous coord.) 158; of line-pair, 21.
Parameter, 10, 216.
—, cross-ratio of, 152, 224, 229-231.
— of conic (latus rectum), 53.
Parametric equations, *see* Freedom-equations.
Pascal's theorem, 235.
Pedal equation, 52.
— triangle, 166.
Pencil of circles, 85.
— of conics, 209-212, 174.
— of lines, 16, 152.
—, harmonic, 22.
Periodic homography, 238.
Perpendicularity (rect. coord.) 12, (oblique) 111, (trilinears), 186, 188; of line-pair, 20, 21; in relation to circular points, 185.
Perspective, 266-268.
— ranges and pencils, 152, 227.
— triangles, 167.
Picquet, H., 40.
Plato, 53.
Point, Frégier, 105.
—, Lemoine or symmedian, 159.
—, mean, 6.
— at infinity, *see* Infinity, *and* Circular points.
Point-circle, 31.
Point-pair, condition for, 137.
Polar w.r.t. system of points on a line, 249.
— axis of a triad, 250.
— circle of triangle, 29, 192.
— conic w.r.t. triangle, 151.
— coordinates, 2.
Pole of coordinate system, 2.
— of a line, 27, 72, 136.
Pole and polar w.r.t. circle, 26-28; ellipse, 37; parabola, 71; general conic, 123, 170; oblique coordinates, 102; from freedom-equations, 220.
— — w.r.t. triangle, 151, 250.
Polygons inscribed and circumscribed to two conics, 56, 293, 294. (*See also* Triangle.)

Poncelet, 145.
Position-ratio, 5.
Power w.r.t. circle, 30, 32, 183-184, 192.
Principal value of angle, 3.
Projection, 152, 266-270.
— of conic into circle, 269.
—, stereographic, 93.
Projective geometry, 145, 179.
— coordinates, 145.
— invariants, 271-274.
Ptolemy, on stereographic projection, 93.
—'s theorem, 8, 95.

Quadrangle, coordinates of vertices, 150; inscribed in two conics, 173.
—, cyclic, 24A.
—, orthocentric, 68, 187, 212.
Quadrilateral, equations of sides, 150; circumscribed about conic, 173; collinearity of mid-points of diagonals, 167, 213.
Quartic equation, 255.

Radical axis, 84.
— centre, 84.
Radius vector, 2.
— of curvature, 51.
— of inversion, 91.
Range, harmonic, 22.
— of conics, 212.
Reciprocal of conic, 278, 286.
— radii, transformation, 93.
Rectangle-theorem for circle, 29, 36.
— — for ellipse, 43.
Rectangular hyperbola, 61; eccentricity, 127; equation referred to asymptotes, 65; condition for, 186; locus apolar to absolute, 188, 290.
Rectilinear coordinates, 3.
Reducing cubic, 256.
Reflexion, 97.
Regions, 13, 29, 37.
Relative position, 1.

Salmon, G., 235.
Section-formulae, 5, 7, 101, 158.
Self-conjugate triangle, 29; of two conics, 173; conic referred to, 172; of a pencil of conics, 210.
Semi-cubical parabola, 77.
Sextic covariant of tetrad, 258, 260.
Sign attached to coordinates, 1, 157.
— of angle, 12.
— of area, 4.
— of distance, 13.
— of position-ratio, 5.
Similar conics, 181, 204-206.
— ranges, 244.
Statical applications, 6, 157, 162.
Steiner, J., 235.
Steiner ellipse, 59, 178, 180, 191.
Stereographic projection, 93.

Straight line, equation, 9-11, 102, 145.
— —, gradient of, 9.
Subnormal of parabola, 74.
Superabundant coordinates, 157.
Supplemental chords, 42.
Symmedian or Lemoine point, 159.
Systems of circles, 84-91.
— of conics, *see* Confocals, Pencils, Ranges.
— of points on a conic, 247-264.

Tact-invariant, 281.
Tangent to circle, 25; ellipse, 37, 43; hyperbola, 61, 64; parabola, 70, 75; general conic, 123, 170; from freedom-equations, 75, 220.
— from a given point, 30, 39, 61, 103, 125, 170; lengths, 29, 44; to confocal conics, 207.
— line-coordinates of, 220.
—, point of contact of, 137.
Tangential coordinates, *see* Line-coordinates.
Tangential equations, 134-136, 169; parametric, 219.
— — of circle, 30, 134, 184; ellipse, 39; hyperbola, 61; parabola, 71.
— properties of ellipse, 49; parabola, 74; confocal conics, 198-200.
Taylor, C., 40.
Tetrad of points on conic, 255.
Thomson, W. (Lord Kelvin), 93.
Tracing of conics, 112-121.
Trammel, 38.
Transformation, conformal, 93.
—, congruent or orthogonal, 265.
— of coordinates, 105-109, 154, 265.
—, imaginary, 270.
—, inverse, 105.
— by inversion or reciprocal radii, 93.
—, linear, 223, 249, 265.
—, projective, 266.
— of equation of second degree, 108.
—, composition of, 107.
Transverse axis, 60.
Triad of points on conic, 250.
Triangle, area, 4, 101; inscribed in ellipse, 45.
—, important points and lines connected with, 158.
— inscribed and circumscribed to two conics, 236, 279, 293, 294.
—, maximum inscribed and minimum circumscribed, 46.
— of reference, 146.
Triangular paper, 164.
Trilinear coordinates, 157.
Tucker circles, 193.

Unit-point, 146.

Vectorial angle, 2.
Vertices of conics, 37.
Virtual circle, 31; conic, 196.

ANALYTICAL CONICS

SOMMERVILLE



BELL