FIFTH
BOOK OF EUCLID

WARD

THE PROPOSITIONS

OF

THE FIFTH BOOK OF EUCLID

PROVED ALGEBRAICALLY:

WITH AN

INTRODUCTION, NOTES, AND QUESTIONS.

BY

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Oxford and London:
JOHN HENRY AND JAMES PARKER.
1862.

Printed by Messrs. Parker, Cornmarket, Oxford.

PREFACE.

THE primary object of the following pages is to provide a text-book for the use of Candidates in the Mathematical Final School of this University, who have the option allowed them of applying Algebra in the demonstration of the propositions of the Fifth Book of Euclid.

An Introduction is prefixed, in which will be found a brief exposition of the principles on which geometrical magnitudes are represented by algebraical symbols, explanations of the terms "ratio" and "compound ratio," and a comparison of the geometrical and algebraical definitions of "proportion."

It is hoped that the explanations here offered may help to remove some of the difficulties usually experienced by learners in the algebraical treatment of propositions relating to geometrical magnitudes, ratios, and proportion; and in particular that the third part of the Introduction will satisfactorily exhibit the agreement of Euclid's Fifth Definition with the assumption of the equality of algebraical fractions as a test of the sameness of ratios.

Definitions of "equimultiples" and "continual proportionals" are inserted.

A few brief Notes on the Propositions are appended.

And Questions are added, the answers to which are contained in the Introduction and Notes.

OXFORD,
Nov. 1, 1862.

CONTENTS.

	PAGE
Introduction	7
Definitions	25
Axioms	33
Propositions	34
Notes	73
Questions	77

INTRODUCTION.

I. Everything that is extended in space, whether infinitely great, infinitesimal (i.e. infinitely small), or of finite greatness, is called a magnitude. There are four kinds of magnitudes, distinguished from one another by the nature of their extension in space; viz. lines, which have length; superficies, which have area; solids, which have volume; and angles, whose greatness consists in the inclination of their including lines. The general criterion of two magnitudes being of the same kind, given by Euclid [a], is that "the less can be multiplied so as to exceed the other." A simpler rule would be, if one can be said to be greater than, equal to, or less than the other.

By the aid of the Science of Number, the greatness or quantity of a finite magnitude, i.e. how much there is of it, can always be expressed by a number or symbol denoting how many times some less magnitude of the same kind must be taken to make it; or, in other words, how often it contains another like magnitude, as when the length of a line is said to be 6 feet. The *abstract number* 6, when so employed, does not express absolute greatness, but in reality institutes a comparison of the length of one line with that of another, stating that it is six times as great; and generally the greatness of any magnitude can be expressed only relatively to some other of the same kind: if, however, the length

[a] Bk. v. Defs. 3, 4.

termed a foot be supposed absolutely known, the *concrete expression* 6 feet expresses absolute length. A greater magnitude which, as in this example, contains another an exact number of times, is said to be a multiple of the less, and the less to be a part, or sub-multiple, of the greater; also, the less is said to measure the greater, and the greater to be measured by the less[b]; or the less is termed an unit, and the greater is said to be expressed in terms of it.

II. When two magnitudes can be expressed as multiples of the same third magnitude (in which case they are said to be commensurable, because capable of being measured by the same part or sub-multiple) the numbers or symbols which express the multiplicity provide means of comparing them with one another, so as to express their relative greatness. Thus, if two lines be 5 feet and 6 feet respectively in length, the numbers 5 and 6, which taken separately represent the lengths of the lines relatively to a third of one foot in length, when combined furnish fractional expressions by which their lengths are compared with one another. For, inasmuch as the one can be divided into five equal parts, and the other into six, each of one foot in length, the less is $\frac{5}{6}$ of the greater, and the greater $\frac{6}{5}$ of the less.

The same fraction would have been obtained by expressing the lengths of the lines in any other common denomination—for example, in inches; for, the one being 60 inches long, and the other 72, the less would by the same reasoning be $\frac{60}{72}$ of the other, which fraction is equal to $\frac{5}{6}$; and it is clear that the fraction which one is of the other must be independent of any par-

[b] Euclid, Bk. v. Defs. 1, 2.

ticular denomination in which the lengths may be expressed. Again, two other magnitudes, such as an area of 30 square yards, and another of 36, may produce the same fraction, $\frac{5}{6}$, when compared with one another. Hence it becomes apparent that two commensurable magnitudes have always an absolute relation to one another in respect of quantity or greatness, which is independent both of their absolute greatness and of the denomination in which they may be expressed. This absolute relation is termed their *ratio.*

If, as in the above instances, the quantities of the magnitudes be expressed numerically in terms of a common unit, the magnitudes themselves are said to have to one another the ratio of the numbers which express them; and in like manner algebraically, if a and b represent two magnitudes of the same kind in a common denomination, say two lines, one of which is a feet and the other b feet in length, the magnitudes are said to have to one another the ratio of a to b, or to be to one another as a to b.

It should be noticed that the same ratio may be variously expressed; for, since it was shewn above that the same line may be said indifferently to be $\frac{5}{6}$ or $\frac{60}{72}$ of another, it follows that the ratio of the one to the other may be expressed by that of 60 to 72 equally as well as by that of 5 to 6; and similarly, magnitudes which are to one another as a to b, are also as $2a$ to $2b$, or $10a$ to $10b$, or generally as ma to mb. In fact, when we say that one of two magnitudes has to the other the ratio of 5 to 6, or of a to b, we are making a particular selection out of an indefinite number of pairs of values which have the same absolute relation to one another that the magnitudes themselves have.

In geometry, no expression of a ratio by representation of the quantities of the magnitudes is required, because ratios are considered only in connection with one another, the ratio of one pair of magnitudes being stated or proved to be the same, or not the same, as that of another pair. But algebraical symbols being capable of representing geometrical as well as other quantities, if the ratios of magnitudes be expressed algebraically, propositions relating to them may be demonstrated on the ordinary principles of algebra.

Hitherto commensurable magnitudes only have been considered. But some geometrical and other magnitudes, although of the same kind, are incommensurable, i.e. do not admit of being expressed as exact multiples of any common part, or in other words, have no common measure. Thus, if the side of a square be expressed in linear units, as in inches, there is not an exact number of such units in the diagonal; neither is there any unit, however small, in terms of which both the side and the diagonal can be expressed by whole numbers. Hence it is not possible to express the one as an arithmetical fraction of the other, nor their ratio as that of two whole numbers. Still they have a ratio, and that not indefinite, but exact and appreciable. The multiplier by which the length of the side of a square must be affected to become equal to the diagonal, is the square root of 2. If we proceed to extract this square root by the ordinary process, it appears in the form of an interminable decimal, in which the figures do not recur, viz. 1·41421, &c.; of which all that we can say arithmetically is, that it lies between 1 and 2; or more nearly, between 1·4 and 1·5; or between 1·41 and 1·42; and so on. Its value, however, is exact: and just as, if an elastic string of the same length as the side of a square were stretched till it became double that length, it would at some time during the process of stretching be exactly of the same length as the diagonal; so if the numerical unit were supposed to become doubled, not by successive addition of finite fractions or decimals, but by gradual and continuous growth, it would pass through that exact value which is the square root of 2. The ratios of such, viz of incommensurable magnitudes, are expressed algebraically by the use of symbols in a manner analogous to that of arithmetic. If the quantities of two magnitudes be symbolized by a and b, we are not restricted, as in arithmetic, to the supposition that a and b represent whole numbers, or even that they are commensurable. a feet may denote the length of the diagonal of a square of which the side contains an exact number of feet, although a is then the symbol of that number multiplied by the square root of 2, and incapable of being expressed arithmetically; and $\frac{a}{b}$ may represent algebraically the fraction which one magnitude is of another, although they be not commensurable. As, then, one of two commensurable magnitudes which is $\frac{5}{6}$ of another is said to have to that other the ratio of 5 to 6; so generally, if $\frac{a}{b}$ be the fraction which any magnitude is of another of the same kind, the former is said to have to the latter the ratio of a to b.

It is convenient to use the notation $a:b$ (read a to b) in such a manner as to say that $a:b$ is the ratio of two magnitudes, meaning that they have to one another the same ratio which a has to b.

III. It has been already stated, that when different pairs of magnitudes are expressed numerically or symbolically, each pair in a common denomination, the fractions formed by the numbers or symbols, and expressing what fractional part one of each pair is of the other, may be identically the same or equal. This identity or equality of the fractions, when it exists, will imply that the absolute mutual relation or ratio of the magnitudes in each pair is identically the same. Thus if a, b, c, d, express the greatness of four magnitudes, so that $\frac{a}{b}$ is any one of the indefinite number of equal fractions which signify what fractional part the first is of the second, and $\frac{c}{d}$ signifies in like manner what fractional part the third is of the fourth; then if $\frac{a}{b} = \frac{c}{d}$, the first magnitude has the same ratio to the second that the third has to the fourth. Four magnitudes which are so related to one another are called *proportionals;* and *proportion* is defined by Euclid to be the "*sameness* or *identity* of ratios," ἡ τῶν λόγων ταυτότης[c].

Euclid's criterion of the sameness of ratios is different from that above, being as follows:—"The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth when, any equimul-

[c] The Oxford edition (1703) has ὁμοιότης, which Dr. Simson translates "the *similitude* of ratios."

tiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth, if the multiple of the first be greater than that of the second, the multiple of the third is greater than that of the fourth; and if equal, equal; and if less, less." These conditions, however, imply the equality of the algebraical or arithmetical fractions corresponding to the ratios, and *vice versâ*. For first, if $\frac{a}{b} = \frac{c}{d}$, $\frac{m\,a}{n\,b} = \frac{m\,c}{n\,d}$, whatever the values of m and n may be; and if $m\,a$ be greater than $n\,b$, $\frac{m\,a}{n\,b}$ is greater than unity; therefore also $\frac{m\,c}{n\,d}$ is greater than unity, and therefore $m\,c$ greater than $n\,d$. Similarly, if $m\,a$ be equal to or less than $n\,b$, $m\,c$ will be simultaneously equal to or less than $n\,d$. The conditions of Euclid's definition are therefore satisfied when the fractions are equal.

And if Euclid's conditions be satisfied, first let the magnitudes in each pair be commensurable. It will then be always possible to take such multipliers that the multiple of the first shall be equal to that of the second (in other words, if two magnitudes have a common measure, a common multiple can also be found); for a and b in this case are symbols of whole numbers, and their product, or any multiple of it, will therefore be a whole number containing each of them an exact number of times. Let m' and n' be values of m and n such that $m'a = n'b$; then if also $m'c = n'd$ (in which case it is evident that the other conditions will necessarily be satisfied); it follows that $\frac{a}{b}$ and $\frac{c}{d}$

are each equal to $\frac{n'}{m'}$, and therefore $\frac{a}{b} = \frac{c}{d}$. But secondly, if the magnitudes in each pair be incommensurable, it will not be possible that any multipliers m and n can be such that $m\,a = n\,b$ or $m\,c = n\,d$. In this case suppose m' to be any integral multiplier of a, and n' to be the greatest integer which multiplied into b will produce a less quantity than $m'a$, and therefore $n' + 1$ the least that will produce a quantity greater than $m'a$; i.e. let $m'a$ be greater than $n'b$ and less than $(n' + 1)\,b$; and therefore $\frac{m'a}{n'b}$ greater than 1 and less than $1 + \frac{1}{n'}$. Now, as greater and greater values of m', and therefore also of n', are taken, the value of the fraction $\frac{1}{n'}$ becomes less and less; and $1 + \frac{1}{n'}$ becomes more and more nearly equal to unity: hence the value of $\frac{m'\,a}{n'\,b}$, which is intermediate between 1 and $1 + \frac{1}{n'}$, also approximates to unity as its limit when m' and n' are indefinitely increased; and therefore the limit to which the value of $\frac{n'}{m'}$ is continually approaching is $\frac{a}{b}$. But if when $m\,a$ is greater than $n\,b$, $m\,c$ is also greater than $n\,d$; and when less, less; we have also $m'c$ greater than $n'd$ and less than $(n' + 1)\,d$: whence it can be shewn in precisely the same manner that the limit to which the value of $\frac{n'}{m'}$ is continually approaching is $\frac{c}{d}$: and the same limit was before shewn to be $\frac{a}{b}$; therefore $\frac{a}{b}=\frac{c}{d}$. Hence universally if magnitudes, related to one another in the manner described in Euclid's fifth definition, be represented algebraically by a, b, c, d; then $\frac{a}{b}=\frac{c}{d}$.

IV. From what has now been advanced, it will be seen how algebra is applicable to the solution of propositions such as those in the fifth book of Euclid. For example, in Proposition B, "Invertendo," where the hypothesis is that four magnitudes are proportionals; the magnitudes are represented algebraically by four symbols, a, b, c, d, supposed to be such that a and b express the quantities of the first two in a common denomination, and c and d of the last two, also in a common denomination. Then the magnitudes being proportionals, a, b, c, d are also proportionals, and therefore $\frac{a}{b}=\frac{c}{d}$. Next an algebraical result, viz. $\frac{b}{a}=\frac{d}{c}$ is derived from the equality of $\frac{a}{b}$ and $\frac{c}{d}$; whence it follows that $b : a :: d : c$. But b and a represent the second and first magnitudes in a common denomination, and d and c represent the fourth and third, also in a common denomination; and therefore $b : a$ is the ratio of the second magnitude to the first, and $d : c$ the ratio of the fourth magnitude to the third. Wherefore the magnitudes are proportionals when taken inversely. In like manner in Proposition 16, "Alternando," where a, b, c, d are taken to represent the magnitudes, it must be understood that *a, b, c, d* are symbols of the quantities of the magnitudes all in one common denomination; for if *a* and *c* expressed the first and third in different denominations, the ratio of *a* to *c* would not be the same as that of the first magnitude to the third; and consequently the algebraical proportion $a : c :: b : d$ would not imply the geometrical result inferred from it. These considerations are suppressed in the statement and conclusion of propositions rendered algebraically, but, in accordance with the usual practice in algebraical problems, must be tacitly understood.

V. If four magnitudes be such that, equimultiples of the first and third and also of the second and fourth being taken, it is possible that the multiple of the first may be greater than that of the second, but the multiple of the third not greater than that of the fourth; or equal, but the other not equal; or less, but the other not less; the magnitudes are clearly not proportionals, since the conditions of the fifth definition are violated. In the first of these cases the first magnitude is said[d] to have a greater ratio to the second than the third has to the fourth; not that one *ratio* can be strictly said to be greater than another, but the first *magnitude* is greater relatively to the second than the third is relatively to the fourth. Algebraically, if the magnitudes be expressed by *a, b, c, d* as before, the fraction $\frac{a}{b}$ will be greater than the fraction $\frac{c}{d}$; for if $m a$ be greater than $n b$ and $m c$ not

[d] Euc., Bk. v. Def. 7.

greater than $n d$, $\frac{m a}{n b}$ is greater and $\frac{m c}{n d}$ not greater than unity, and therefore $\frac{m a}{n b}$ is greater than $\frac{m c}{n d}$; whence $\frac{a}{b}$ is also greater than $\frac{c}{d}$. Ratios cannot properly be said to be either equal or unequal; but if the ratios are *the same*, the corresponding fractions are equal; and if the ratios are not the same, the fractions are unequal. It is only for abbreviation, as stated above, that *a* is said to have to *b* a greater ratio than *c* has to *d*. Euclid is careful to state, not that the first magnitude *has*, but that it *is said to have* a greater ratio to the second than the third has to the fourth.

VI. No definition of "compound ratio" is to be found in any copy of the Greek text of Euclid, though it is clear that a technical use of the term was intended in the Enunciation of Prop. 23, Bk. VI., "Equiangular parallelograms have to one another the ratio which is compounded of the ratios of their sides,"—*λόγον τὸν συγκείμενον ἐκ τῶν πλευρῶν.* It may easily be seen that the ratio of two equiangular parallelograms is in some way dependent on the ratios of their sides; but without a definition it would be impossible to infer from the above enunciation what precise relation between them it was proposed to establish. To supply this defect Dr. Simson has framed a definition, founded upon the demonstrated result from which the truth of the enunciation is inferred in Euclid's proposition.

The analysis of Euclid's proof is as follows:—Lines marked *L, M, N* are found by a previous proposition, such that *L* has to *M* the same ratio that a side of the first parallelogram has to a side of the other, and M to N the same ratio that an adjacent side of the first has to an adjacent side of the other; it is next proved that the first parallelogram has to the other the same ratio that L has to N, and this is then said to be "the ratio which is compounded of the ratios of their sides." Hence it appears that the ratio of two magnitudes is said by Euclid to be compounded of two other ratios when it is dependent on them in such a manner that, if three magnitudes be found which have, the first to the second and the second to the third, those same ratios, the ratio of the first to the last of these three magnitudes is the same as that of the other two.

In order, apparently, to simplify this expression and to extend the idea of compounding to a greater number of ratios, Dr. Simson has given the following definition: "When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio which is compounded of the ratios of the first to the second, of the second to the third, the third to the fourth, and so on unto the last magnitude." This he has explained and accommodated to Euclid's proposition somewhat as follows:—If A, B, C, D be four magnitudes of the same kind, A is said to have to D the ratio which is compounded of the ratios of A to B, B to C, and C to D: and if E, F, G, H, K, L be other magnitudes, such that $E : F :: A : B$, $G : H :: B : C$, $K : L :: C : D$; then A is said to have to D the ratio which is compounded of the ratios of E to F, G to H, and K to L: and further, if there be two other magnitudes, M and N, such that $M : N :: A : D$, then M is said to have to N the ratio which is compounded of those ratios. This last extension of Dr. Simson's definition is the sense in which compound ratio is used by Euclid. The capital letters, it should be observed, used here, are not algebraical symbols of quantity, but are as it were names of the magnitudes themselves; just as a line is called *A* or a triangle *C* in the other books of Euclid.

If the corresponding small letters be used to express algebraically the above magnitudes, the definition will be applied thus: if $e : f :: a : b$, $g : h :: b : c$, $k : l :: c : d$, and also $m : n :: a : d$, then *M* is said to have to *N* the ratio which is compounded of the ratios of *E* to *F*, *G* to *H*, and *K* to *L*. But the algebraical proportions imply the following equalities: $\frac{e}{f} = \frac{a}{b}$, $\frac{g}{h} = \frac{b}{c}$, $\frac{k}{l} = \frac{c}{d}$, $\frac{m}{n} = \frac{a}{d}$, and therefore $\frac{e}{f} \times \frac{g}{h} \times \frac{k}{l} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{d} = \frac{m}{n}$: whence it appears that when a magnitude is said to have to another the ratio which is compounded of the ratios of several other magnitudes, the fraction corresponding to the former ratio is equal to the product of the fractions corresponding to the others; or a compound ratio in geometry is analogous to a compound fraction in arithmetic or algebra.

Here as elsewhere the algebraical relations are more extensive in their signification than the geometrical; for if *m* and *n* represented any quantities not geometrical, the ratio of *m* to *n* would be said to be compounded of the ratios of any quantities represented by the other letters when $\frac{m}{n} = \frac{e}{f} \times \frac{g}{h} \times \frac{k}{l}$, which is abundantly illustrated under the rule of "Compound Proportion" in every Arithmetic.

The algebraical form also provides the following rule: If the ratio of two quantities (including geometrical quantities, or magnitudes) depends on the ratios of several others in pairs, in such a manner that if the quantities in each of all the pairs except any one were equal to one another (i.e. if all the ratios but any one were ratios of equality), the ratio of the two quantities would be that of the remaining pair (i.e. would be the same as the remaining ratio); then the ratio of those two quantities will always be that which is 'compounded' of those ratios. For just as in the case of two rectangles, if their lengths were equal and their breadths as 2 to 3, the one would be to the other as 2 to 3, i.e. would be $\frac{2}{3}$ of it; and if their breadths were equal and lengths as 5 to 7, the same would be to the other as 5 to 7, i.e. would be $\frac{5}{7}$ of it; but if at the same time that the breadths are as 2 to 3 the lengths are as 5 to 7, then the former will be $\frac{5}{7}$ of what it otherwise would have been, i.e. will be $\frac{5}{7} \times \frac{2}{3}$ of the other; so of quantities in general, such as those above, if when $e=f$ and $g=h$, $m : n :: k : l$ and therefore $\frac{m}{n}=\frac{k}{l}$; and when $e=f$ and $k=l$, $m : n :: g : h$, and therefore $\frac{m}{n} \; \frac{g}{h}$; and when $g=h$ and $k=l$, $m : n :: e : f$, and therefore $\frac{m}{n}=\frac{e}{f}$; then if none of the quantities are equal, $\frac{m}{n}$ will be $\frac{e}{f} \times$ what it would have been if e had been equal to f, that is, $\frac{e}{f} \times \frac{g}{h} \times$ what it would have been if both e had been equal to f and g to h, that is, $\frac{e}{f} \times \frac{g}{h} \times \frac{k}{l}$.

An illustration of the truth of this rule, applied to magnitudes, is seen in Euclid, Bk. XI. Prop. D, (Dr. Simson's): "Solid parallelopipeds which are contained by parallelograms equiangular to one another, each to each,—that is, of which the solid angles are equal, each to each,—have to one another the ratio which is (the same with the ratio) compounded of the ratios of their sides." The same parallelopipeds may also be shewn to have to one another the ratio which is compounded of the ratios of their edges.

When a ratio is compounded of several ratios, all of which are the same, it is termed a duplicate ratio, or triplicate, quadruplicate, &c., according to the number of the ratios of which it is compounded. Thus if $a : b :: b : c$, the ratio of a to c, which is compounded of the ratios of a to b and b to c, is said to be "the duplicate ratio of that which a has to b;" *τὸ πρῶτον πρὸς τὸ τρίτον διπλασίονα λόγον ἔχειν λέγεται ἤπερ πρὸς τὸ δεύτερον.* In this case the compound fraction corresponding to the compound ratio gives the following result: $\frac{a}{c} = \frac{a}{b} \times \frac{b}{c}$; but $\frac{a}{b} = \frac{b}{c}$; therefore $\frac{a}{c} = \left(\frac{a}{b}\right)^2$. Similarly, if $a : b :: b : c :: c : d$, a is said to have to d the triplicate ratio of a to b; and $\frac{a}{d} = \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{a}{b} \times \frac{a}{b} \times \frac{a}{b} = \left(\frac{a}{b}\right)^3$. In like manner, if one magnitude have to another the ratio which is compounded of the ratios of g to h, k to l, and m to n; when $g : h :: k : l$

$:: m : n$, the ratio of the magnitudes is the triplicate ratio of g to h, and the fraction corresponding to their ratio is equal to $\left(\frac{g}{h}\right)^3$.

If a, b, c represent straight lines, and $a : b :: b : c$, the ratio of a to c is the duplicate ratio of the lines represented by a and b; and, since $\frac{a}{c} = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$, $a : c :: a^2 : b^2$; also a^2 and c^2 represent algebraically the squares described upon the lines a and c; therefore the duplicate ratio of two lines is the ratio of the squares described upon them. In like manner the triplicate ratio of two lines is the ratio of the cubes of which they are the edges. These results might also be inferred from Prop. 20, Bk. VI., and Prop. 33, Bk. XI.

THE FIFTH BOOK OF EUCLID.

DEFINITIONS.

I.

A LESS magnitude is said to be a part of a greater magnitude when the less measures the greater; that is, "when the less is contained a certain number of times exactly in the greater."

If $ma = b$ or $a = \frac{b}{m}$, m being any whole number, a is said to be a *part* of b.

II. *a.*

A greater magnitude is said to be a multiple of a less when the greater is measured by the less; that is, "when the greater contains the less a certain number of times exactly."

If $a = mb$, m being any whole number, a is said to be a *multiple* of b.

II. β.

Magnitudes which contain other less magnitudes the same number of times exactly are said to be equimultiples of them.

If $a = mb$ and $c = md$, a and c are said to be equimultiples of b and d.

III.

"Ratio is a mutual relation of two magnitudes of the same kind to one another, in respect of quantity."

If a and b express two magnitudes of the same kind in a common denomination, so that $\frac{a}{b}$ is the fraction which one is of the other, the former is said to have to the latter the ratio of a to b. Vide Introduction, § II.

IV.

Magnitudes are said to have a ratio to one another when the less can be multiplied so as to exceed the other.

This is the criterion of magnitudes being of the same kind.

V.

The first of four magnitudes is said to have the same ratio to the second which the third has to the fourth when, any equimultiples whatsoever of the first and third being taken, and any equimultiples whatsoever of the second and fourth, if the multiple of the first be less than that of the second, the multiple of the third is also less than that of the fourth; or, if the multiple of the first be equal to that of the second, the multiple of the third is also equal to that of the fourth; or, if the multiple of the first be greater than that of the second, the multiple of the third is also greater than that of the fourth.

Of a, b, c, d, a is said to have to b the same ratio which c has to d when, ma and mc being any equimultiples whatsoever of a and c, and nb and nd any equimultiples whatsoever of b and d, no values can possibly be given to m and n which will make ma less than nb but mc not less than nd, or ma equal to nb but mc not equal to nd, or ma greater than nb but mc not greater than nd. The above conditions imply that $\frac{a}{b}=\frac{c}{d}$, and *vice versâ*. Vide Introduction, § III.

VI. α.

Magnitudes which have the same ratio are called proportionals. "N.B. When four magnitudes are proportionals, it is usually expressed by saying, the first is to the second as the third to the fourth."

If the ratio of a to b be the same as that of c to d, and the same as that of e to f; a, b, c, d, e, f are called proportionals. This relation is expressed by

$$a : b :: c : d :: e : f,$$

which is read, a is to b as c to d and as e to f.

VI. β.

Magnitudes of the same kind are said to be continued or continual proportionals when the ratios of the first to the second, of the second to the third, of the third to the fourth, and so on, are all the same.

If $a : b :: b : c :: c : d$, &c.; a, b, c, d, &c. are said to be continued or continual proportionals.

VII.

When of the equimultiples of four magnitudes (taken as in the fifth definition) the multiple of the first is greater than that of the second, but the multiple of the third is not greater than that of the fourth; then the first magnitude is said to have to the second a greater ratio than the third has to the fourth: and, on the contrary, the third is said to have to the fourth a less ratio than the first has to the second.

If any values can be given to the multipliers m and n, applied as in the fifth definition, which will make ma greater than nb but mc not greater than nd, a is said to have to b a greater ratio than c has to d, and c is said to have to d a less ratio than a has to b. In this case $\frac{a}{b}$ is greater than $\frac{c}{d}$ and $\frac{c}{d}$ less than $\frac{a}{b}$. Vide Introduction, § v.

VIII.

"Analogy, or proportion, is the similitude of ratios."

The similitude (Gk. ταυτότης) of ratios implies the equality of the corresponding fractions. Vide Def. v, and Introduction, § III.

IX.

Proportion consists in three terms at least.

X.

When three magnitudes are proportionals, the first is said to have to the third the duplicate ratio of that which it has to the second.

If $a : b :: b : c$, a is said to have to c the duplicate ratio of that which it has to b. Vide Introduction, pp. 21 and 22.

XI.

When four magnitudes are continual proportionals, the first is said to have to the fourth the triplicate ratio of that which it has to the second, and so on, quadruplicate, &c., increasing the denomination still by unity, in any number of proportionals.

If $a : b :: b : c :: c : d :: d : e$, &c., a is said to have to d the triplicate ratio of that which it has to b, and to e the quadruplicate ratio of that which it has to b, &c. Vide Introduction, pp. 21 and 22.

Definition *A*, to wit, of compound ratio.

When there are any number of magnitudes of the same kind, the first is said to have to the last of them the ratio compounded of the ratio which the first has to the second, and of the ratio which the second has to the third, and of the ratio which the third has to the fourth, and so on unto the last magnitude.

For example, if A, B, C, D be four magnitudes of the same kind, the first, A, is said to have to the last, D, the ratio compounded of the ratio of A to B, and of the ratio of B to C, and of the ratio of C to D; or, the ratio of A to D is said to be compounded of the ratios of A to B, B to C, and C to D.

And if A has to B the same ratio which E has to F; and B to C the same ratio that G has to H; and C to D the same that K has to L; then, by this definition, A is said to have to D the ratio compounded of ratios which are the same with the ratios of E to F, G to H, and K to L. And the same thing is to be understood when it is more briefly expressed by saying, A has to D the ratio compounded of the ratios of E to F, G to H, and K to L.

In like manner, the same things being supposed, if M has to N the same ratio which A has to D; then, for shortness' sake, M is said to have to N the ratio compounded of the ratios of E to F, G to H, and K to L.

Vide Introduction, § VI.

XII.

In proportionals, the antecedent terms are called homologous to one another, as also the consequents to one another.

If a, b, c, d be proportionals, i.e. if $a : b :: c : d$; a and c, the antecedents or former terms of the ratios, are said to be homologous to one another; and b and d, the consequent or latter terms, are homologous.

"Geometers make use of the following technical words, to signify certain ways of changing either the order or magnitude of proportionals, so that they continue still to be proportionals."

XIII.

Permutando, or alternando, by permutation or alternately. This word is used when there are four proportionals, and it is inferred that the first has the same ratio to the third which the second has to the fourth; or that the first is to the third as the second to the fourth: as is shewn in Prop. XVI. of this fifth book.

If $a : b :: c : d$, "permutando" or "alternando" $a : c :: b : d$.

XIV.

Invertendo, by inversion; when there are four proportionals, and it is inferred that the second is to the first as the fourth to the third. Prop. B.

If $a : b :: c : d$, "invertendo" $b : a :: d : c$.

XV.

Componendo, by composition; when there are four proportionals, and it is inferred that the first together with the second is to the second, as the third together with the fourth is to the fourth. Prop. XVIII.

If $a : b :: c : d$, "componendo" $a + b : b :: c + d : d$.

XVI.

Dividendo, by division; when there are four proportionals, and it is inferred that the excess of the first above the second is to the second, as the excess of the third above the fourth is to the fourth. Prop. XVII.

If $a : b :: c : d$, "dividendo" $a - b : b :: c - d : d$; or, which is the same thing (vide Prop. XVII.), if $a + b : b :: c + d : d$, $a : b :: c : d$.

XVII.

Convertendo, by conversion; when there are four proportionals, and it is inferred that the first is to its excess above the second, as the third to its excess above the fourth. Prop. E.

If $a : b :: c : d$, "convertendo" $a : a - b :: c : c - d$.

XVIII.

Ex æquali (sc. distantiâ), or ex æquo, from equality of distance: when there is any number of magnitudes more than two, and as many others, such that they are proportionals when taken two and two of each rank, and it is inferred that the first is to the last of the first rank of magnitudes, as the first is to the last of the others: "Of this there are the two following kinds, which arise from the different order in which the magnitudes are taken, two and two."

XIX.

Ex æquali, from equality. This term is used simply by itself, when the first magnitude is to the second of the first rank, as the first to the second of the other rank; and as the second is to the third of the first rank, so is the second to the third of the other; and so on in order: and the inference is as mentioned in the preceding definition; whence this is called ordinate proportion. Proposition XXII.

$$a, b, c, d \ldots . k, l, m, n;$$
$$p, q, r, s \ldots . w, x, y, z.$$

If $a : b :: p : q$,
$b : c :: q : r$,
$c : d :: r : s$,

. . . . .

$$k : l :: w : x,$$
$$l : m :: x : y,$$
$$m : n :: y : z,$$
"Ex æquali" $a : n :: p : z.$

XX.

Ex æquali in proportione perturbatâ seu inordinatâ, from equality in perturbate or disorderly proportion [a]. This term is used when the first magnitude is to the second of the first rank, as the last but one is to the last of the second rank; and as the second is to the third of the first rank, so is the last but two to the last but one of the second rank; and as the third is to the fourth of the first rank, so is the third from the last to the last but two of the second rank; and so on in a cross order: and the inference is as in the 18th Definition. Prop. XXIII.

$$a, b, c, d \ldots . k, l, m, n;$$
$$p, q, r, s \ldots . w, x, y, z.$$
If $a : b :: y : z,$
$$b : c :: x : y,$$
$$c : d :: w : x,$$
$$\cdot \quad \cdot \quad \cdot \quad \cdot$$
$$k : l :: r : s,$$
$$l : m :: q : r,$$
$$m : n :: p : q,$$
"Ex æquali in proportione perturbatâ" $a : n :: p : z.$

[a] 4 Prop. lib. II. Archimedis de sphærâ et cylindro.

AXIOMS.

I.

Equimultiples of the same, or of equal magnitudes, are equal to one another.

If $a = m b$ and $c = m b$, $a = c$; or if $a = m b$, $c = m d$, and $b = d$, then $a = c$.

II.

Those magnitudes, of which the same or equal magnitudes are equimultiples, are equal to one another.

If $a = m b$ and $a = m c$, $b = c$; or if $a = m b$, $d = m c$, and $a = d$, then $b = c$.

III.

A multiple of a greater magnitude is greater than the same multiple of a less.

If a be greater than b, $m a$ is greater than $m b$.

IV.

That magnitude, of which a multiple is greater than the same multiple of another, is greater than that other magnitude.

If $m a$ be greater than $m b$, a is greater than b.

THE FIFTH BOOK OF EUCLID.

PROPOSITION I.

If any number of magnitudes be equimultiples of as many, each of each; what multiple soever any one of them is of its part, the same multiple shall all the first magnitudes be of all the other.

$$a, b, c; \; d, e, f.$$

Let a, b, c be equimultiples of d, e, f, each of each; so that $a = md$, $b = me$, and $c = mf$.

Then shall $a + b + c = m(d + e + f)$.

Because $a = md$, $b = me$, and $c = mf$,

therefore $a + b + c = md + me + mf$;

but $m(d + e + f) = md + me + mf$,

therefore $a + b + c = m(d + e + f)$;

that is, $a + b + c$ is the same multiple of $d + e + f$ that each of a, b, c is of its part $d, e,$ or f.

The same demonstration may be applied to any number of magnitudes.

Therefore, if any number of magnitudes, &c.—*Q.E.D.*

Or, generally,

$$a, b, c \ldots h, k, l;$$

$$p, q, r \ldots x, y, z.$$

Let $a, b, c \ldots h, k, l$ be equimultiples of $p, q, r \ldots x, y, z$; so that $a = mp$, $b = mq$, $c = mr \ldots h = mx$, $k = my$, $l = mz$.

$$\text{Then shall } a + b + c \ldots\ldots + h + k + l$$
$$= m(p + q + r \ldots\ldots + x + y + z).$$
$$\text{Because } a = mp,\ b = mq,\ c = mr \ldots\ldots$$
$$h = mx,\ k = my,\ l = mz,$$
$$\text{therefore } a + b + c \ldots\ldots + h + k + l$$
$$= mp + mq + mr \ldots\ldots + mx + my + mz;$$
$$\text{but } m(p + q + r \ldots\ldots + x + y + z)$$
$$= mp + mq + mr \ldots\ldots + mx + my + mz,$$
$$\text{therefore } a + b + c \ldots\ldots + h + k + l$$
$$= m(p + q + r \ldots\ldots + x + y + z);$$

that is, $a + b + c \ldots\ldots + h + k + l$ is the same multiple of $p + q + r \ldots\ldots + x + y + z$ that each of $a, b, c, \ldots\ldots h, k, l$ is of its part $p, q, r \ldots\ldots x, y, z$.

Therefore, if any number of magnitudes, &c.—*Q.E.D.*

PROPOSITION II.

If the first magnitude be the same multiple of the second that the third is of the fourth, and the fifth the same multiple of the second that the sixth is of the fourth; then shall the first together with the fifth be the same multiple of the second that the third together with the sixth is of the fourth.

$$a, b, c, d, e, f.$$

Let a be the same multiple of b that c is of d, and e the same multiple of b that f is of d; so that,

$$a = mb,\ c = md,$$
$$e = nb,\ f = nd.$$

Then shall $a + e$ be the same multiple of b that $c + f$ is of d.

$$\text{Because } a = mb, \text{ and } e = nb,$$
$$\text{therefore } a + e = mb + nb = (m + n)\,b.$$

And in like manner because $c = m\,d$, and $f = n\,d$,

therefore $c + f = m\,d + n\,d = (m + n)\,d$.

Therefore $a + e$ and $c + f$ contain b and d respectively $m + n$ times; that is, $a + e$ is the same multiple of b that $c + f$ is of d.

Therefore, if the first, &c.—*Q. E. D.*

COR. From this it is plain that if any number of magnitudes a, e, g, k be multiples of b, and as many c, f, h, l be the same multiples of d, each of each; then $a + e + g + k$ is the same multiple of b that $c + f + h + l$ is of d.

PROPOSITION III.

If the first be the same multiple of the second that the third is of the fourth, and if of the first and third there be taken any equimultiples; these shall be equimultiples, the one of the second, and the other of the fourth.

$$a, b, c, d;\ e, f.$$

Let a be the same multiple of b that c is of d; and e and f be equimultiples of a and c; so that,

$$a = m\,b,\ c = m\,d,$$

$$\text{and}\quad e = n\,a,\ f = n\,c.$$

Then shall e be the same multiple of b that f is of d.

Because $a = m\,b$, and $e = n\,a$,

therefore $e = n \times m\,b = m\,n\,b$;

and because $c = m\,d$, and $f = n\,c$,

therefore $f = n \times m\,d = m\,n\,d$.

Therefore e and f contain b and d respectively $m\,n$ times; that is, e is the same multiple of b that f is of d.

Therefore, if the first, &c.—*Q. E. D.*

PROPOSITION IV.

If the first of four magnitudes has the same ratio to the second which the third has to the fourth; then any equimultiples whatever of the first and third shall have the same ratio to any equimultiples of the second and fourth, viz.: "the multiple of the first shall have the same ratio to that of the second, which the multiple of the third has to that of the fourth."

$$a, b, c, d; e, f, g, h.$$

Let $a : b :: c : d$[a]; and let e, g be equimultiples of a, c, and f, h of b, d; so that

$$e = ma, f = nb,$$

$$g = mc, h = nd.$$

Then shall $e : f :: g : h$.

Because $a : b :: c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$.

Multiply each of these equals by $\frac{m}{n}$;

then $\frac{ma}{nb} = \frac{mc}{nd}$:

but $e = ma, f = nb, g = mc$, and $h = nd$;

therefore $\frac{e}{f} = \frac{g}{h}$,

and therefore $e : f :: g : h$.

Therefore, if the first, &c.—*Q. E. D.*

Cor. Likewise, if the first has the same ratio to the second which the third has to the fourth, then also, any equimultiples whatever of the first and third have the same ratio to the second and fourth: and in like

[a] This expression is to be read here and elsewhere, Let a be to b as c is to d.

manner, the first and the third have the same ratio to any equimultiples whatever of the second and fourth.

First,

$$\text{If } a : b :: c : d,$$
$$\text{and } e = m\,a,\ g = m\,c;$$
$$\text{then } \frac{a}{b} = \frac{c}{d};$$
$$\text{therefore } \frac{m\,a}{b} = \frac{m\,c}{d},$$
$$\text{that is } \frac{e}{b} = \frac{g}{d};$$
$$\text{and therefore } e : b :: g : d.$$

Secondly,

$$\text{If } a : b :: c : d,$$
$$\text{and } f = n\,b,\ h = n\,d;$$
$$\text{then } \frac{a}{b} = \frac{c}{d};$$
$$\text{therefore } \frac{a}{n\,b} = \frac{c}{n\,d},$$
$$\text{that is } \frac{a}{f} = \frac{c}{h};$$
$$\text{and therefore } a : f :: c : h.$$

PROPOSITION V.

If one magnitude be the same multiple of another which a magnitude taken from the first is of a magnitude taken from the other; the remainder shall be the same multiple of the remainder that the whole is of the whole.

$$a, b;\ c, d.$$

Let a be the same multiple of b which c, taken from a, is of d, taken from b; so that $a = m\,b$, $c = m\,d$.

Then shall $a - c = m(b - d)$.
Because $a = mb$, and $c = md$,
therefore $a - c = mb - md$;
but $m(b - d) = mb - md$,
therefore also $a - c = m(b - d)$;

that is, $a - c$ is the same multiple of $b - d$ that a is of b.

Therefore, if one magnitude, &c.—*Q. E. D.*

PROPOSITION VI.

If two magnitudes be equimultiples of two others, and if equimultiples of these be taken from the first two; the remainders are either equal to these others or equimultiples of them.

$$a, b;\ c, d;\ e, f.$$

Let a, b be equimultiples of c, d; and e, f, taken from a, b, also equimultiples of c, d; so that

$$a = mc,\ b = md,$$
$$e = nc,\ f = nd.$$

Then shall either $a - e = c$ and $b - f = d$, or $a - e$ and $b - f$ shall be equimultiples of c and d.

Because $a = mc$ and $e = nc$,
therefore $a - e = mc - nc = (m - n)c$;
and because $b = md$ and $f = nd$,
therefore $b - f = md - nd = (m - n)d$.

Wherefore if the difference between m and n be unity, i.e. if $m - n = 1$; $a - e = c$ and $b - f = d$.

Otherwise $a - e$ and $b - f$ contain c and d respectively $m - n$ times; that is, they are equimultiples of c and d.

Therefore, if two magnitudes, &c.—*Q. E. D.*

PROPOSITION A.

If the first of four magnitudes has to the second the same ratio which the third has to the fourth; then, if the first be greater than the second, the third is also greater than the fourth; if equal, equal; and if less, less.

$$a, b, c, d.$$

Let $a : b :: c : d$,

Then if a be greater than b, c shall be also greater than d; and if equal, equal; and if less, less.

Because $a : b :: c : d$,

$$\text{therefore } \frac{a}{b} = \frac{c}{d}.$$

But if a be greater than b, $\frac{a}{b}$ is greater than unity;

therefore also $\frac{c}{d}$ is greater than unity,

and therefore c greater than d.

And if a be equal to b, $\frac{a}{b}$ is equal to unity;

therefore also $\frac{c}{d}$ is equal to unity,

and therefore c equal to d.

And in like manner if a be less than b, it may be shewn that c is less than d.

Therefore, if the first, &c.—*Q. E. D.*

PROPOSITION B.

"INVERTENDO."

If four magnitudes be proportionals, they are proportionals also when taken inversely.

$$a, b, c, d.$$

Let $a : b :: c : d$.

Then shall $b : a :: d : c$.

Because $a : b :: c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$.

Divide unity by each of these equals:

then $1 \div \frac{a}{b} = 1 \div \frac{c}{d}$;

therefore $1 \times \frac{b}{a} = 1 \times \frac{d}{c}$,

i.e. $\frac{b}{a} = \frac{d}{c}$;

and therefore $b : a :: d : c$.

Therefore, if four magnitudes, &c.—*Q. E. D.*

PROPOSITION C.

If the first be the same multiple or part of the second, that the third is of the fourth; the first is to the second, as the third is to the fourth.

$$a, b, c, d.$$

First, let a be the same multiple of b that c is of d; so that $a = m\,b$, $c = m\,d$.

Then shall $a : b :: c : d$.

Because $a = m\,b$, therefore $\frac{a}{b} = m$;

and because $c = m\,d$, therefore $\frac{c}{d} = m$.

Therefore $\frac{a}{b} = \frac{c}{d}$,

and therefore $a : b :: c : d$.

Secondly, let a be the same part of b that c is of d; so that, $a = \frac{b}{n}$, $c = \frac{d}{n}$.

Then shall $a : b :: c : d$.

Because $a = \frac{b}{n}$, therefore $\frac{a}{b} = \frac{1}{n}$;

and because $c = \frac{d}{n}$, therefore $\frac{c}{d} = \frac{1}{n}$.

Wherefore $\frac{a}{b} = \frac{c}{d}$;

and therefore $a : b :: c : d$.

Therefore, if the first be, &c.—*Q. E. D.*

PROPOSITION D.

If the first be to the second, as the third to the fourth; and if the first be a multiple or part of the second; the third is the same multiple or part of the fourth.

$a, b, c, d.$

Let $a : b :: c : d$.

And first, let a be a multiple of b; so that, $a = m\,b$.

Then shall $c = m\,d$.

Because $a : b :: c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$;

and because $a = m\,b$, therefore $\frac{a}{b} = m$:

wherefore also $\frac{c}{d} = m$,

and therefore $c = m\,d$;

i.e. c is the same multiple of d that a is of b.

Secondly, let a be a part of b; so that, $a = \frac{b}{n}$.

Then shall $c = \frac{d}{n}$.

As before $\frac{a}{b} = \frac{c}{d}$;

and because $a = \frac{b}{n}$, therefore $\frac{a}{b} = \frac{1}{n}$:

wherefore also $\frac{c}{d} = \frac{1}{n}$,

and therefore $c = \frac{d}{n}$;

i.e. c is the same part of d that a is of b.

Therefore, if the first, &c.—*Q. E. D.*

PROPOSITION VII.

Equal magnitudes have the same ratio to the same magnitude; and the same has the same ratio to equal magnitudes.

a, b, c.

Let $a = b$.

Then shall $a : c :: b : c$,

and $c : a :: c : b$.

First, because $a = b$,

therefore $\frac{a}{c} = \frac{b}{c}$,

and therefore $a : c :: b : c$.

Secondly, because $a = b$,

therefore $\frac{c}{a} = \frac{c}{b}$,

and therefore $c : a :: c : b$.

Therefore equal magnitudes have, &c.—*Q. E. D.*

PROPOSITION VIII.

Of unequal magnitudes, the greater has a greater ratio to the same than the less has; and the same magnitude has a greater ratio to the less, than it has to the greater.

a, b, c.

Let *a* be greater than *b*.

Then shall *a* have to *c* a greater ratio than *b* has to *c*; and *c* shall have to *b* a greater ratio than *c* has to *a*.

First, because *a* is greater than *b*,

therefore $\frac{a}{c}$ is greater than $\frac{b}{c}$;

for if two fractions have the same denominator, that which has the greater numerator is the greater of the two.

Therefore *a* has to *c* a greater ratio than *b* has to *c*.

Secondly, because *b* is less than *a*,

therefore $\frac{c}{b}$ is greater than $\frac{c}{a}$;

for if two fractions have the same numerator, that which has the less denominator is the greater of the two.

Therefore *c* has to *b* a greater ratio than *c* has to *a*.

Therefore, of unequal magnitudes, &c.—*Q. E. D.*

PROPOSITION IX.

Magnitudes which have the same ratio to the same magnitude, are equal to one another; and those to which the same magnitude has the same ratio, are equal to one another.

a, b, c.

First, let $a : c :: b : c$.

Then shall $a = b$.

Because $a : c :: b : c$,

therefore $\frac{a}{c} = \frac{b}{c}$;

and therefore $a = b$.

Secondly, let $c : a :: c : b$.

Then shall $a = b$.

Because $c : a :: c : b$,

therefore "invertendo" $a : c :: b : c$; Prop. B.

and therefore $a = b$ by the former case.

Therefore magnitudes which have, &c.—*Q. E. D.*

Or, in the second case,

Because $c : a :: c : b$,

therefore $\frac{c}{a} = \frac{c}{b}$;

and therefore $a = b$.

PROPOSITION X.

That magnitude which has a greater ratio than another has to the same magnitude, is the greater of the two; and that magnitude to which the same has a greater ratio than it has to another magnitude, is the less of the two.

$a, b, c.$

First, let a have to c a greater ratio than b has to c.

Then shall a be greater than b.

Because a has to c a greater ratio than b has to c.

therefore $\frac{a}{c}$ is greater than $\frac{b}{c}$:

but if two fractions have the same denominator, the greater has the greater numerator;

therefore a is greater than b.

Secondly, let c have to b a greater ratio than c has to a.

Then shall b be less than a.

Because c has to b a greater ratio than c has to a,

therefore $\frac{c}{b}$ is greater than $\frac{c}{a}$:

but if two fractions have the same numerator, the greater has a less denominator than the other;

therefore b is less than a.

Therefore, that magnitude, &c.—*Q. E. D.*

PROPOSITION XI.

Ratios that are the same to the same ratio are the same to one another.

$a, b; \; c, d; \; e, f.$

Let $a : b :: c : d$, and $c : d :: e : f$.

Then shall $a : b :: e : f$.

Because $a : b :: c : d$,

$$\text{therefore } \frac{a}{b} = \frac{c}{d};$$

and because $c : d :: e : f$,

$$\text{therefore } \frac{c}{d} = \frac{e}{f}:$$

$$\text{but } \frac{a}{b} = \frac{c}{d};$$

$$\text{therefore } \frac{a}{b} = \frac{e}{f},$$

and therefore $a : b :: e : f$.

Therefore, ratios that are, &c.—*Q. E. D.*

PROPOSITION XII.

If any number of magnitudes be proportionals, as one of the antecedents is to its consequent, so shall all the antecedents taken together be to all the consequents.

$$a, b;\ c, d;\ e, f.$$

Let $a : b :: c : d :: e : f$.

Then shall $a : b :: a + c + e : b + d + f$.

Because $a : b :: c : d :: e : f$,

$$\text{therefore } \frac{a}{b} = \frac{c}{d} = \frac{e}{f}.$$

Since these fractions are equal to one another, let k be a quantity to which each of them is equal: then

$$\text{because } \frac{a}{b} = k, \text{ therefore } a = bk;$$

$$\text{,, } \frac{c}{d} = k, \quad \text{,,} \quad c = dk;$$

$$\text{and ,, } \frac{e}{f} = k, \quad \text{,,} \quad e = fk:$$

therefore $a + c + e = bk + dk + fk$;

but $(b + d + f)k = bk + dk + fk$,

therefore $a + c + e = (b + d + f)k$;

$$\text{therefore } \frac{a + c + e}{b + d + f} = k:$$

$$\text{but } \frac{a}{b} = k,$$

$$\text{therefore } \frac{a}{b} = \frac{a + c + e}{b + d + f},$$

and therefore $a : b :: a + c + e : b + d + f$.

The same proof may be extended to any number of proportionals.

Therefore, if any number, &c.—*Q. E. D.*

Or, generally,

$$a, b;\ c, d;\ e, f;\ \ldots\ y, z.$$

Let $a : b :: c : d :: e : f \ldots :: y : z$.

Then shall

$$a : b :: a + c + e \ldots + y : b + d + f \ldots + z.$$

Because $a : b :: c : d :: e : f \ldots :: y : z$,

$$\text{therefore } \frac{a}{b} = \frac{c}{d} = \frac{e}{f} \ldots = \frac{y}{z}.$$

Let k be a quantity to which each of these fractions is equal: then

$$\text{because } \frac{a}{b} = k, \text{ therefore } a = bk;$$

$$\text{,, } \frac{c}{d} = k, \text{ ,, } c = dk;$$

$$\text{,, } \frac{e}{f} = k, \text{ ,, } e = fk;$$

. . . .

$$\text{,, } \frac{y}{z} = k, \text{ ,, } y = zk:$$

$$\text{therefore } a + c + e \ldots + y = bk + dk + fk \ldots + zk$$
$$= (b + d + f \ldots + z)\, k;$$

$$\text{therefore } \frac{a + c + e \ldots + y}{b + d + f \ldots + z} = k:$$

$$\text{but } \frac{a}{b} = k,$$

$$\text{therefore } \frac{a}{b} = \frac{a + c + e \ldots + y}{b + d + f \ldots + z};$$

and therefore

$$a : b :: a + c + e \ldots + y : b + d + f \ldots + z.$$

PROPOSITION XIII.

If the first has to the second the same ratio which the third has to the fourth, but the third to the fourth a greater ratio than the fifth to the sixth; the first shall also have to the second a greater ratio than the fifth to the sixth.

a, *b*, *c*, *d*, *e*, *f*.

Let $a : b :: c : d$; and let c have to d a greater ratio than e has to f.

Then shall a have to b a greater ratio than e has to f.

Because $a : b :: c : d$,

$$\text{therefore } \frac{a}{b} = \frac{c}{d};$$

but because c has to d a greater ratio than e has to f,

$$\text{therefore } \frac{c}{d} \text{ is greater than } \frac{e}{f};$$

$$\text{wherefore also } \frac{a}{b} \text{ is greater than } \frac{e}{f},$$

and therefore a has to b a greater ratio than e has to f.

Therefore, if the first, &c.—*Q. E. D.*

Cor. And if the first have a greater ratio to the second than the third has to the fourth, but the third the same ratio to the fourth which the fifth has to the sixth; it may be demonstrated, in like manner, that the first has a greater ratio to the second than the fifth has to the sixth.

In this case $\frac{a}{b}$ is greater than $\frac{c}{d}$, and $\frac{c}{d}=\frac{e}{f}$; therefore $\frac{a}{b}$ is greater than $\frac{e}{f}$;

and therefore a has to b a greater ratio than e has to f.

PROPOSITION XIV.

If the first has to the second the same ratio which the third has to the fourth: then if the first be greater than the third, the second shall be greater than the fourth; if equal, equal; and if less, less.

$a, b, c, d.$

Let $a : b :: c : d$.

Then if a be greater than c, b shall be greater than d; and if equal, equal; and if less, less.

Because $a : b :: c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$.

But if a be greater than c; since of two equal fractions, the one which has the greater numerator must also have the greater denominator[b];

therefore b is greater than d.

And if $a = c$; since equal fractions which have equal numerators must also have equal denominators;

therefore $b = d$.

And if a be less than c; since of two equal fractions, the one which has the less numerator must also have the less denominator;

[b] This property of fractions may be proved as follows:—

If $\frac{a}{b} = \frac{c}{d}$, let each of these be equal to k; then

because $\frac{a}{b} = k$, therefore $a = b\,k$,

and because $\frac{c}{d} = k$, therefore $c = d\,k$;

but a is greater than c, therefore $b\,k$ is greater than $d\,k$,

and therefore b is greater than d.

In like manner if a be equal to c, b is equal to d; and if less, less.

therefore b is less than d.

Therefore, if the first has, &c.—*Q. E. D.*

PROPOSITION XV.

Magnitudes have the same ratio to each other which their equimultiples have.

$a, b, c, d.$

Let c and d be equimultiples of a and b; so that $c = m\,a$ and $d = m\,b$.

Then shall $a : b :: c : d$.

Because $c = m\,a$, and $d = m\,b$,

$$\text{therefore } \frac{c}{d} = \frac{m\,a}{m\,b};$$

$$\text{but } \frac{a}{b} = \frac{m\,a}{m\,b};$$

$$\text{therefore } \frac{a}{b} = \frac{c}{d},$$

and therefore $a : b :: c : d$.

Therefore, magnitudes have, &c.—*Q. E. D.*

PROPOSITION XVI.

"PERMUTANDO," OR "ALTERNANDO."

If four magnitudes of the same kind be proportionals, they shall also be proportionals when taken alternately.

$a, b, c, d.$

Let $a : b :: c : d$.

Then shall $a : c :: b : d$.

Because $a : b :: c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$.

Multiply these equal fractions, each by $\frac{b}{c}$;

then $\frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c}$,

whence, by dividing out the *b*'s on the left hand side and the *c*'s on the right,

$$\frac{a}{c} = \frac{b}{d};$$

and therefore $a : c :: b : d$.

Therefore, if four magnitudes, &c.—*Q. E. D.*

PROPOSITION XVII.

"DIVIDENDO."

If magnitudes, taken jointly, be proportionals, they shall also be proportionals when taken separately; that is, if two magnitudes together have to one of them the same ratio which two others have to one of these, the remaining one of the first two shall have to the other the same ratio which the remaining one of the last two has to the other of these.

$a, b, c, d.$

Let $a + b : b :: c + d : d$;

Then shall $a : b :: c : d$.

Because $a + b : b :: c + d : d$,

therefore $\frac{a+b}{b} = \frac{c+d}{d}$;

$$\text{therefore } \frac{a}{b} + 1 = \frac{c}{d} + 1;$$

and, subtracting unity from each of these equals,

$$\frac{a}{b} = \frac{c}{d};$$

therefore $a : b :: c : d$.

Therefore, if magnitudes, &c.—*Q. E. D.*

In Def. XVI. the term "dividendo" is said to be used, "when four magnitudes are proportionals, and it is inferred that the excess of the first above the second is to the second as the excess of the third above the fourth is to the fourth."

This is substantially the same proposition as that enuntiated and proved in the text; for if $a + b$, b, $c + d$, and d be assumed as the four magnitudes, the excess of the first above the second is a, and the excess of the third above the fourth is c; and the hypothesis $a + b : b :: c + d : d$ will lead to the conclusion $a : b :: c : d$ as before.

It may also be proved directly as follows:—

$a, b, c, d.$

Let $a : b :: c : d$.

Then shall $a - b :: b :: c - d : d$.

Because $a : b :: c : d$,

$$\text{therefore } \frac{a}{b} = \frac{c}{d};$$

whence, by subtracting unity from each of these equals,

$$\frac{a}{b} - 1 = \frac{c}{d} - 1;$$

$$\text{therefore } \frac{a-b}{b} = \frac{c-d}{d},$$

and therefore $a - b : b :: c - d : d$.

Therefore, if four magnitudes, &c.—*Q. E. D.*

PROPOSITION XVIII.

"COMPONENDO."

If magnitudes, taken separately, be proportionals, they shall also be proportionals when taken jointly; that is, if the first be to the second, as the third to the fourth, the first and second together shall be to the second as the third and fourth together to the fourth.

$a, b, c, d.$

Let $a : b :: c : d$.

Then shall $a + b : b :: c + d : d$.

Because $a : b :: c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$.

To each of these equals add unity;

then $\frac{a}{b} + 1 = \frac{c}{d} + 1$;

therefore $\frac{a+b}{b} = \frac{c+d}{d}$,

and therefore $a + b : b :: c + d : d$.

Therefore, if magnitudes taken separately, &c.—*Q. E. D.*

PROPOSITION XIX.

If a whole magnitude be to a whole as a magnitude taken from the first is to a magnitude taken from the other, the remainder shall be to the remainder as the whole to the whole.

a, b, c, d.

Let *c* and *d* be taken from *a* and *b*; leaving the remainders $a - c$ and $b - d$: and let $a : b :: c : d$.

Then shall $a - c : b - d :: a : b$.

Because $a : b :: c : d$,

therefore, alternando, $a : c :: b : d$;

therefore, dividendo, $a - c : c :: b - d : d$;

therefore, alternando, $a - c : b - d :: c : d$;

but, by hypothesis, $a : b :: c : d$;

and ratios that are the same to the same ratio are the same to one another, Prop. XI.

therefore, $a - c : b - d :: a : b$.

Therefore, if a whole magnitude, &c.—*Q. E. D.*

Otherwise,

Because $a : b :: c : d$,

$$\text{therefore } \frac{a}{b} = \frac{c}{d};$$

$$\text{Multiply these equals, each by } \frac{b}{c};$$

$$\text{then } \frac{a}{b} \times \frac{b}{c} = \frac{c}{d} \times \frac{b}{c};$$

$$\text{whence } \frac{a}{c} = \frac{b}{d}, \text{ as in Prop. XVI.}$$

From each of these equals subtract unity;

$$\text{then } \frac{a}{c} - 1 = \frac{b}{d} - 1;$$

$$\text{therefore } \frac{a - c}{c} = \frac{b - d}{d}.$$

$$\text{Multiply these equals, each by } \frac{c}{b - d};$$

$$\text{Then } \frac{a - c}{c} \times \frac{c}{b - d} = \frac{b - d}{d} \times \frac{c}{b - d};$$

$$\text{whence } \frac{a-c}{b-d}=\frac{c}{d}:$$

$$\text{but } \frac{a}{b}=\frac{c}{d};$$

$$\text{therefore } \frac{a-c}{b-d}=\frac{a}{b};$$

and therefore $a-c : b-d :: a : b$.

COR. If the whole be to the whole, as a magnitude taken from the first is to a magnitude taken from the other; the remainder likewise is to the remainder as the magnitude taken from the first is to that taken from the other.

This has been proved in the former demonstration, viz.,

$$a-c : b-d :: c : d.$$

And in the latter it was shewn that

$$\frac{a-c}{b-d}=\frac{c}{d};$$

whence $a-c : b-d :: c : d$.

PROPOSITION E.

"CONVERTENDO."

If four magnitudes be proportionals, they are also proportionals by conversion; that is, the first is to its excess above the second, as the third to its excess above the fourth.

$$a, b, c, d.$$

Let $a : b :: c : d$.

Then shall $a : a-b :: c : c-d$.

Because $a : b :: c : d$,

therefore, dividendo, $a-b : b :: c-d : d$;

therefore, invertendo, $b : a-b :: d : c-d$;

and therefore, componendo, $a : a - b :: c : c - d$.
Therefore, if four magnitudes, &c.—*Q. E. D.*

Otherwise,

Because $a : b :: c : d$,

$$\text{therefore } \frac{a}{b} = \frac{c}{d}.$$

From each of these equals subtract unity,

$$\text{then } \frac{a}{b} - 1 = \frac{c}{d} - 1;$$

$$\text{therefore } \frac{a-b}{b} = \frac{c-d}{d}:$$

$$\text{but } \frac{a}{b} = \frac{c}{d};$$

$$\text{therefore } \frac{a}{b} \div \frac{a-b}{b} = \frac{c}{d} \div \frac{c-d}{d};$$

$$\text{therefore } \frac{a}{b} \times \frac{b}{a-b} = \frac{c}{d} \times \frac{d}{c-d};$$

$$\text{whence } \frac{a}{a-b} = \frac{c}{c-d},$$

and therefore $a : a - b :: c : c - d$.

PROPOSITION XX.

If there be three magnitudes, and other three, which taken two and two, have the same ratio; if the first be greater than the third, the fourth shall be greater than the sixth; if equal, equal; and if less, less.

$a, b, c; \ d, e, f.$

Let $a : b :: d : e$,

and $b : c :: e : f$.

Then if a be greater than c, d shall be greater than f; and if equal, equal; and if less, less.

Because $a : b :: d : e$,

therefore $\frac{a}{b} = \frac{d}{e}$;

and because $b : c :: e : f$,

therefore $\frac{b}{c} = \frac{e}{f}$.

Therefore $\frac{a}{b} \times \frac{b}{c} = \frac{d}{e} \times \frac{e}{f}$;

whence $\frac{a}{c} = \frac{d}{f}$.

And if a be greater than c, $\frac{a}{c}$ is greater than unity; therefore $\frac{d}{f}$ is also greater than unity, and therefore d greater than f.

Similarly, if a be equal to c, d is equal to f; and if less, less.

Therefore, if there be, &c.—*Q. E. D.*

PROPOSITION XXII.

"Ex æquali," or "Ex æquo."

If there be any number of magnitudes, and as many others, which, taken two and two in order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first has to the last of the others.

$a, b, c, d; \ e, f, g, h.$

Let $a : b :: e : f$,

$b : c :: f : g$,

and $c : d :: g : h$.

Then shall $a : d :: e : h$.

Because $a : b :: e : f$, therefore $\frac{a}{b} = \frac{e}{f}$;

" $b : c :: f : g$, " $\frac{b}{c} = \frac{f}{g}$;

and " $c : d :: g : h$, " $\frac{c}{d} = \frac{g}{h}$.

Therefore $\frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{e}{f} \times \frac{f}{g} \times \frac{g}{h}$;

whence, dividing out the b's and c's on the left-hand side, and the f's and g's on the right,

$$\frac{a}{d} = \frac{e}{h};$$

and therefore $a : d :: e : h$.

The same proof may be extended to any number of magnitudes.

Therefore, if there be, &c.—*Q. E. D.*

Or, generally,

$a, b, c, d \dots k, l, m, n$;
$p, q, r, s \dots w, x, y, z$.

Let $a : b :: p : q$,
$b : c :: q : r$,
$c : d :: r : s$,
. . . .
$k : l :: w : x$,
$l : m :: x : y$,
and $m : n :: y : z$.

Then shall $a : n :: p : z$.

By reason of the above proportions,

$$\frac{a}{b}=\frac{p}{q},\ \frac{b}{c}=\frac{q}{r},\ \frac{c}{d}=\frac{r}{s},\ \&c.$$

. . . .

$$\frac{k}{l}=\frac{w}{x},\ \frac{l}{m}=\frac{x}{y},\ \text{and}\ \frac{m}{n}=\frac{y}{z}.$$

Therefore $\frac{a}{b}\times\frac{b}{c}\times\frac{c}{d}\ \ldots\ldots\times\frac{k}{l}\times\frac{l}{m}\times\frac{m}{n}$

$$=\frac{p}{q}\times\frac{q}{r}\times\frac{r}{s}\ \ldots\ldots\times\frac{w}{x}\times\frac{x}{y}\times\frac{y}{z}:$$

Whence, dividing out the b's, c's, d's . . . k's, l's, and m's on the left-hand side; and the q's, r's, s's . . . w's, x's, and y's on the right;

$$\frac{a}{n}=\frac{p}{z},$$

and therefore $a : n :: p : z$.

PROPOSITION XXIII.

"Ex æquali in proportione perturbata," or "Ex æquo perturbato."

If there be any number of magnitudes, and as many others, which, taken two and two in a cross order, have the same ratio; the first shall have to the last of the first magnitudes the same ratio which the first has to the last of the others.

$a, b, c, d;\ e, f, g, h.$

Let $a : b :: g : h,$

$b : c :: f : g,$

and $c : d :: e : f.$

Then shall $a : d :: e : h.$

Because $a : b :: g : h$, therefore $\frac{a}{b}=\frac{g}{h};$

$$\text{because } b : c :: f : g, \text{ therefore } \frac{b}{c} = \frac{f}{g};$$

$$\text{and } ,, \quad c : d :: e : f, \quad ,, \quad \frac{c}{d} = \frac{e}{f}.$$

$$\text{Therefore } \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} = \frac{g}{h} \times \frac{f}{g} \times \frac{e}{f};$$

whence, dividing out the b's and c's on the left-hand side, and the g's and f's on the right,

$$\frac{a}{d} = \frac{e}{h};$$

and therefore $a : d :: e : h$.

The same proof may be extended to any number of magnitudes.

Therefore, if there be, &c.—*Q. E. D.*

Or, generally,

$$a, b, c, d, \ldots . k, l, m, n;$$
$$p, q, r, s, \ldots . w, x, y, z.$$

Let $a : b :: y : z$,
$b : c :: x : y$,
$c : d :: w : x$,
. . . .
$k : l :: r : s$,
$l : m :: q : r$,
and $m : n :: p : q$.

Then shall $a : n :: p : z$.

By reason of the above proportions

$$\frac{a}{b} = \frac{y}{z}, \ \frac{b}{c} = \frac{x}{y}, \ \frac{c}{d} = \frac{w}{x}, \text{ &c.}$$

. . . .

$$\frac{k}{l} = \frac{r}{s}, \ \frac{l}{m} = \frac{q}{r}, \text{ and } \frac{m}{n} = \frac{p}{q}.$$

$$\text{Therefore } \frac{a}{b} \times \frac{b}{c} \times \frac{c}{d} \ldots . \times \frac{k}{l} \times \frac{l}{m} \times \frac{m}{n}$$

$$=\frac{y}{z}\times\frac{x}{y}\times\frac{w}{x}\ldots\ldots\times\frac{r}{s}\times\frac{q}{r}\times\frac{p}{q}:$$

whence, dividing out the b's, c's, d's .... k's, l's, and m's on the left-hand side; and the y's, x's, w's, ... s's, r's, and q's on the right;

$$\frac{a}{n}=\frac{p}{z},$$

and therefore $a : n :: p : z$.

PROPOSITION XXIV.

If the first has to the second the same ratio which the third has to the fourth, and the fifth to the second the same ratio which the sixth has to the fourth; the first and fifth together shall have the same ratio to the second which the third and sixth together have to the fourth.

$a, b, c, d, e, f.$

Let $a : b :: c : d$,

and $e : b :: f : d$.

Then shall $a + e : b :: c + f : d$.

Because $a : b :: c : d$,

therefore $\frac{a}{b}=\frac{c}{d}$;

and because $e : b :: f : d$,

therefore $\frac{e}{b}=\frac{f}{d}$.

Whence $\frac{a}{b}+\frac{e}{b}=\frac{c}{d}+\frac{f}{d}$;

therefore $\frac{a+e}{b}=\frac{c+f}{d}$,

and therefore $a + e : b :: c + f : d$.

Therefore, if the first, &c.—*Q. E. D.*

Cor. 1. If the same hypothesis be made as in the proposition, the excess of the first and fifth shall be to the second as the excess of the third and sixth to the fourth.

$$\text{Because } \frac{a}{b} = \frac{c}{d}, \text{ and } \frac{e}{b} = \frac{f}{d}, \text{ as before;}$$

$$\text{therefore } \frac{a}{b} - \frac{e}{b} = \frac{c}{d} - \frac{f}{d};$$

$$\text{whence } \frac{a-e}{b} = \frac{c-f}{d};$$

and therefore $a - e : b :: c - f : d$.

Cor. 2. The proposition holds true of two ranks of magnitudes, whatever be their number, of which each of the first rank has to the second magnitude the same ratio that the corresponding one of the second rank has to a fourth magnitude.

$a, b, c, d;\ e, g,$ &c., $f, h,$ &c.

Let $a : b :: c : d$,
$e : b :: f : d$,
$g : b :: h : d$, &c.

$$\text{Then } \frac{a}{b} = \frac{c}{d}, \frac{e}{b} = \frac{f}{d}, \frac{g}{b} = \frac{h}{d}, \text{ \&c.}$$

$$\text{Therefore } \frac{a}{b} + \frac{e}{b} + \frac{g}{b} + \text{\&c.} = \frac{c}{d} + \frac{f}{d} + \frac{h}{d} + \text{\&c.}$$

$$\text{Therefore } \frac{a + e + g + \text{\&c.}}{b} = \frac{c + f + h + \text{\&c.}}{d},$$

and therefore $a + e + g +$ &c. $: b :: c + f + h +$ &c. $: d$.

PROPOSITION XXV.

If four magnitudes of the same kind be proportionals, the greatest and least of them together are greater than the other two together.

$a, b, c, d.$

Let $a : b :: c : d$;

and let a be the greatest of the four, and therefore d the least. (Props. XIV. and A.)

Then shall $a + d$ be greater than $b + c$.

Because $a : b :: c : d$,

therefore, dividendo, $a - b : b :: c - d : d$;

therefore, invertendo, $b : a - b :: d : c - d$:

but b is greater than d,

therefore also $a - b$ is greater than $c - d$. (Prop. XIV.)

To each of these unequals add $b + d$;

then $a - b + b + d$ is greater than $c - d + b + d$,

or $a + d$ is greater than $b + c$.

Therefore, if four magnitudes, &c.—*Q. E. D.*

Otherwise,

Because $a : b :: c : d$,

therefore $\frac{a}{b} = \frac{c}{d}$.

From each of these equals subtract unity:

then $\frac{a}{b} - 1 = \frac{c}{d} - 1$;

wherefore $\frac{a - b}{b} = \frac{c - d}{d}$:

but b is greater than d; and of two equal fractions the one which has the greater denominator has also the greater numerator;

therefore $a - b$ is greater than $c - d$.

To each of these unequals add $b + d$;
then $a - b + b + d$ is greater than $c - d + b + d$,
or $a + d$ is greater than $b + c$.

PROPOSITION F.

Ratios which are compounded of the same ratios are the same to one another.

$a, b, c;\ d, e, f.$

Let $a : b :: d : e$,
and $b : c :: e : f$.

Then the ratio which is compounded of $a : b$ and $b : c$ shall be the same to the ratio which is compounded of $d : e$ and $e : f$.

Because $a : b :: d : e$,
and $b : c :: e : f$,
therefore, ex æquali, $a : c :: d : f$:

but by the definition of compound ratio, $a : c$ is the ratio compounded of $a : b$ and $b : c$; and $d : f$ is the ratio compounded of $d : e$ and $e : f$; therefore the ratio compounded of $a : b$ and $b : c$ is the same to the ratio compounded of $d : e$ and $e : f$.

The same proof may be extended to any number of ratios.

Therefore, ratios which are, &c.—*Q. E. D.*

Otherwise,

Because $a : b :: d : e$,
and $b : c :: e : f$;

therefore $\frac{a}{b} = \frac{d}{e}$, and $\frac{b}{c} = \frac{e}{f}$:

therefore $\frac{a}{b} \times \frac{b}{c} = \frac{d}{e} \times \frac{e}{f}$;

and, dividing out the b's on the left-hand side, and the e's on the right,

$$\frac{a}{c} = \frac{d}{f}.$$

Therefore $a : c :: d : f$, &c.

PROPOSITION G.

If several ratios be the same to several ratios, each to each; the ratio which is compounded of ratios which are the same to the first ratios, each to each, shall be the same to the ratio compounded of ratios which are the same to the other ratios, each to each.

$a, b, c, d; \; k, l, m:$

$e, f, g, h; \; p, q, r.$

Let $a : b :: e : f$,

and $c : d :: g : h$.

Then the ratio which is compounded of ratios which are the same to the ratios $a : b$ and $c : d$, shall be the same to the ratio which is compounded of ratios which are the same to $e : f$ and $g : h$.

Take k, l, m, such that

$a : b :: k : l$,

and $c : d :: l : m$;

and p, q, r, such that

$e : f :: p : q$,

and $g : h :: q : r$.

Then $k : m$ is the ratio compounded of ratios which are the same to the ratios $a : b$ and $c : d$; and $p : r$ is the ratio compounded of ratios which are the same to the ratios $e : f$ and $g : h$;

and it is to be shewn that $k : m :: p : r$.

Because $k : l :: a : b$,
$a : b :: e : f$,
and $e : f :: p : q$;
therefore $k : l :: p : q$. Prop. XI.
And because $l : m :: c : d$,
$c : d :: g : h$,
and $g : h :: q : r$;
therefore $l : m :: q : r$. Prop. XI.
And since $k : l :: p : q$,
and $l : m :: q : r$;
therefore, ex æquali, $k : m :: p : r$.

The same proof may be extended to any number of ratios.

Therefore, if several ratios, &c.—*Q. E. D.*

Otherwise,

Because $a : b :: e : f$, therefore $\frac{a}{b} = \frac{e}{f}$;

and „ $c : d :: g : h$, „ $\frac{c}{d} = \frac{g}{h}$:

wherefore $\frac{a}{b} \times \frac{c}{d} = \frac{e}{f} \times \frac{g}{h}$.

But because $a : b :: k : l$, therefore $\frac{a}{b} = \frac{k}{l}$;

and „ $c : d :: l : m$ „ $\frac{c}{d} = \frac{l}{m}$:

wherefore $\frac{a}{b} \times \frac{c}{d} = \frac{k}{l} \times \frac{l}{m} = \frac{k}{m}$.

Also, because $e : f :: p : q$, therefore $\frac{e}{f} = \frac{p}{q}$;

and „ $g : h :: q : r$, „ $\frac{g}{h} = \frac{q}{r}$:

$$\text{wherefore } \frac{e}{f} \times \frac{g}{h} = \frac{p}{q} \times \frac{q}{r} = \frac{p}{r}.$$

$$\text{Therefore } \frac{k}{m} = \frac{p}{r},$$

and therefore $k : m :: p : r$.

PROPOSITION H.

If a ratio which is compounded of several ratios be the same to a ratio which is compounded of several other ratios; and if one of the first ratios, or the ratio which is compounded of several of them, be the same to one of the last ratios, or to the ratio which is compounded of several of them; then the remaining ratio of the first, or, if there be more than one, the ratio compounded of the remaining ratios, shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio compounded of these remaining ratios.

$$a, b, c, d, e, f;$$
$$g, h, k, l, m.$$

Let the ratio which is compounded of several ratios, $a : b$, $b : c$, $c : d$, $d : e$, and $e : f$, be the same to the ratio which is compounded of several other ratios, $g : h$, $h : k$, $k : l$, and $m : n$; and let the ratio which is compounded of some of the former, viz. of $a : b$, $b : c$, and $c : d$, be the same to the ratio which is compounded of some of the others, viz. of $g : h$ and $h : k$.

Then the ratio which is compounded of the remaining first ratios, viz. of $d : e$ and $e : f$, shall be the same to the ratio which is compounded of the other remaining ratios, viz. of $k : l$ and $m : n$.

Since $a : f$ is the ratio which is compounded of the first ratios, and $g : m$ is that which is compounded of the others;

by hypothesis $a : f :: g : m$.

Also, since $a : d$ is the ratio which is compounded of $a : b$, $b : c$, and $c : d$; and $g : k$ is that which is compounded of $g : h$ and $h : k$;

by hypothesis $a : d :: g : k$.

And because $a : d :: g : k$,

therefore, invertendo, $d : a :: k : g$;

but $a : f :: g : m$:

therefore, ex æquali, $d : f :: k : m$.

But $d : f$ is the ratio which is compounded of the ratios $d : e$ and $e : f$; and $k : m$ is the ratio which is compounded of the ratios $k : l$ and $l : m$.

Therefore, if a ratio which is compounded, &c.—*Q.E.D.*

PROPOSITION K.

If there be any number of ratios, and any number of other ratios, such that the ratio which is compounded of ratios which are the same to the first ratios, each to each, is the same to the ratio which is compounded of ratios which are the same, each to each, to the last ratios; and if one of the first ratios, or the ratio which is compounded of ratios which are the same to several of the first ratios, each to each, be the same to one of the last ratios, or to the ratio which is compounded of ratios which are the same, each to each, to several of the last ratios; then the remaining ratio of the first, or, if there be more than one, the ratio which is compounded of ratios which are the same, each to each, to the remaining ratios of the first shall be the same to the remaining ratio of the last, or, if there be more than one, to the ratio which is compounded of ratios which are the same, each to each, to these remaining ratios.

$$a, b;\ c, d;\ e, f. \quad q, r, s, t.$$
$$g, h;\ i, j;\ k, l;\ m, n;\ o, p. \quad u, v, w, x, y, z.$$

Let the ratio which is compounded of ratios which are the same to the ratios $a : b$, $c : d$, and $e : f$ be the same to the ratio which is compounded of ratios which are the same to the ratios $g : h$, $i : j$, $k : l$, $m : n$, $o : p$.

And let $a : b$, one of the first ratios, be the same to the ratio which is compounded of ratios which are the same to several of the last ratios, viz. to $g : h$ and $i : j$.

Then shall the ratio which is compounded of ratios which are the same to $c : d$ and $e : f$, the remaining first ratios, be the same to the ratio which is compounded of ratios which are the same to the other remaining ratios, viz. to $k : l$, $m : n$, and $o : p$.

Take q, r, s, t, such that

$$a : b :: q : r,$$
$$c : d :: r : s,$$
$$\text{and } e : f :: s : t;$$

and u, v, w, x, y, z, such that

$$g : h :: u : v,$$
$$i : j :: v : w,$$
$$k : l :: w : x,$$
$$m : n :: x : y,$$
$$o : p :: y : z.$$

Then $q : t$ is the ratio which is compounded of ratios which are the same to the first ratios, and $u : z$ is the ratio which is compounded of ratios which are the same to the last ratios; and by hypothesis

$$q : t :: u : z.$$

Also $u : w$ is the ratio which is compounded of ratios which are the same to the ratios $g : h$ and $i : j$; and by hypothesis

$$a : b :: u : w.$$

And it is to be shewn that $r : t$, which is the ratio compounded of ratios which are the same to $c : d$ and $e : f$, is the same to $w : z$, which is the ratio compounded of ratios which are the same to $k : l$, $m : n$, and $o : p$.

Because $a : b :: u : w$,
and $a : b :: q : r$,
therefore $q : r :: u : w$,
therefore, invertendo, $r : q :: w : u$;
but $q : t :: u : z$;
therefore, ex æquali, $r : t :: w : z$.

Therefore, if there be, &c.—*Q. E. D.*

Otherwise,

Because $q : t :: u : z$, therefore $\frac{q}{t} = \frac{u}{z}$:

but $\frac{q}{t} = \frac{q}{r} \times \frac{r}{t}$ and $\frac{u}{z} = \frac{u}{w} \times \frac{w}{z}$;

therefore $\frac{q}{r} \times \frac{r}{t} = \frac{u}{w} \times \frac{w}{z}$.

And because $a : b :: q : r$, therefore $\frac{a}{b} = \frac{q}{r}$;

„ „ $a : b :: u : w$, „ $\frac{a}{b} = \frac{u}{w}$;

wherefore $\frac{q}{r} = \frac{u}{w}$.

Therefore also $\frac{r}{t} = \frac{w}{z}$,

and therefore $r : t :: w : z$.

NOTES.

Prop. I.—This proposition has been previously assumed by Euclid to be true in two particular cases: viz. in Bk. III., Prop. 20, Case 1, where the sum of the doubles of two angles was taken to be double of the sum of the angles themselves; and in Bk. II., Prop. 8, where the sum of the quadruples of two figures was taken to be quadruple of the sum of the figures themselves.

The algebraical demonstration includes the property of numbers, that the sum of the products of any number and several others is equal to the product of that number and the sum of the others: e.g. $3 \times 7 + 3 \times 5 + 3 \times 4 = 3 \times (7 + 5 + 4)$. Also if m and all the other quantities be taken to represent straight lines, it affords an algebraical proof of Prop. 1, Bk. II.; the products of the quantities representing the rectangles contained by the lines.

Prop. II.—The algebraical demonstration, as generalized in the corollary, includes the property of numbers, that if two numbers be multiplied by several others, each by the same, the sums of the products formed from each are equal to the products of those two numbers and a common multiplier: e.g. $7 \times 3 + 5 \times 3 = 12 \times 3$, and $7 \times 4 + 5 \times 4 = 12 \times 4$.

Prop. III.—The algebraical demonstration includes the property of numbers, that if two numbers be multiplied by any number, and the products so formed multiplied again by another number, each by the same; the final products are equal to the products of the original numbers and a common multiplier: $3 \times 4 \times 5 = 12 \times 5$, and $3 \times 4 \times 9 = 12 \times 9$.

Prop. V.—This proposition has been previously assumed by Euclid to be true in a particular case in Bk. III., Prop. 20, Case 2, where the difference of the doubles of two angles was taken to be double of the difference of the angles themselves.

The algebraical demonstration includes the property of numbers, that the difference of the products of any number and two others is equal to the product of that number and the difference of the others: e.g. $3 \times 7 - 3 \times 5 = 3 \times (7 - 5)$. Compare this with Prop. I., subtraction taking the place of addition.

Prop. VI.—The algebraical demonstration includes the property of numbers, that if two numbers be multiplied by two others, each by the same, the differences of the products formed from each are either equal to the numbers themselves or to the products of those two numbers and a common multiplier: e.g. $7 \times 3 - 6 \times 3 = 3$, and $7 \times 4 - 6 \times 4 = 4$; and $7 \times 3 - 5 \times 3 = 2 \times 3$, and $7 \times 4 - 5 \times 4 = 2 \times 4$.

Prop. D.—The exact converse of the preceding proposition would be, If the first be to the second as the third to the fourth; the first shall be the same multiple of the second, or the same part of it, that the third is of the fourth. But that would not be always true; for the first need not be a multiple or part of the second, nor the third of the fourth. This proposition may be termed a conditional converse of the preceding.

Prop. XII.—The algebraical demonstration proves the property of fractions, that if several fractions be equal to one another, each of them is equal to the fraction which has the sum of their numerators for a numerator and the sum of their denominators for a denominator.

Prop. XVI.—It has been shewn in the Introduction (p. 15) that the algebraical demonstration proves the geometrical proposition only on the supposition that a, b, c, d represent magnitudes of the same kind and in a common denomination. It is however always true of four *numbers* which are proportionals; and consequently in arithmetical examples on the Rule of Three the second and third terms are often interchanged without error in the result, even when the quantities are not all of the same kind.

Thus the statement 3 cwt. : 6 cwt. : : £4 : the answer, will give Ans. £8; and 3 : 4 : : 6 : the answer, will give Ans. 8; although it cannot be rightly said that 3 cwt. has any ratio to £4, or 6 cwt. to £8.

Prop. XX.—It is further true, that if there be *any number* of magnitudes and as many others, which taken two and two in order, have the same ratio; if the first be greater than the last of the first magnitudes, the first is also greater than the last of the others; and if equal, equal; and if less, less.

The magnitudes being the same as in Prop. XXII. "Ex æquali," it may be shewn, as in that proposition, that $\frac{a}{n} = \frac{p}{z}$; whence the result may be inferred.

Prop. XXI.—This may be extended to *any number* of magnitudes in the same manner as the preceding proposition; the equality of the fractions being proved as in Prop. XXIII., "Ex æquali in proportione perturbatâ," and the inference drawn as before.

Prop. G.—Euclid has made no distinction between the ratio compounded of other ratios and the ratio compounded of ratios which are the same to those ratios; and on the principle laid down in the second paragraph of Dr. Simson's explanations of his Def. A, the ratio of K to M in this proposition may be said to be the ratio compounded of the ratios of A to B and C to D; and the ratio of N to P may in like manner be said to be compounded of E to F and G to H. Hence Propositions F and G might have been included under the enunciation of the former.

Prop. K.—This proposition might have been included under the enunciation of Prop. H, in the same manner that Prop. G might have been included in that of Prop. F. In the algebraical proof given in the text, a less number of magnitudes have been represented than in Dr. Simson's demonstration, because the proof of the theorem does not appear to require that the ratio compounded of the same ratios should be expressed in more ways than one.

Dr. Simson's demonstration is adapted to the case in which certain magnitudes h, k, l are known to have the same ratios to one another that *C* has to *D* and *E* to *F*; and m, n, o, p are known to have the same ratios to one another that *M* has to *N*, *O* to *P*, and *Q* to *R*; whilst *T*, *V*, *X*, and a, b, c, d are other magnitudes arbitrarily chosen which have to one another those same ratios.

QUESTIONS.

1. WHAT is a magnitude?

2. How many different kinds of magnitudes are there? and how are they distinguished generally from one another?

3. Specify the different kinds of magnitudes and the characteristic distinction of each.

4. What is Euclid's criterion of magnitudes being of the same kind? Is there any other?

5. Compare the meanings of the word "part" in the fifth and other books of Euclid.

6. How is a magnitude expressed algebraically as a part of another? and how as a multiple?

7. How may the correlative relations of a multiple and its part be variously described?

8. When are two or more magnitudes said to be equimultiples of as many others?

9. How are equimultiples expressed algebraically?

10. When are two magnitudes said to be commensurable? and when incommensurable? Give examples.

11. What is meant by the quantity of a magnitude? and how is the quantity of a magnitude expressed algebraically or arithmetically?

12. How is the ratio of two magnitudes expressed algebraically?

13. Can the ratio of the same two magnitudes be expressed algebraically in more ways than one?

14. Compare the geometrical and algebraical definitions of the sameness (or similitude) of ratios.

15. When equimultiples of four magnitudes are taken as in the fifth definition; if when the multiple of the first is greater than that of the second, the multiple of the third must be greater than that of the fourth; does that condition alone prove the magnitudes to be proportionals?

16. If, particular equimultiples being taken, the multiple of the first is equal to that of the second, and the multiple of the third equal to that of the fourth; are the magnitudes necessarily proportionals?

17. Can equimultiples be always taken, so that the multiple of the first is equal to that of the second, and the multiple of the third equal to that of the fourth? If not, why?

18. If, particular equimultiples being taken, the multiple of the first is greater than that of the second, but the multiple of the third not greater than that of the fourth; has the first necessarily a greater ratio to the second than the third has to the fourth?

19. What is the algebraical criterion that the first of four magnitudes has a greater ratio to the second than the third has to the fourth?

20. Distinguish between proportionals and continual proportionals.

21. If four magnitudes which are proportionals be represented algebraically by a, b, c, d; and it is assumed that $a : b :: c : d$; what condition is suppressed?

22. If the lengths of four lines which are proportionals be a inches, b feet, c yards, and d fathoms severally; what is the relation between a, b, c, and d?

23. If four magnitudes which are continual proportionals be represented algebraically by a, b, c, d; and it is assumed that $a : b :: b : c :: c : d$; what condition is suppressed?

24. If the lines in Question 22 are continual proportionals, what is the relation between a, b, c, and d?

25. How does Euclid apply the term "compound ratio" in Prop. 23, Bk. VI.?

26. If A, B, C, D be four magnitudes, what is meant by the ratio compounded of the ratios of A to B and C to D?

27. If a, b, c, d be four algebraical quantities, what is the ratio compounded of the ratios of a to b and b to c?

28. If $A : B :: B : C :: C : D :: D : E$, how is the ratio of A to B related to the ratios of A to C, A to D, and A to E severally?

29. Are duplicate ratios, triplicate, quadruplicate, &c., compound ratios?

30. How are the duplicate and triplicate ratios of a to b expressed algebraically? and what do they imply geometrically, if a and b express the lengths of two lines?

31. If $a : b :: c : d$, what proportions are inferred by taking the quantities "permutando" or "alternando," "invertendo," "componendo," "dividendo," and "convertendo?" Which of them is applicable only when the magnitudes are all of the same kind? How many must be of the same kind when the others are applied?

32. What properties of numbers are included in the algebraical demonstrations of Props. 1, 2, 3, 5, and 6?

33. How can Prop. 1, Bk. II. be inferred from the algebraical demonstration of Prop. 1?

34. What properties of fractions are proved in the algebraical demonstrations of Props. B, 12, 16, 17, 18, 19, E, 22, and 23?

35. Describe generally the process by which propositions, such as those of the fifth book of Euclid, relating to geometrical magnitudes, are proved algebraically.

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