CHAPTER VII.

DIFFERENTIAL EQUATIONS.

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ART. 1. EQUATIONS OF FIRST ORDER AND DEGREE.

In the Integral Calculus, supposing y to denote an unknown function of the independent variable x, the derivative of y with respect to x is given in the form of a function of x, and it is required to find the value of y as a function of x. In other words, given an equation of the form

$$\frac{dy}{dx} = f(x), \quad \text{or} \quad dy = f(x)dx, \tag{1}$$

of which the general solution is written in the form

$$y = \int f(x)dx, \tag{2}$$

it is the object of the Integral Calculus to reduce the expression in the second member of equation (2) to the form of a known function of x. When such reduction is not possible, the equation serves to define a new function of x.

In the extension of the processes of integration of which the following pages give a sketch the given expression for the derivative may involve not only x, but the unknown function y; or, to write the equation in a form analogous to equation (1), it may be

$$Mdx + Ndy = 0, \tag{3}$$

in which M and N are functions of x and y. This equation is in fact the general form of the differential equation of the first order and degree; either variable being taken as the independent variable, it gives the first derivative of the other variable in terms of x and y. So also the solution is not necessarily an expression of either variable as a function of the other, but is generally a relation between x and y which makes either an implicit function of the other.

When we recognize the left member of equation (3) as an "exact differential," that is, the differential of some function of x and y, the solution is obvious. For example, given the equation

$$xdy + ydx = 0, \qquad (4)$$

the solution

$$xy = C, \qquad (5)$$

where C is an arbitrary constant, is obtained by "direct integration." When a particular value is attributed to C, the result is a "particular integral;" thus $y = x^{-1}$ is a particular integral of equation (4), while the more general relation expressed by equation (5) is known as the "complete integral."

In general, the given expression $Mdx + Ndy$ is not an exact differential, and it is necessary to find some less direct method of solution.

The most obvious method of solving a differential equation of the first order and degree is, when practicable, to "separate the variables," so that the coefficient of dx shall contain x only, and that of dy, y only. For example, given the equation

$$(1 - y)dx + (1 + x)dy = 0, \qquad (6)$$

the variables are separated by dividing by $(1 + x)(1 - y)$.

Thus

$$\frac{dx}{1 + x} + \frac{dy}{1 - y} = 0.$$

Each term is now directly integrable, and hence

$$\log(1 + x) - \log(1 - y) = c.$$

The solution here presents itself in a transcendental form, but it is readily reduced to an algebraic form. For, taking the exponential of each member, we find

$$\frac{1 + x}{1 - y} = e^c = C, \quad \text{whence} \quad 1 + x = C(1 - y), \qquad (7)$$

where C is put for the constant e^c.

To verify the result in this form we notice that differentiation gives $dx = -Cdy$, and substituting in equation (6) we find

$$-C(1-y)+1+x=0,$$

which is true by equation (7).

Prob. 1. Solve the equation $dy + y \tan x\, dx = 0$.

(Ans. $y = C \cos x$.)

Prob. 2. Solve $\dfrac{dy}{dx} + b^2y^2 = a^2$. $\left(\text{Ans. } \dfrac{by+a}{by-a} = ce^{2abx}.\right)$

Prob. 3. Solve $\dfrac{dy}{dx} = \dfrac{y^2+1}{x^2+1}$. $\left(\text{Ans. } y = \dfrac{x+c}{1-cx}.\right)$

Prob. 4. Helmholtz's equation for the strength of an electric current C at the time t is

$$C = \frac{E}{R} - \frac{l}{R}\frac{dC}{dt},$$

where E, R, and L are given constants. Find the value of C, determining the constant of integration by the condition that its initial value shall be zero.

ART. 2. GEOMETRICAL REPRESENTATION.

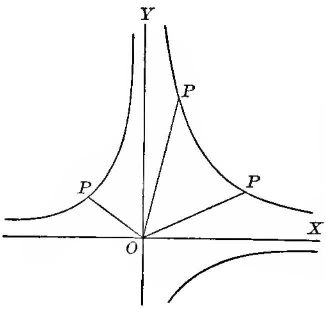
The meaning of a differential equation may be graphically illustrated by supposing simultaneous values of x and y to be the rectangular coordinates of a variable point. It is convenient to put p for the value of the ratio $dy : dx$. Then P being the moving point (x, y) and ϕ denoting the inclination of its path to the axis of x, we have

$$p = \frac{dy}{dx} = \tan \phi.$$

The given differential equation of the first order is a relation between p, x, and y, and, being of the first degree with respect to p, determines in general a single value of p for any assumed values of x and y. Suppose in the first place that, in addition to the differential equation, we were given one pair of simultaneous values of x and y, that is, one position of the point P. Now let P start from this fixed initial point and begin to move in either direction along the straight line whose inclination is determined by the value of p corresponding to the initial values of x and y. We thus have a moving point satisfying the given differential equation. As the point P moves the values of x and y vary, and we must suppose the direction of its motion to vary in such a way that the simultaneous values of x, y, and p continue to satisfy the differential equation. In that case, the path of the moving point is said to satisfy the differential equation. The point P may return to its initial position, thus describing a closed curve, or it may pass to infinity in each direction from the initial point describing an infinite branch of a curve.* The ordinary cartesian equation of the path of P is a particular integral of the differential equation.

If no pair of associated values of x and y be known, P may be assumed to start from any initial point, so that there is an unlimited number of curves representing particular integrals of the equation. These form a "system of curves," and the complete integral is the equation of the system in the usual form of a relation between x, y, and an arbitrary "parameter." This parameter is of course the constant of integration. It is constant for any one curve of the system, and different values of it determine different members of the system of curves, or different particular integrals.

As an illustration, let us take equation (4) of Art. 1, which may be written



$$\frac{dy}{dx} = -\frac{y}{x}.$$

Denoting by θ the inclination to the axis of x of the line joining P with the origin, the equation is equivalent to $\tan\phi = -\tan\theta$, and therefore expresses that P moves in a direction inclined equally with OP to either axis, but on the other

* When the form of the functions M and N is unrestricted, there is no reason why either of these cases should exist, but they commonly occur among such differential equations as admit of solution.

side. Starting from any position in the plane, the point P thus moving must describe a branch of an hyperbola having the two axes as its asymptotes; accordingly, the complete integral $xy = C$ is the equation of the system consisting of these hyperbolas.

Prob. 5. Write the differential equation which requires P to move in a direction always perpendicular to OP, and thence derive the equation of the system of curves described.

(Ans. $\frac{dy}{dx} = -\frac{x}{y}$; $x^2 + y^2 = C$.)

Prob. 6. What is the system described when ϕ is the complement of θ? (Ans. $x^2 - y^2 = C$.)

Prob. 7. If $\phi = 2\theta$, show geometrically that the system described consists of circles, and find the differential equation.

(Ans. $2xydx = (x^2 - y^2)dy$.)

ART. 3. PRIMITIVE OF A DIFFERENTIAL EQUATION.

Let us now suppose an ordinary relation between x and y, which may be represented by a curve, to be given. By differentiation we may obtain an equation of which the given equation is of course a solution or particular integral. But by combining this with the given equation any number of differential equations of which the given equation is a solution may be found. For example, from

$$y^2 = m(x - a) \tag{1}$$

we obtain directly

$$2ydy = mdx, \tag{2}$$

of which equation (1) is an integral; again, dividing (2) by (1) we have

$$\frac{2dy}{y} = \frac{dx}{x - a}, \tag{3}$$

and of this equation also (1) is an integral.

If in equation (1) m be regarded as an arbitrary parameter, it is the equation of a system of parabolas having a common axis and vertex. The differential equation (3), which does not contain m, is satisfied by every member of this system of curves.

Hence equation (1) thus regarded is the complete integral of equation (3), as will be found by solving the equation in which the variables are already separated.

Now equation (3) is obviously the only differential equation independent of m which could be derived from (1) and (2), since it is the result of eliminating m. It is therefore the "differential equation of the system;" and in this point of view the integral equation (1) is said to be its "primitive."

Again, if in equation (1) a be regarded as the arbitrary constant, it is the equation of a system of equal parabolas having a common axis. Now equation (2) which does not contain a is satisfied by every member of this system of curves; hence it is the differential equation of the system, and its primitive is equation (1) with a regarded as the arbitrary constant.

Thus, a primitive is an equation containing as well as x and y an arbitrary constant, which we may denote by C, and the corresponding differential equation is a relation between x, y, and p, which is found by differentiation, and elimination of C if necessary. This is therefore also a method of verifying the complete integral of a given differential equation. For example, in verifying the complete integral (7) in Art. 1 we obtain by differentiation $1 = -Cp$. If we use this to eliminate C from equation (7) the result is equation (6); whereas the process before employed was equivalent to eliminating p from equation (6), thereby reproducing equation (7).

Prob. 8. Write the equation of the system of circles in Prob. 7, Art. 2, and derive the differential equation from it as a primitive.

Prob. 9. Write the equation of the system of circles passing through the points $(0, b)$ and $(0, -b)$, and derive from it the differential equation of the system.

ART. 4. EXACT DIFFERENTIAL EQUATIONS.

In Art. 1 the case is mentioned in which $Mdx + Ndy$ is an "exact differential," that is, the differential of a function of x and y. Let u denote this function; then

$$du = Mdx + Ndy, \tag{1}$$

and in the notation of partial derivatives

$$M = \frac{\partial u}{\partial x}, \qquad N = \frac{\partial u}{\partial y}.$$

Then, since by a theorem of partial derivatives $\frac{\partial^2 u}{\partial y \partial x} = \frac{\partial^2 u}{\partial x \partial y}$,

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}. \tag{2}$$

This condition must therefore be fulfilled by M and N in order that equation (1) may be possible. When it is fulfilled $Mdx + Ndy = 0$ is said to be an "exact differential equation," and its complete integral is

$$u = C. \tag{3}$$

For example, given the equation

$$x(x + 2y)dx + (x^2 - y^2)dy = 0,$$

$M = x(x + 2y)$, $N = x^2 - y^2$, $\frac{\partial M}{\partial y} = 2x$, and $\frac{\partial N}{\partial x} = 2x$; the condition (2) is fulfilled, and the equation is exact. To find the function u, we may integrate Mdx, treating y as a constant; thus,

$$\tfrac{1}{3}x^3 + x^2 y = Y,$$

in which the constant of integration Y may be a function of y. The result of differentiating this is

$$x^2 dx + 2xy\,dx + x^2 dy = dY,$$

which should be identical with the given equation; therefore, $dY = y^2\,dy$, whence $Y = \frac{1}{3}y^3 + C$, and substituting, the complete integral may be written

$$x^3 + 3x^2 y = y^3 + C.$$

The result is more readily obtained if we notice that all terms containing x and dx only, or y and dy only, are exact differentials; hence it is only necessary to examine the terms containing both x and y. In the present case, these are $2xy\,dx + x^2 dy$, which obviously form the differential of $x^2 y$; whence, integrating and multiplying by 3, we obtain the result above.

The complete integral of any equation, in whatever way it was found, can be put in the form $u = C$, by solving for C. Hence an exact differential equation $du = 0$ can be obtained, which must be equivalent to the given equation

$$Mdx + Ndy = 0, \tag{4}$$

here supposed not to be exact. The exact equation $du = 0$ must therefore be of the form

$$\mu(Mdx + Ndy) = 0, \tag{5}$$

where μ is a factor containing at least one of the variables x and y. Such a factor is called an "integrating factor" of the given equation. For example, the result of differentiating equation (7), Art. 1, when put in the form $u = C$, is

$$\frac{(1 - y)dx + (1 + x)dy}{(1 - y)^2} = 0,$$

so that $(1 - y^2)^{-2}$ is an integrating factor of equation (6). It is to be noticed that the factor by which we separated the variables, namely, $(1 - y)^{-1}(1 - x)^{-1}$, is also an integrating factor.

It follows that if an integrating factor can be discovered, the given differential equation can at once be solved.* Such a factor is sometimes suggested by the form of the equation. Thus, given

$$(y - x)dy + ydx = 0,$$

the terms $ydx - xdy$, which contain both x and y, are not exact, but become so when divided by either x^2 or y^2; and because the remaining term contains y only, y^{-2} is an integrating factor of the whole expression. The resulting integral is

$$\log y + \frac{x}{y} = C.$$

Prob. 10. Show from the integral equation in Prob. 9, Art. 3, that x^{-2} is an integrating factor of the differential equation.

Prob. 11. Solve the equation $x(x^2 + 3y^2)dx + y(y^2 + 3x^2)dy = 0$.
(Ans. $x^4 + 6x^2y^2 + y^4 = c$.)

* Since μM and μN in the exact equation (5) must satisfy the condition (2), we have a partial differential equation for μ; but as a general method of finding μ this simply comes back to the solution of the original equation.

Prob. 12. Solve the equation $y\,dy + x\,dx + \frac{x\,dy - y\,dx}{x^2 + y^2}$.

(Ans. $\frac{x^2 + y^2}{2} + \tan^{-1}\frac{y}{x} = c$.)

Prob. 13. If $u = c$ is a form of the complete integral and μ the corresponding integrating factor, show that $\mu f(u)$ is the general expression for the integrating factors.

Prob. 14. Show that the expression $x^\alpha y^\beta(mydx + nxdy)$ has the integrating factor $x^{km-1-\alpha}y^{kn-1-\beta}$; and by means of such a factor solve the equation $y(y^3 + 2x^4)dx + x(x^4 - 2y^3)dy = 0$.

(Ans. $2x^4y - y^4 = cx^2$.)

Prob. 15. Solve $(x^2 + y^2)dx - 2xy\,dy = 0$. (Ans. $x^2 - y^2 = cx$.)

ART. 5. HOMOGENEOUS EQUATION.

The differential equation $Mdx + Ndy = 0$ is said to be homogeneous when M and N are homogeneous functions of x and y of the same degree; or, what is the same thing, when $\frac{dy}{dx}$ is expressible as a function of $\frac{y}{x}$. If in such an equation the variables are changed from x and y to x and v, where

$$v = \frac{y}{x}; \qquad \text{whence} \quad y = xv \quad \text{and} \quad dy = xdv + vdx,$$

the variables x and v will be separable. For example, the equation

$$(x - 2y)dx + ydy = 0$$

is homogeneous; making the substitutions indicated and dividing by x,

$$(1 - 2v)dx + v(xdv + vdx) = 0,$$

whence
$$\frac{dx}{x} + \frac{vdv}{(v - 1)^2} = 0.$$

Integrating,
$$\log x + \log(v - 1) - \frac{1}{v - 1} = C;$$

and restoring y,

$$\log(y - x) - \frac{x}{y - x} = C.$$

The equation $Mdx + Ndy = 0$ can always be solved when

M and N are functions of the first degree, that is, when it is of the form

$$(ax + by + c)dx + (a'x + b'y + c')dy = 0.$$

For, assuming $x = x' + h$, $y = y' + k$, it becomes

$$(ax' + b'y' + ah + bk + c)dx' + (a'x' + b'y' + a'h + b'k + c')dy' = 0,$$

which, by properly determining h and k, becomes

$$(ax' + by')dx' + (a'x' + b'y')dy',$$

a homogeneous equation.

This method fails when $a : b = a' : b'$, that is, when the equation takes the form

$$(ax + by + c)dx + [m(ax + by) + c']dy = 0;$$

but in this case if we put $z = ax + by$, and eliminate y, it will be found that the variables x and z can be separated.

Prob. 16. Show that a homogeneous differential equation represents a system of similar and similarly situated curves, the origin being the center of similitude, and hence that the complete integral may be written in a form homogeneous in x, y, and c.

Prob. 17. Solve $x\,dy - y\,dx - \sqrt{(x^2 + y^2)}dx = 0$.
(Ans. $x^2 = c^2 - 2cy$.)

Prob. 18. Solve $(3y - 7x + 7)dx + (7y - 3x + 3)dy = 0$.
(Ans. $(y - x + 1)^2(y + x - 1)^5 = c$.)

Prob. 19. Solve $(x^2 + y^2)dx - 2xy\,dy = 0$. (Ans. $x^2 - y^2 = cx$.)

Prob. 20. Solve $(1 + xy)y\,dx + (1 - xy)x\,dy = 0$ by introducing the new variable $z = xy$. (Ans. $x = Cye^{\frac{1}{xy}}$.)

Prob. 21. Solve $\dfrac{dy}{dx} = ax + by + c$. (Ans. $abx + b^2y + a + bc = Ce^{bx}$.)

ART. 6. THE LINEAR EQUATION.

A differential equation is said to be "linear" when (one of the variables, say x, being regarded as independent,) it is of the first degree with respect to y, and its derivatives. The linear equation of the first order may therefore be written in the form

$$\frac{dy}{dx} + Py = Q, \tag{1}$$

where P and Q are functions of x only. Since the second member is a function of x, an integrating factor of the first member will be an integrating factor of the equation provided it contains x only. To find such a factor, we solve the equation

$$\frac{dy}{dx} + Py = 0, \tag{2}$$

which is done by separating the variables; thus, $\dfrac{dy}{y} = -Pdx$;

whence $\log y = c - \int Pdx$ or

$$y = Ce^{-\int Pdx}. \tag{3}$$

Putting this equation in the form $u = c$, the corresponding exact equation is

$$e^{\int Pdx}(dy + Pydx) = 0,$$

whence $e^{\int Pdx}$ is the integrating factor required. Using this factor, the general solution of equation (1) is

$$e^{\int Pdx}y = \int e^{\int Pdx}Qdx + C. \tag{4}$$

In a given example the integrating factor should of course be simplified in form if possible. Thus

$$(1 + x^2)dy = (m + xy)dx$$

is a linear equation for y; reduced to the form (1), it is

$$\frac{dy}{dx} - \frac{x}{1 + x^2}y = \frac{m}{1 + x^2},$$

from which

$$\int Pdx = -\int \frac{x\,dx}{1 + x^2} = -\frac{1}{2}\log(1 + x^2).$$

The integrating factor is, therefore,

$$e^{\int Pdx} = \frac{1}{\sqrt{(1 + x^2)}};$$

whence the exact equation is

$$\frac{dy}{\sqrt{(1 + x^2)}} - \frac{xy\,dx}{(1 + x^2)^{\frac{3}{2}}} = \frac{mdx}{(1 + x^2)^{\frac{3}{2}}}.$$

Integrating, there is found

$$\frac{y}{\sqrt{(1+x^2)}} = \frac{mx}{\sqrt{(1+x^2)}} + C,$$

or

$$y = mx + C\sqrt{(1+x^2)}.$$

An equation is sometimes obviously linear, not for y, but for some function of y. For example, the equation

$$\frac{dy}{dx} + \tan y = x \sec y$$

when multiplied by $\cos y$ takes a form linear for $\sin y$; the integrating factor is e^x, and the complete integral

$$\sin y = x - 1 + ce^{-x}.$$

In particular, the equation $\frac{dy}{dx} + Py = Qy^n$, which is known as "the extension of the linear equation," is readily put in a form linear for y^{1-n}.

Prob. 22. Solve $x^2\frac{dy}{dx} + (1 - 2x)y = x^2$. (Ans. $y = x^2(1 + ce^{\frac{1}{x}})$.)

Prob. 23. Solve $\cos x \frac{dy}{dx} + y - 1 + \sin x = 0$.

(Ans. $y(\sec x + \tan x) = x + c$.)

Prob. 24. Solve $\frac{dy}{dx} \cos x + y \sin x = 1$.

(Ans. $y = \sin x + c \cos x$.)

Prob. 25. Solve $\frac{dy}{dx} = x^3y^3 - xy$. (Ans. $\frac{1}{y^2} = x^2 + 1 + ce^{x^2}$.)

Prob. 26. Solve $\frac{dy}{dx} = \frac{1}{xy + x^2y^3}$. (Ans. $\frac{1}{x} = 2 - y^2 + ce^{-\frac{1}{2}y^2}$.)

ART. 7. FIRST ORDER AND SECOND DEGREE.

If the given differential equation of the first order, or relation between x, y, and p, is a quadratic for p, the first step in the solution is usually to solve for p. The resulting value of p will generally involve an irrational function of x and y; so that an equation expressing such a value of p, like some of those solved in the preceding pages, is not properly to be regarded as an equation of the first degree. In the exceptional case when the expression whose root is to be extracted is a perfect square, the equation is decomposable into two equations properly of the first degree. For example, the equation

$$xy(1 + 4p^2) = 2p(x^2 + y^2)$$

when solved for p gives $2p = \dfrac{y}{x}$, or $2p = \dfrac{x}{y}$; it may therefore be written in the form

$$(2px - y)(2py - x) = 0,$$

and is satisfied by putting either

$$\frac{dy}{dx} = \frac{y}{2x}, \quad \text{or} \quad \frac{dy}{dx} = \frac{x}{2y}.$$

The integrals of these equations are

$$y^2 = cx \quad \text{and} \quad 2y^2 - x^2 = C,$$

and these form two entirely distinct solutions of the given equation.

As an illustration of the general case, let us take the equation

$$xp^2 = y^2, \quad \text{or} \quad \frac{dy}{dx} = \pm \frac{\sqrt{y}}{\sqrt{x}}. \qquad (1)$$

Separating the variables and integrating,

$$\sqrt{x} \pm \sqrt{y} = \pm \sqrt{c}, \qquad (2)$$

and this equation rationalized becomes

$$(x - y)^2 - 2c(x + y) + c^2 = 0. \qquad (3)$$

There is thus a single complete integral containing one arbitrary constant and representing a single system of curves; namely, in this case, a system of parabolas touching each axis at the same distance c from the origin. The separate equations given in the form (2) are merely branches of the same parabola.

Recurring now to the geometrical interpretation of a differential equation, as given in Art. 2, it was stated that an equation of the first degree determines, in general, for assumed values of x and y, that is, at a selected point in the plane, a single value of p. The equation was, of course, then supposed rational in x and y.* The only exceptions occur at points for which the value of p takes the indeterminate form; that is, the equation being $Mdx + Ndy = 0$, at points (if any exist) for which $M = 0$ and $N = 0$. It follows that, except at such points, no two curves of the system representing the complete integral intersect, while through such points an unlimited number of the curves may pass, forming a "pencil of curves."†

On the other hand, in the case of an equation of the second degree, there will in general be two values of p for any given point. Thus from equation (1) above we find for the point (4, 1), $p = \pm \frac{1}{2}$; there are therefore two directions in which a point starting from the position (4, 1) may move while satisfying the differential equation. The curves thus described represent two of the particular integrals. If the same values of x and y be substituted in the complete integral (3), the result is a quadratic for c, giving $c = 9$ and $c = 1$, and these determine the two particular integral curves, $\sqrt{x} + \sqrt{y} = 3$, and $\sqrt{x} - \sqrt{y} = 1$.

In like manner the general equation of the second degree, which may be written in the form

$$Lp^2 + Mp + N = 0,$$

where L, M, and N are one-valued functions of x and y, represents a system of curves of which two intersect in any given point for which p is found to have two real values. For these points, therefore, the complete integral should generally give two real values of c. Accordingly we may assume, as the standard form of its equation,

$$Pc^2 + Qc + R = 0,$$

* In fact p was supposed to be a one-valued function of x and y; thus, $p = \sin^{-1}x$ would not in this connection be regarded as an equation of the first degree.

† In Prob. 6, Art. 3, the integral equation represents the pencil of circles passing through the points $(0, b)$ and $(0, -b)$; accordingly p in the differential equation is indeterminate at these points. In some cases, however, such a point is merely a node of one particular integral. Thus in the illustration given in Art. 2, p is indeterminate at the origin, and this point is a node of the only particular integral, $xy = 0$, which passes through it.

where P, Q, and R are also one-valued functions of x and y. If there are points which make p imaginary in the differential equation, they will also make c imaginary in the integral.

Prob. 27. Solve the equation $p^2 + y^2 = 1$ and reduce the integral to the standard form.

(Ans. $(y + \cos x)c^2 - 2c \sin x + y - \cos x = 0$.)

Prob. 28. Solve $yp^2 + 2xp - y = 0$, and show that the intersecting curves at any given point cut at right angles.

Prob. 29. Solve $(x^2 + 1)p^2 = 1$. (Ans. $c^2e^{2y} - 2cxe^y = 1$.)

ART. 8. SINGULAR SOLUTIONS.

A differential equation not of the first degree sometimes admits of what is called a "singular solution;" that is to say, a solution which is not included in the complete integral. For suppose that the system of curves representing the complete integral has an envelope. Every point A of this envelope is a point of contact with a particular curve of the complete integral system; therefore a point moving in the envelope when passing through A has the same values of x, y, and p as if it were moving through A in the particular integral curve. Hence such a point satisfies the differential equation and will continue to satisfy it as long as it moves in the envelope. The equation of the envelope is therefore a solution of the equation.

As an illustration, let us take the system of straight lines whose equation is

$$y = cx + \frac{a}{c}, \tag{1}$$

where c is the arbitrary parameter. The differential equation derived from this primitive is

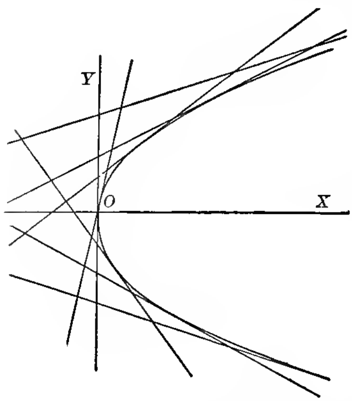
$$y = px + \frac{a}{p}, \tag{2}$$

of which therefore (1) is the complete integral.

Now the lines represented by equation (1), for different values of c, are the tangents to the parabola

$$y^2 = 4ax. \tag{3}$$

A point moving in this parabola has the same value of p as if it were moving in one of the tangents, and accordingly equation (3) will be found to satisfy the differential equation (2).



It will be noticed that for any point on the convex side of the parabola there are two real values of p; for a point on the other side the values of p are imaginary, and for a point on the curve they are equal. Thus its equation (3) expresses the relation between x and y which must exist in order that (2) regarded as a quadratic for p may have equal roots, as will be seen on solving that equation.

In general, writing the differential equation in the form

$$Lp^2 + Mp + N = 0, \tag{4}$$

the condition of equal roots is

$$M^2 - 4LN = 0. \tag{5}$$

The first member of this equation, which is the "discriminant" of equation (4), frequently admits of separation into factors rational in x and y. Hence, if there be a singular solution, its equation will be found by putting the discriminant of the differential equation, or one of its factors, equal to zero.

It does not follow that every such equation represents a solution of the differential equation. It can only be inferred that it is a locus of points for which the two values of p become equal. Now suppose that two distinct particular integral curves touch each other. At the point of contact, the two values of p, usually distinct, become equal. The locus of such points is called a "tac-locus." Its equation plainly satisfies the discriminant, but does not satisfy the differential equation. An illustration is afforded by the equation

$$y^2(p^2 + 1) = a^2,$$

of which the complete integral is $y^2 + (x - c)^2 = a^2$, and the discriminant, see equation (5), is $y^2(y^2 - a^2) = 0$.

This is satisfied by $y = a$, $y = -a$, and $y = 0$, the first two of which satisfy the differential equation, while $z = 0$ does not. The complete integral represents in this case all circles of radius a with center on the axis of x. Two of these circles touch at every point of the axis of x, which is thus a tac-locus, while $y = a$ and $y = -a$ constitute the envelope.

The discriminant is the quantity which appears under the radical sign when the general equation (4) is solved for p, and therefore it changes sign as we cross the envelope. But the values of p remain real as we cross the tac-locus, so that the discriminant cannot change sign. Accordingly the factor which indicates a tac-locus appears with an even exponent (as y^2 in the example above), whereas the factor indicating the singular solution appears as a simple factor, or with an odd exponent.

A simple factor of the discriminant, or one with an odd exponent, gives in fact always the boundary between a region of the plane in which p is real and one in which p is imaginary; nevertheless it may not give a singular solution. For the two arcs of particular integral curves which intersect in a point on the real side of the boundary may, as the point is brought up to the boundary, become tangent to each other, but not to the boundary curve. In that case, since they cannot cross the boundary, they become branches of the same particular integral forming a cusp. A boundary curve of this character is called a "cusp-locus"; the value of p for a point moving in it is of course different from the equal values of p at the cusp, and therefore its equation does not satisfy the differential equation.*

Prob. 30. To what curve is the line $y = mx + a\sqrt{(1 - m^2)}$ always tangent? (Ans. $y^2 - x^2 = a^2$.)

Prob. 31. Show that the discriminant of a decomposable differ-

* Since there is no reason why the values of p referred to should be identical, we conclude that the equation $Lp^2 + Mp + N = 0$ has not in general a singular solution, its discriminant representing a cusp-locus except when a certain condition is fulfilled.

ential equation cannot be negative. Interpret the result of equating it to zero in the illustrative example at the beginning of Art. 7.

Prob. 32. Show that the singular solutions of a homogeneous differential equation represent straight lines passing through the origin.

Prob. 33. Solve the equation $xp^2 - 2yp + ax = 0$.
(Ans. $x^2 - 2cy + ac^2 = 0$; singular solution $y^2 = ax^2$.)

Prob. 34. Show that the equation $p^2 + 2xp - y = 0$ has no singular solution, but has a cusp-locus, and that the tangent at every cusp passes through the origin.

ART. 9. SINGULAR SOLUTION FROM COMPLETE INTEGRAL.

When the complete integral of a differential equation of the second degree has been found in the standard form

$$Pc^2 + Qc + R = 0 \qquad (1)$$

(see the end of Art. 7), the substitution of special values of x and y in the functions P, Q, and R gives a quadratic for c whose roots determine the two particular curves of the system which pass through a given point. If there is a singular solution, that is, if the system of curves has an envelope, the two curves which usually intersect become identical when the given point is moved up to the envelope. Every point on the envelope therefore satisfies the condition of equal roots for equation (1), which is

$$Q^2 - 4PR = 0; \qquad (2)$$

and, reasoning exactly as in Art. 8, we infer that the equation of the singular solution will be found by equating to zero the discriminant of the equation in c or one of its factors. Thus the discriminant of equation (1), Art. 8, or "c-discriminant," is the same as the "p-discriminant," namely, $y^2 - 4ax$, which equated to zero is the equation of the envelope of the system of straight lines.

But, as in the case of the p-discriminant, it must not be inferred that every factor gives a singular solution. For example, suppose a squared factor appears in the c-discriminant. The locus on which this factor vanishes is not a curve in crossing which c and p become imaginary. At any point of it there will be two distinct values of p, corresponding to arcs of particular integral curves passing through that point; but, since there is but one value of c, these arcs belong to the same particular integral, hence the point is a double point or node. The locus is therefore called a "node-locus." The factor representing it does not appear in the p-discriminant, just as that representing a tac-locus does not appear in the c-discriminant.

Again, at any point of a cusp-locus, as shown at the end of Art. 8, the two branches of particular integrals become arcs of the same particular integral; the values of c become equal, so that a cusp-locus also makes the c-discriminant vanish.

The conclusions established above obviously apply also to equations of a degree higher than the second. In the case of the c-equation the general method of obtaining the condition for equal roots, which is to eliminate c between the original and the derived equation, is the same as the process of finding the envelope or "locus of the ultimate intersections" of a system of curves in which c is the arbitrary parameter.

Now suppose the system of curves to have for all values of c* a double point, it is obvious that among the intersections of two neighboring curves there are two in the neighborhood of the nodes, and that ultimately they coincide with the node, which accounts for the node-locus appearing twice in the discriminant or locus of ultimate intersections. In like manner,

* It is noticed in the second foot-note to Art. 7 that for an equation of the first degree p takes the indeterminate form, not only at a point through which all curves of the system pass (where the value of c would also be found indeterminate), but at a node of a particular integral. So also when the equation is of the nth degree, if there is a node for a particular value of c, the n values of c at the point (which is not on a node-locus where two values of c are equal) determine $n + 1$ arcs of particular integrals passing through the point; and therefore there are $n + 1$ distinct values of p at the point, which can only happen when p takes the indeterminate form, that is to say, when all the coefficients of the p-equation (which is of the nth degree) vanish. See Cayley on Singular Solutions in the Messenger of Mathematics, New Series, Vol. II, p. 10 (Collected Mathematical Works, Vol. VIII, p. 529). The present theory of Singular Solutions was established by Cayley in this paper and its continuation, Vol. VI, p. 23. See also a paper by Dr. Glaisher, Vol. XII, p. 1.

if there is a cusp for all values of c, there are three intersections of neighboring curves (all of which may be real) which ultimately coincide with the cusp; therefore a cusp-locus will appear as a cubed factor in the discriminant.*

Prob. 35. Show that the singular solutions of a homogeneous equation must be straight lines passing through the origin.

Prob. 36. Solve $3p^2y^2 - 2xyp + 4y^2 - x^2 = 0$, and show that there is a singular solution and a tac-locus.

Prob. 37. Solve $yp^2 + 2xp - y = 0$, and show that there is an imaginary singular solution. (Ans. $y^2 = 2cx + c^2$.)

Prob. 38. Show that the equation $(1 - x^2)p^2 = 1 - y^2$ represents a system of conics touching the four sides of a square.

Prob. 39. Solve $yp^2 - 4xp + y = 0$; examine and interpret both discriminants. (Ans. $c^2 + 2cx(3y^2 - 8x^2) - 3x^2y^4 + y^6 = 0$.)

ART. 10. SOLUTION BY DIFFERENTIATION.

The result of differentiating a given differential equation of the first order is an equation of the second order, that is, it contains the derivative $\dfrac{d^2y}{dx^2}$; but, if it does not contain y explicitly, it may be regarded as an equation of the first order for the variables x and p. If the integral of such an equation can be obtained it will be a relation between x, p, and a constant of integration c, by means of which p can be eliminated from the original equation, thus giving the relation between x, y, and c which constitutes the complete integral. For example, the equation

$$\frac{dy}{dx} + 2xy = x^2 + y^2, \tag{1}$$

* The discriminant of $Pc^2 + Qc + R = 0$ represents in general an envelope, no further condition requiring to be fulfilled as in the case of the discriminant of $Lp^2 + Mp + N = 0$. Compare the foot-note to Art. 8. Therefore where there is an integral of this form there is generally a singular solution, although $Lp^2 + Mp + N = 0$ has not in general a singular solution. We conclude, therefore, that this equation (in which L, M, and N are one-valued functions of x and y) has not in general an integral of the above form in which P, Q, and R are one-valued functions of x and y. Cayley, Messenger of Mathematics, New Series, Vol. VI, p. 23.

when solved for y, becomes

$$y = x + \sqrt{p}; \tag{2}$$

whence by differentiation

$$p = 1 + \frac{1}{2\sqrt{p}}\frac{dp}{dx}. \tag{3}$$

The variables can be separated in this equation, and its integral is

$$\sqrt{p} = \frac{C + e^{2x}}{C - e^{2x}}.$$

Substituting in equation (2), we find

$$y = x + \frac{C + e^{2x}}{C - e^{2x}},$$

which is the complete integral of equation (1).

This method sometimes succeeds with equations of a higher degree when the solution with respect to p is impossible or leads to a form which cannot be integrated. A differential equation between p and one of the two variables will be obtained by direct integration when only one of the variables is explicitly present in the equation, and also when the equation is of the first degree with respect to x and y. In the latter case after dividing by the coefficient of y, the result of differentiation will be a linear equation for x as a function of p, so that an expression for x in terms of p can be found, and then by substitution in the given equation an expression for y in terms of p. Hence, in this case, any number of simultaneous values of x and y can be found, although the elimination of p may be impracticable.

In particular, a homogeneous equation which cannot be solved for p may be soluble for the ratio $y : x$, so as to assume the form $y = x\phi(p)$. The result of differentiation is

$$p = \phi(p) + \phi'(p)\frac{dp}{dx};$$

in which the variables x and p can be separated.

Another special case is of the form

$$y = px + f(p), \tag{1}$$

which is known as Clairaut's equation. The result of differentiation is

$$p = p + x\frac{dp}{dx} + f'(p)\frac{dp}{dx},$$

which implies either

$$x + f'(p) = 0, \qquad \text{or} \qquad \frac{dp}{dx} = 0.$$

The elimination of p from equation (1) by means of the first of these equations * gives a solution containing no arbitrary constant, that is, a singular solution. The second is a differential equation for p; its integral is $p = c$, which in equation (1) gives the complete integral

$$y = cx + f(c). \tag{2}$$

This complete integral represents a system of straight lines, the singular solution representing the curve to which they are all tangent. An example has already been given in Art. 8.

A differential equation is sometimes reducible to Clairaut's form by means of a more or less obvious transformation of the variables. It may be noticed in particular that an equation of the form

$$y = nxp + \phi(x, p)$$

is sometimes so reducible by transformation to the independent variable z, where $x = z^n$; and an equation of the form

$$y = nxp + \phi(y, p),$$

by transformation to the new dependent variable $v = y^n$. A double transformation of the form indicated may succeed when the last term is a function of both x and y as well as of p.

Prob. 40. Solve the equation $3y = 2p^3 + 3p^2$; find a singular solution and a cusp-locus. (Ans. $(x + y + c - \frac{1}{3})^2 = \frac{4}{9}(x+c)^3$.)

Prob. 41. Solve $2y = xp + \frac{a}{p}$, and find a cusp-locus.

(Ans. $a^2c^2 - 12acxy + 8cy^3 - 12x^2y^2 + 16ax^3 = 0$.)

* The equation is in fact the same that arises in the general method for the condition of equal roots. See Art. 9.

Prob. 42. Solve $(x^2 - a^2)p^2 - 2xyp + y^2 - a^2 = 0$.
(Ans. The circle $x^2 + y^2 = 0$, and its tangents.)

Prob. 43. Solve $y = -xp + x^4p^2$.
(Ans. $c^2x + c - xy = 0$, and $1 + 4x^3y = 0$.)

Prob. 44. Solve $p^3 - 4xyp + 8y^2 = 0$.
(Ans. $y = c(x - c)^2$; $27y = 4x^3$ and $y = 0$ are singular solutions; $y = 0$ is also a particular integral.)

Prob. 45. Solve $x^2(y - px) = yp^2$. (Ans. $y^2 = cx^2 + c^2$.)

ART. 11. GEOMETRIC APPLICATIONS; TRAJECTORIES.

Every property of a curve which involves the direction of its tangents admits of statement in the form of a differential equation. The solution of such an equation therefore determines the curve having the given property. Thus, let it be required to determine the curve in which the angle between the radius vector and the tangent is n times the vectorial angle. Using the expression for the trigonometric tangent of that angle, the expression of the property in polar coordinates is

$$\frac{rd\theta}{dr} = \tan n\theta.$$

Separating the variables and integrating, the complete integral is

$$r^n = c^n \sin n\theta.$$

The mode in which the constant of integration enters here shows that the property in question is shared by all the members of a system of similar curves.

The solution of a question of this nature will thus in general be a system of curves, the complete integral of a differential equation, but it may be a singular solution. Thus, if we express the property that the sum of the intercepts on the axes made by the tangent to a curve is equal to the constant a, the straight lines making such intercepts will themselves constitute the complete integral system, and the curve required is the singular solution, which, in accordance with Art. 8, is the envelope of these lines. The result in this case will be found to be the parabola $\sqrt{x} + \sqrt{y} = \sqrt{a}$.

An important application is the determination of the "orthogonal trajectories" of a given system of curves, that is to say, the curves which cut at right angles every curve of the given system. The differential equation of the trajectory is readily derived from that of the given system; for at every point of the trajectory the value of p is the negative reciprocal of its value in the given differential equation. We have therefore only to substitute $-p^{-1}$ for p to obtain the differential equation of the trajectory. For example, let it be required to determine the orthogonal trajectories of the system of parabolas

$$y^2 = 4ax$$

having a common axis and vertex. The differential equation of the system found by eliminating a is

$$2\,xdy = y\,dx.$$

Putting $-\dfrac{dx}{dy}$ in place of $\dfrac{dy}{dx}$, the differential equation of the system of trajectories is

$$2x\,dx + y\,dy = 0,$$

whence, integrating,

$$2x^2 + y^2 = c^2.$$

The trajectories are therefore a system of similar ellipses with axes coinciding with the coordinate axes.

Prob. 46. Show that when the differential equation of a system is of the second degree, its discriminant and that of its trajectory system will be identical; but if it represents a singular solution in one system, it will constitute a cusp locus of the other.

Prob. 47. Determine the curve whose subtangent is constant and equal to a. (Ans. $ce^x = y^a$.)

Prob. 48. Show that the orthogonal trajectories of the curves $r^n = c^n \sin n\theta$ are the same system turned through the angle $\dfrac{\pi}{2n}$ about the pole. Examine the cases $n = 1$, $n = 2$, and $n = \frac{1}{2}$.

Prob. 49. Show that the orthogonal trajectories of a system of circles passing through two given points is another system of circles having a common radical axis.

Prob. 50. Determine the curve such that the area inclosed by any two ordinates, the curve and the axis of x, is equal to the product of the arc and the constant line a. Interpret the singular solution.

(Ans. The catenary $y = \frac{1}{2}a(e^{\frac{x}{a}} - e^{-\frac{x}{a}})$.)

Prob. 51. Show that a system of confocal conics is self-orthogonal.

ART. 12. SIMULTANEOUS DIFFERENTIAL EQUATIONS.

A system of n equations between $n + 1$ variables and their differentials is a "determinate" differential system, because it serves to determine the n ratios of the differentials; so that, taking any one of the variables as independent, the others vary in a determinate manner, and may be regarded as functions of the single independent variable. Denoting the variables by x, y, z, etc., the system may be written in the symmetrical form

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \ldots,$$

where X, Y, Z . . . may be any functions of the variables.

If any one of the several equations involving two differentials contains only the two corresponding variables, it is an ordinary differential equation; and its integral, giving a relation between these two variables, may enable us by elimination to obtain another equation containing two variables only, and so on until n integral equations have been obtained. Given, for example, the system

$$\frac{dx}{x} = \frac{dy}{z} = \frac{dz}{y}. \qquad (1)$$

The relation between dy and dz above contains the variables y and z only, and its integral is

$$y^2 - z^2 = a. \qquad (2)$$

Employing this to eliminate z from the relation between dx and dy it becomes

$$\frac{dx}{x} = \frac{dy}{\sqrt{(y^2 + a)}},$$

of which the integral is

$$y + \sqrt{(y^2 + a)} = bx. \tag{3}$$

The integral equations (2) and (3), involving two constants of integration, constitute the complete solution. It is in like manner obvious that the complete solution of a system of n equations should contain n arbitrary constants.

Confining ourselves now to the case of three variables, an extension of the geometrical interpretation given in Art. 2 presents itself. Let x, y, and z be rectangular coordinates of P referred to three planes. Then, if P starts from any given position A, the given system of equations, determining the ratios $dx : dy : dz$, determines the direction in space in which P moves. As P moves, the ratios of the differentials (as determined by the given equations) will vary, and if we suppose P to move in such a way as to continue to satisfy the differential equations, it will describe in general a curve of double curvature which will represent a particular solution. The complete solution is represented by the system of lines which may be thus obtained by varying the position of the initial point A. This system is a "doubly infinite" one; for the two relations between x, y, and z which define it analytically must contain two arbitrary parameters, by properly determining which we can make the line pass through any assumed initial point.*

Each of the relations between x, y and z, or integral equations, represents by itself a surface, the intersection of the two surfaces being a particular line of the doubly infinite system. An equation like (2) in the example above, which contains only one of the constants of integration, is called an *integral* of the differential system, in contradistinction to an "integral equa-

* It is assumed in the explanation that X, Y, and Z are one-valued functions of x, y, and z. There is then but one direction in which P can move when passing a given point, and the system is a non-intersecting system of lines. But if this is not the case, as for example when one of the equations giving the ratio of the differentials is of higher degree, the lines may form an intersecting system, and there would be a theory of singular solutions, into which we do not here enter.

tion" like (3), which contains both constants. An integral represents a surface which contains a singly infinite system of lines representing particular solutions selected from the doubly infinite system. Thus equation (2) above gives a surface on which lie all those lines for which a has a given value, while b may have any value whatever; in other words, a surface which passes through an infinite number of the particular solution lines.

The integral of the system which corresponds to the constant b might be found by eliminating a between equations (2) and (3). It might also be derived directly from equation (1); thus we may write

$$\frac{dx}{x} = \frac{dy}{z} = \frac{dz}{y} = \frac{dy + dz}{y + z} = \frac{du}{u},$$

in which a new variable $u = y + z$ is introduced. The relation between dx and du now contains but two variables, and its integral,

$$y + z = bx, \tag{4}$$

is the required integral of the system; and this, together with the integral (2), presents the solution of equations (1) in its standard form. The form of the two integrals shows that in this case the doubly infinite system of lines consists of hyperbolas, namely, the sections of the system of hyperbolic cylinders represented by (2) made by the system of planes represented by (4).

A system of equations of which the members possess a certain symmetry may sometimes be solved in the following manner. Since

$$\frac{dx}{X} = \frac{dy}{Y} = \frac{dz}{Z} = \frac{\lambda dx + \mu dy + \nu dz}{\lambda X + \mu Y + \nu Z},$$

if we take multipliers λ, μ, ν such that

$$\lambda X + \mu Y + \nu Z = 0,$$

we shall have $$\lambda dx + \mu dy + \nu dz = 0.$$

If the expression in the first member is an exact differential, direct integration gives an integral of the given system. For example, let the given equations be

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx};$$

l, m and n form such a set of multipliers, and so also do x, y and z. Hence we have

$$l dx + m dy + n dz = 0,$$

and also
$$x\,dx + y\,dy + z\,dz = 0.$$

Each of these is an exact equation, and their integrals

$$lx + my + nz = a$$

and
$$x^2 + y^2 + z^2 = b^2$$

constitute the complete solution. The doubly infinite system of lines consists in this case of circles which have a common axis, namely, the line passing through the origin and whose direction cosines are proportional to l, m, and n.

Prob. 52. Solve the equations $\frac{dx}{x^2 - y^2 - z^2} = \frac{dy}{2xy} = \frac{dz}{2xz}$, and interpret the result geometrically. (Ans. $y = az$, $x^2 + y^2 + z^2 = bz$.)

Prob. 53. Solve $\frac{dx}{y + z} = \frac{dy}{z + x} = \frac{dz}{x + y}$.

$$\left(\text{Ans. } \sqrt{(x + y + z)} = \frac{a}{z - y} = \frac{b}{x - z}.\right)$$

Prob. 54. Solve $\frac{dx}{(b - c)yz} = \frac{dy}{(c - a)zx} = \frac{dz}{(a - b)xy}$.

(Ans. $x^2 + y^2 + z^2 = A$, $ax^2 + by^2 + cz^2 = B$.)

ART. 13. EQUATIONS OF THE SECOND ORDER.

A relation between two variables and the successive derivatives of one of them with respect to the other as independent variable is called a differential equation of the order indicated by the highest derivative that occurs. For example,

$$(1 + x^2)\frac{d^2y}{dx^2} + x\frac{dy}{dx} + mx = 0$$

is an equation of the second order, in which x is the independent variable. Denoting as heretofore the first derivative by p, this equation may be written

$$(1 + x^2)\frac{dp}{dx} + xp + mx = 0, \tag{1}$$

and this, in connection with

$$\frac{dy}{dx} = p, \tag{2}$$

which defines p, forms a pair of equations of the first order, connecting the variables x, y, and p. Thus any equation of the second order is equivalent to a pair of simultaneous equations of the first order.

When, as in this example, the given equation does not contain y explicitly, the first of the pair of equations involves only the two variables x and p; and it is further to be noticed that, when the derivatives occur only in the first degree, it is a linear equation for p. Integrating equation (1) as such, we find

$$p = -m + \frac{c_1}{\sqrt{(1 + x^2)}}; \tag{3}$$

and then using this value of p in equation (2), its integral is

$$y = c_2 - mx + c_1 \log [x + \sqrt{(1 + x^2)}], \tag{4}$$

in which, as in every case of two simultaneous equations of the first order, we have introduced two constants of integration.

An equation of the first order is readily obtained also when the independent variable is not explicitly contained in the equation. The general equation of rectilinear motion in dynamics affords an illustration. This equation is $\frac{d^2s}{dt^2} = f(s)$, where s denotes the distance measured from a fixed center of force upon the line of motion. It may be written $\frac{dv}{dt} = f(s)$, in connection with $\frac{ds}{dt} = v$, which defines the velocity. Eliminating dt from these equations, we have $vdv = f(s)ds$, whose integral is $\frac{1}{2}v^2 = \int f(s)ds + c$, the "equation of energy" for the unit mass. The substitution of the value found for v in the second equation gives an equation from which t is found in terms of s by direct integration.

The result of the first integration, such as equation (3) above, is called a "first integral" of the given equation of the second order; it contains one constant of integration, and its complete integral, which contains a second constant, is also the "complete integral" of the given equation.

A differential equation of the second order is "exact" when, all its terms being transposed to the first member, that member is the derivative with respect to x of an expression of the first order, that is, a function of x, y and p. It is obvious that the terms containing the second derivative, in such an exact differential, arise solely from the differentiation of the terms containing p in the function of x, y and p. For example, let it be required to ascertain whether

$$(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + y = 0 \tag{5}$$

is an exact equation. The terms in question are $(1 - x^2)\frac{dp}{dx}$, which can arise only from the differentiation of $(1 - x^2)p$. Now subtract from the given expression the complete derivative of $(1 - x^2)p$, which is

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx};$$

the remainder is $x\frac{dy}{dx} + y$, which is an exact derivative, namely, that of xy. Hence the given expression is an exact differential, and

$$(1 - x^2)\frac{dy}{dx} + xy = c_1 \tag{6}$$

is the first integral of the given equation. Solving this linear equation for y, we find the complete integral

$$y = c_1x + c_2\sqrt{(1 - x^2)}. \tag{7}$$

Prob. 55. Solve $(1 - x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2$.

(Ans. $y = (\sin^{-1} x)^2 + c_1 \sin^{-1} x + c_2$.)

Prob. 56. Solve $\frac{d^2y}{dx^2} = \frac{2y}{x^2}$. $\left(\text{Ans. } y = \frac{c_1}{x} + c_2x^2.\right)$

Prob. 57. Solve $\frac{d^2y}{dx^2} = a^2x - b^2y$.

(Ans. $a^2x - b^2y = A \sin bx + B \cos bx$.)

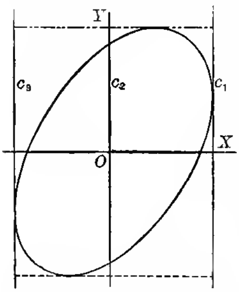
Prob. 58. Solve $y\frac{d^2y}{dx^2} + \left(\frac{dy}{dx}\right)^2 = 1$. (Ans. $y^2 = x^2 + c_1x + c_2$.)

ART. 14. THE TWO FIRST INTEGRALS.

We have seen in the preceding article that the complete integral of an equation of the second order is a relation between x, y and two constants c_1 and c_2. Conversely, any relation between x, y and two arbitrary constants may be regarded as a primitive, from which a differential equation free from both arbitrary constants can be obtained. The process consists in first obtaining, as in Art. 3, a differential equation of the first order independent of one of the constants, say c_2, that is, a relation between x, y, p and c_1, and then in like manner eliminating c_1 from the derivative of this equation. The result is the equation of the second order or relation between x, y, p and q (q denoting the second derivative), of which the original equation is the complete primitive, the equation of the first order being the first integral in which c_1 is the constant of integration. It is obvious that we can, in like manner, obtain from the primitive a relation between x, y, p and c_2, which will also be a first integral of the differential equation. Thus, to a given form of the primitive or complete integral there corresponds two first integrals.

Geometrically the complete integral represents a doubly infinite system of curves, obtained by varying the values of c_1 and of c_2 independently. If we regard c_1 as fixed and c_2 as arbitrary, we select from that system a certain singly infinite system; the first integral containing c_1 is the differential equation of this system, which, as explained in Art. 2, is a relation between the coordinates of a moving point and the direction of its motion common to all the curves of the system. But the equation of the second order expresses a property involving curvature as well as direction of path, and this property being independent of c_1 is common to all the systems corresponding to different values of c_1, that is, to the entire doubly infinite system. A moving point, satisfying this equation, may have any position and move in any direction, provided its path has the proper curvature as determined by the value of q derived from the equation, when the selected values of x, y and p have been substituted therein.*

For example, equation (7) of the preceding article represents an ellipse having its center at the origin and touching the lines $x = \pm 1$, as in the diagram; c_1 is the ordinate of the point of contact with $x = 1$, and c_2 that of the point in which the ellipse cuts the axis of y. If we regard c_1 as fixed and c_2 as arbitrary, the equation represents the system of ellipses touching the two lines at fixed points, and equation (6) is the differential equation of this system. In like manner, if c_2 is fixed and c_1 arbitrary, equation (7) represents a system of ellipses cutting the axis of y in fixed points and touching the lines $x = \pm 1$. The corresponding differential equation will be found to be



$$(y - xp)\sqrt{(1 - x^2)} = c_2.$$

Finally, the equation of the second order, independent of c_1 and c_2 [(5) of the preceding article] is the equation of the doubly infinite system of conics † with center at the origin, and touching the fixed lines $x = \pm 1$.

* If the equation is of the second or higher degree in q, the condition for equal roots is a relation between x, y and p, which may be found to satisfy the given equation. If it does, it represents a system of singular solutions; each of the curves of this system, at each of its points, not only touches but osculates with a particular integral curve. It is to be remembered that a singular solution of a first integral is not generally a solution of the given differential equation; for it represents a curve which simply touches but does not osculate a set of curves belonging to the doubly infinite system.

† Including hyperbolas corresponding to imaginary values of c_2.

But, starting from the differential equation of second order, we may find other first integrals than those above which correspond to c_1 and c_2. For instance, if equation (5) be multiplied by p, it becomes

$$(1 - x^2)p\frac{dp}{dx} - xp^2 + yp = 0,$$

which is also an exact equation, giving the first integral

$$(1 - x^2)p^2 + y^2 = c_3^2,$$

in which c_3 is a new constant of integration.

Whenever two first integrals have thus been found independently, the elimination of p between them gives the complete integral without further integration.* Thus the result of eliminating p between this last equation and the first integral containing c_1 [equation (6), Art. 13] is

$$y^2 - 2c_1xy + c_3^2x^2 = c_3^2 - c_1^2,$$

which is therefore another form of the complete integral. It is obvious from the first integral above that c_3 is the maximum value of y, so that it is the differential equation of the system of ellipse inscribed in the rectangle drawn in the diagram. A comparison of the two forms of the complete integral shows that the relation between the constants is $c_3^2 = c_1^2 + c_2^2$.

If a first integral be solved for the constant, that is, put in the form $\phi(x, y, p) = c$, the constant will disappear on differentiation, and the result will be the given equation of second order multiplied, in general, by an integrating factor. We can thus find any number of integrating factors of an equation already solved, and these may suggest the integrating factors of more general equations, as illustrated in Prob. 59 below.

* The principle of this method has already been applied in Art. 10 to the solution of certain equations of the first order; the process consisted of forming the equation of the second order of which the given equation is a first integral (but with a particular value of the constant), then finding another first integral and deriving the complete integral by elimination of p.

Prob. 59. Solve the equation $\frac{d^2y}{dx^2} + a^2y = 0$ in the form

$$y = A \cos ax + B \sin ax;$$

and show that the corresponding integrating factors are also integrating factors of the equation

$$\frac{d^2y}{dx^2} + a^2y = X,$$

where X is any function of x; and thence derive the integral of this equation.

$$(\text{Ans. } ay = \sin ax \int \cos ax \,.\, Xdx - \cos ax \int \sin ax \,.\, Xdx).$$

Prob. 60. Find the rectangular and also the polar differential equation of all circles passing through the origin.

$$\left(\text{Ans. } (x^2 + y^2)\frac{d^2y}{dx^2} = 2\left[1 + \left(\frac{dy}{dx}\right)^2\right]\left(x\frac{dy}{dx} - y\right), \quad \text{and} \quad r + \frac{d^2r}{d\theta^2} = 0.\right)$$

ART. 15. LINEAR EQUATIONS.

A linear differential equation of any order is an equation of the first degree with respect to the dependent variable y and each of its derivatives, that is, an equation of the form

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_ny = X, \tag{1}$$

where the coefficients $P_0, \ldots P_n$ and the second member X are functions of the independent variable only.

The solution of a linear equation is always supposed to be in the form $y = f(x)$; and if y_1 is a function which satisfies the equation, it is customary to speak of the function y_1, rather than of the equation $y = y_1$, as an "integral" of the linear equation. The general solution of the linear equation of the first order has been given in Art. 6. For orders higher than the first the general expression for the integrals cannot be effected by means of the ordinary functional symbols and the integral sign, as was done for the first order in Art. 6.

The solution of equation (1) depends upon that of

$$P_0\frac{d^ny}{dx^n} + P_1\frac{d^{n-1}y}{dx^{n-1}} + \ldots + P_ny = 0. \tag{2}$$

The complete integral of this equation will contain n arbitrary constants, and the mode in which these enter the expression for y is readily inferred from the form of the equation. For let y_1 be an integral, and c_1 an arbitrary constant; the result of putting $y = c_1y_1$ in equation (2) is c_1 times the result of putting $y = y_1$; that is, it is zero; therefore c_1y_1 is an integral. So too, if y_2 is an integral, c_2y_2 is an integral; and obviously also $c_1y_1 + c_2y_2$ is an integral. Thus, if n distinct integrals y_1, $y_2, \ldots y_n$ can be found,

$$y = c_1y_1 + c_2y_2 + \ldots + c_ny_n \tag{3}$$

will satisfy the equation, and, containing, as it does, the proper number of constants, will be the complete integral.

Consider now equation (1); let Y be a particular integral of it, and denote by u the second member of equation (3), which is the complete integral when $X = 0$. If

$$y = Y + u \tag{4}$$

be substituted in equation (1), the result will be the sum of the results of putting $y = Y$ and of putting $y = u$; the first of these results will be X, because Y is an integral of equation (1), and the second will be zero because u is an integral of equation (2). Hence equation (4) expresses an integral of (2); and since it contains the n arbitrary constants of equation (3), it is the complete integral of equation (1). With reference to this equation Y is called "the particular integral," and u is called "the complementary function." The particular integral contains no arbitrary constant, and any two particular integrals may differ by any multiple of a term belonging to the complementary function.

If one term of the complementary function of a linear equation of the second order be known, the complete solution can be found. For let y_1 be the known term; then, if $y = y_1v$ be substituted in the first member, the coefficient of v in the result will be the same as if v were a constant: it will therefore be zero, and v being absent, the result will be a linear equation of the first order for v', the first derivative of v. Under the same circumstances the order of any linear equation can in like manner be reduced by unity.

A very simple relation exists between the coefficients of an exact linear equation. Taking, for example, the equation of the second order, and indicating derivatives by accents, if

$$P_0 y'' + P_1 y' + P_2 y = X$$

is exact, the first term of the integral will be $P_0 y'$ Subtracting the derivative of this from the first member, the remainder is $(P_1 - P_0')y' + P_2 y$. The second term of the integral must therefore be $(P_1 - P_0')y$; subtracting the derivative of this expression, the remainder, $(P_2 - P_1' + P_0'')y$, must vanish. Hence $P_2 - P_1' + P_0'' = 0$ is the criterion for the exactness of the given equation. A similar result obviously extends to equations of higher orders.

Prob. 61. Solve $x\frac{d^2y}{dx} - (3 + x)\frac{dy}{dx} + 3y = 0$, noticing that e^x is an integral. (Ans. $y = c_1 e^x + c_2(x^3 + 3x^2 + 6x + 6$.)

Prob. 62. Solve $(x^2 - x)\frac{d^2y}{dx^2} + 2(2x + 1)\frac{dy}{dx} + 2y = 0$.

(Ans. $(x - 1)^5 y = c_1(x^4 - 6x^2 + 2x - \frac{1}{3} - 4x^3 \log x) + c_2 x^3$.)

Prob. 63. Solve $\frac{d^3y}{d\theta^3} + \cos\theta\frac{d^2y}{d\theta^2} - 2\sin\theta\frac{dy}{d\theta} - y\cos\theta = \sin 2\theta$.

$\left(\text{Ans. } y = e^{-\sin\theta}\int e^{\sin\theta}(c_1\theta + c_2)d\theta + c_3 e^{-\sin\theta} - \frac{\sin\theta - 1}{2}.\right)$

ART. 16. LINEAR EQUATIONS WITH CONSTANT COEFFICIENTS.

The linear equation with constant coefficients and second member zero may be written in the form

$$A_0 D^n y + A_1 D^{n-1} y + \ldots + A_n y = \tag{1}$$

in which D stands for the operator $\frac{d}{dx}$, D^2 for $\frac{d^2}{dx^2}$, etc., so that D^n indicates that the operator is to be applied n times. Then, since $De^{mx} = me^{mx}$, $D^2 e^{mx} = m^2 e^{mx}$, etc., it is evident that if

$y = e^{mx}$ be substituted in equation (1), the result after rejecting the factor e^{mx} will be

$$A_0 m^n + A_1 m^{n-1} + \dots + A_n = 0. \qquad (2)$$

Hence, if m satisfies equation (2), e^{mx} is an integral of equation (1); and if $m_1, m_2, \dots m_n$ are n distinct roots of equation (2), the complete integral of equation (1) will be

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x} + \dots + c_n e^{m_n x}. \qquad (3)$$

For example, if the given equation is

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = 0,$$

the equation to determine m is

$$m^2 - m - 2 = 0,$$

of which the roots are $m_1 = 2$, $m_2 = -1$; therefore the integral is

$$y = c_1 e^{2x} + c_2 e^{-x}.$$

The general equation (1) may be written in the symbolic form $f(D) \cdot y = 0$, in which f denotes a rational integral function. Then equation (2) is $f(m) = 0$, and, just as this last equation is equivalent to

$$(m - m_1)(m - m_2) \dots (m - m_n) = 0, \qquad (4)$$

so the symbolic equation $f(D) \cdot y = 0$ may be written

$$(D - m_1)(D - m_2) \dots (D - m_n)y = 0. \qquad (5)$$

This form of the equation shows that it is satisfied by each of the quantities which satisfy the separate equations

$$(D - m_1)y = 0, \quad (D - m_2)y = 0 \dots (D - m_n)y = 0; \qquad (6)$$

that is to say, by the separate terms of the complete integral.

If two of the roots of equation (2) are equal, say to m_1, two of the equations (6) become identical, and to obtain the full number of integrals we must find two terms corresponding to the equation

$$(D - m_1)^2 y = 0; \qquad (7)$$

in other words, the complete integral of this equation of which $y_1 = e^{m_1 x}$ is known to be one integral. For this purpose we put, as explained in the preceding article, $y = y_1 v$. By differentiation, $Dy = De^{m_1 x}v = e^{m_1 x}(m_1 v + Dv)$; therefore

$$(D - m_1)e^{m_1 x}v = e^{m_1 x}Dv. \tag{8}$$

In like manner we find

$$(D - m_1)^2 e^{m_1 x}v = e^{m_1 x}D^2 v. \tag{9}$$

Thus equation (7) is transformed to $D^2 v = 0$, of which the complete integral is $v = c_1 x + c_2$; hence that of equation (7) is

$$y = e^{m_1 x}(c_1 x + c_2). \tag{10}$$

These are therefore the two terms corresponding to the squared factor $(D - m_1)^2$ in $f(D)y = 0$.

It is evident that, in a similar manner, the three terms corresponding to a case of three equal roots can be shown to be $e^{m_1 x}(c_1 x^2 + c_2 x + c_3)$, and so on.

The pair of terms corresponding to a pair of imaginary roots, say $m_1 = \alpha + i\beta$, $m_2 = \alpha - i\beta$, take the imaginary form

$$c_1 e^{(\alpha + i\beta)x} + c_2 e^{(\alpha - i\beta)x} = e^{\alpha x}(c_1 e^{i\beta x} + c_2 e^{-i\beta x}).$$

Separating the real and imaginary parts of $e^{i\beta x}$ and $e^{-i\beta x}$, and changing the constants, the expression becomes

$$e^{\alpha x}(A \cos \beta x + B \sin \beta x). \tag{11}$$

For a multiple pair of imaginary roots the constants A and B must be replaced by polynomials as above shown in the case of real roots.

When the second member of the equation with constant coefficients is a function of X, the particular integral can also be made to depend upon the solution of linear equations of the first order. In accordance with the symbolic notation introduced above, the solution of the equation

$$\frac{dy}{dx} - ay = X, \quad \text{or} \quad (D - a)y = X \tag{12}$$

is denoted by $y = (D - a)^{-1}X$, so that, solving equation (12), we have

$$\frac{1}{D - a}X = e^{ax}\int e^{-ax}X\,dx \tag{13}$$

as the value of the inverse symbol whose meaning is "that function of x which is converted to X by the direct operation expressed by the symbol $D - a$." Taking the most convenient special value of the indefinite integral in equation (13), it gives the particular integral of equation (12). In like manner, the particular integral of $f(D)y = X$ is denoted by the inverse symbol $\frac{1}{f(D)}X$. Now, with the notation employed above, the symbolic fraction may be decomposed into partial fractions with constant numerators thus:

$$\frac{1}{f(D)}X = \frac{N_1}{D - m_1}X + \frac{N_2}{D - m_2}X + \ldots + \frac{N_n}{D - m_n}X,* \quad (14)$$

in which each term is to be evaluated by equation (13), and may be regarded (by virtue of the constant involved in the indefinite integral) as containing one term of the complementary function. For example, the complete solution of the equation

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} - 2y = X$$

is thus found to be

$$y = \tfrac{1}{3}e^{2x}\int e^{-2x}X\,dx - \tfrac{1}{3}e^{-x}\int e^{x}X\,dx.$$

When X is a power of x the particular integral may be found as follows, more expeditiously than by the evaluation of the integrals in the general solution. For example, if $X = x^2$ the particular integral in this example may be evaluated by development of the inverse symbol, thus:

$$\begin{aligned} y = \frac{1}{D^2 - D - 2}x^2 &= -\frac{1}{2}\,\frac{1}{1 + \frac{1}{2}(D - D^2)}x^2 \\ &= -\tfrac{1}{2}[1 - \tfrac{1}{2}(D - D^2) + \tfrac{1}{4}(D - D^2)^2 - \ldots]x^2 \\ &= -\tfrac{1}{2}[1 - \tfrac{1}{2}D + \tfrac{3}{4}D^2 - \ldots]x^2 = -\tfrac{1}{2}x^2 + \tfrac{1}{2}x - \tfrac{3}{4}. \end{aligned}$$

* The validity of this equation depends upon the fact that the operations expressed in the second member of

$$f(D) = (D - m_1)(D - m_2) + \ldots + (D - m_n)$$

are commutative, hence the process of verification is the same as if the equation were an algebraic identity. This general solution was published by Boole in the Cambridge Math. Journal, First Series, vol. II, p. 114. It had, however, been previously published by Lobatto, Théorie des Charactéristiques, Amsterdam, 1837.

The form of the operand shows that, in this case, it is only necessary to carry the development as far as the term containing D^2.

For other symbolic methods applicable to special forms of X we must refer to the standard treatises on this subject.

Prob. 64. Solve $4\frac{d^3y}{dx^3} - 3\frac{dy}{dx} + y = 0$.

(Ans. $y = e^{\frac{1}{2}x}(Ax + B) + ce^{-x}$.)

Prob. 65. Show that $\frac{1}{f(D)}e^{ax} = \frac{1}{f(a)}e^{ax}$

and that $\frac{1}{f(D^2)}\sin(ax + \beta) = \frac{1}{f(-a^2)}\sin(ax + \beta)$.

Prob. 66. Solve $(D^2 + 1)y = e^x + \sin 2x + \sin x$. (Compare Prob. 59, Art. 14.)

(Ans. $y = A \sin x + B \cos x + \frac{1}{2}e^x - \frac{1}{3}\sin 2x - \frac{1}{2}x \cos x$.)

ART. 17. HOMOGENEOUS LINEAR EQUATIONS.

The linear differential equation

$$A_0 x^n \frac{d^n y}{dx^n} + A_1 x^{n-1}\frac{d^{n-1}y}{dx^{n-1}} + \ldots + A^n y = 0, \qquad (1)$$

in which A_0, A_1, etc., are constants, is called the "homogeneous linear equation." It bears the same relation to x^m that the equation with constant coefficients does to e^{mx}. Thus, if $y = x^m$ be substituted in this equation, the factor x^m will divide out from the result, giving an equation for determining m, and the n roots of this equation will in general determine the n terms of the complete integral. For example, if in the equation

$$x^2\frac{d^2y}{dx^2} + 2x\frac{dy}{dx} - 2y = 0$$

we put $y = x^m$, the result is $m(m-1) + 2m - 2 = 0$, or $(m-1)(m+2) = 0$.

The roots of this equation are $m_1 = 1$ and $m_2 = -2$. Hence
$$y = c_1 x + c_2 x^{-2}$$
is the complete integral.

Equation (1) might in fact have been reduced to the form with constant coefficients by changing the independent variable to θ, where $x = e^{\theta}$, or $\theta = \log x$. We may therefore at once infer from the results established in the preceding article that the terms corresponding to a pair of equal roots are of the form

$$(c_1 + c_2 \log x)x^m, \tag{2}$$

and also that the terms corresponding to a pair of imaginary roots, $\alpha \pm i\beta$, are

$$x^{\alpha}[A \cos (\beta \log x) + B \sin (\beta \log x)]. \tag{3}$$

The analogy between the two classes of linear equations considered in this and the preceding article is more clearly seen when a single symbol $\vartheta = xD$ is used for the operation of taking the derivative and then multiplying by x, so that $\vartheta x^m = mx^m$. It is to be noticed that the operation x^2D^2 is not the same as ϑ^2 or $xDxD$, because the operations of taking the derivative and multiplying by a variable are not "commutative," that is, their order is not indifferent. We have, on the contrary, $x^2D^2 = \vartheta(\vartheta - 1)$; then the equation given above, which is

$$(x^2D^2 + 2xD - 2)y = 0,$$

becomes

$$[\vartheta(\vartheta - 1) + 2\vartheta - 2]y = 0, \quad \text{or} \quad (\vartheta - 1)(\vartheta + 2)y = 0,$$

the function of ϑ produced being the same as the function of m which is equated to 0 in finding the values of m.

A linear equation of which the first member is homogeneous and the second member a function of x may be reduced to the form

$$f(\vartheta) \cdot y = X. \tag{4}$$

The particular integral may, as in the preceding article (see eq. (14)), be separated into parts each of which depends upon the solution of a linear equation of the first order. Thus, solving the equation

$$x\frac{dy}{dx} - ay = X, \quad \text{or} \quad (\vartheta - a)y = X, \tag{5}$$

we find

$$\frac{1}{\vartheta - a}X = x^a \int x^{-a-1} X dx. \tag{6}$$

The more expeditious method which may be employed when X is a power of x is illustrated in the following example: Given $x^2\dfrac{d^3y}{dx^3} - 2\dfrac{dy}{dx} = x^2$. The first member becomes homogeneous when multiplied by x, and the reduced equation is $(\vartheta^3 - 3\vartheta^2)y = x^3$.

The roots of $f(\vartheta) = 0$ are 3 and the double root zero, hence the complementary function is $c_1x^3 + c_2 + c_3 \log x$. Since in general $f(\vartheta)x^r = f(r)x^r$, we infer that in operating upon x^3 we may put $\vartheta = 3$. This gives for the particular integral

$$\frac{1}{\vartheta - 3}\frac{1}{\vartheta^2}x^3 = \frac{1}{9}\frac{1}{\vartheta - 3}x^3,$$

but fails with respect to the factor $\vartheta - 3$.* We therefore now fall back upon equation (6), which gives

$$\frac{1}{\vartheta - 3}x^3 = x^3\int x^{-1}dx = x^3 \log x.$$

The complete integral therefore is

$$y = c_1x^3 + c_2 + c_3 \log x + \tfrac{1}{9}x^3 \log x.$$

Prob. 67. Solve $2x^2\dfrac{d^2y}{dx^2} + 3x\dfrac{dy}{dx} - 3y = x^2$.

(Ans. $y = c_1x + c_2x^{-\frac{3}{2}} + \frac{1}{7}x^2$.)

Prob. 68. Solve $(x^2D^3 + 3xD^2 + D)y = \dfrac{1}{x}$.

(Ans. $y = c_1 + c_2 \log x + c_3(\log x)^2 + \frac{1}{6}(\log x)^3$.)

ART. 18. SOLUTIONS IN INFINITE SERIES.

We proceed in this article to illustrate the method by which the integrals of a linear equation whose coefficients are algebraic functions of x may be developed in series whose terms are powers of x. For this purpose let us take the equation

$$x^2\frac{d^2y}{dx^2} + x\frac{dy}{dx} + (x^4 - n^2)y = 0, \tag{1}$$

* The failure occurs because x^3 is a term of the complementary function having an indeterminate coefficient; accordingly the new term is of the same form as the second term necessary when 3 is a double root, but of course with a determinate coefficient.

which is known as "Bessel's Equation," and serves to define the "Besselian Functions."

If in the first member of this equation we substitute for y the single term Ax^m the result is

$$A(m^2 - n^2)x^m + Ax^{m+2}, \qquad (2)$$

the first term coming from the homogeneous terms of the equation and the second from the term x^2y which is of higher degree. If this last term did not exist the equation would be satisfied by the assumed value of y, if m were determined so as to make the first term vanish, that is, in this case, by Ax^n or Bx^{-n}. Now these are the first terms of two series each of which satisfies the equation. For, if we add to the value of y a term containing x^{m+2}, thus $y = A_0x^m + A_1x^{m+2}$, the new term will give rise, in the result of substitution, to terms containing x^{m+2} and x^{m+4} respectively, and it will be possible so to take A_1 that the entire coefficient of x^{m+2} shall vanish. In like manner the proper determination of a third term makes the coefficient of x^{m+4} in the result of substitution vanish, and so on. We therefore at once assume

$$y = \sum_0^\infty A_r x^{m+2r} = A_0x^m + A_1 x^{m+2} + A_2x^{m+4} + \ldots, \qquad (3)$$

in which r has all integral values from 0 to ∞. Substituting in equation (1)

$$\sum_0^\infty[\{(m + 2r)^2 - n^2\}A_rx^{m+2r} + A_rx^{m+2(r+1)}] = 0. \qquad (4)$$

The coefficient of each power of x in this equation must separately vanish; hence, taking the coefficient of x^{m+2r}, we have

$$[(m + 2r)^2 - n^2]A_r + A_{r-1} = 0. \qquad (5)$$

When $r = 0$, this reduces to $m^2 - n^2 = 0$, which determines the values of m, and for other values of r it gives

$$A_r = -\frac{1}{(m + 2r + n)(m + 2r - n)}A_{r-1}, \qquad (6)$$

the relation between any two successive coefficients.

For the first value of m, namely n, this relation becomes

$$A_r = -\frac{1}{2^2(n + r)r} A_{r-1};$$

whence, determining the successive coefficients in equation (3), the first integral of the equation is

$$A_0 y_1 = A_0 x^n \left[1 - \frac{1}{n+1}\frac{x^2}{2^2} + \frac{1}{(n+1)(n+2)}\frac{x^4}{2^4 . 2!} - \dots \right]. \quad (7)$$

In like manner, the other integral is found to be

$$B_0 y_2 = B_0 x^{-n} \left[1 + \frac{1}{n-1}\frac{x^2}{2^2} + \frac{1}{(n-1)(n-2)}\frac{x^4}{2^4 . 2!} + \dots \right], \quad (8)$$

and the complete integral is $y = A_0 y_1 + B_0 y_2$.*

This example illustrates a special case which may arise in this form of solution. If n is a positive integer, the second series will contain infinite coefficients. For example, if $n = 2$, the third coefficient, or B_2, is infinite, unless we take $B_0 = 0$, in which case B_2 is indeterminate and we have a repetition of the solution y_1. This will always occur when the same powers of x occur in the two series, including, of course, the case in which m has equal roots. For the mode of obtaining a new integral in such cases the complete treatises must be referred to.†

It will be noticed that the simplicity of the relation between consecutive coefficients in this example is due to the fact that equation (1) contained but two groups of terms producing different powers of x, when Ax^m is substituted for y as in expression (2). The group containing the second derivative necessarily gives rise to a coefficient of the second degree in m, and from it we obtained two values of m. Moreover, because the other group was of a degree higher by two units, the assumed series was an ascending one, proceeding by powers of x^2.

* The Besselian function of the nth order usually denoted by J_n is the value of y_1 above, divided by $2^n n!$ if n is a positive integer, or generally by $2^n \Gamma(n+1)$. For a complete discussion of these functions see Lommel's Studien über die Bessel'schen Functionen, Leipzig, 1868; Todhunter's Treatise on Laplace's, Lamé's and Bessel's Functions, London, 1875, etc.

† A solution of the kind referred to contains as one term the product of the regular solution and log x, and is sometimes called a "logarithmic solution." See also American Journal of Mathematics, Vol. XI, p. 37. In the case of Bessel's equation, the logarithmic solution is the "Besselian Function of the second kind."

In the following example,

$$\frac{d^2y}{dx^2} + a\frac{dy}{dx} - 2\frac{y}{x^2} = 0, \tag{9}$$

there are also two such groups of terms, and their difference of degree shows that the series must ascend by simple powers. We assume therefore at once

$$y = \sum_0^\infty A_r x^{m+r}. \tag{10}$$

The result of substitution is

$$\sum_0^\infty [\{(m+r)(m+r-1)-2\}A_r x^{m+r-2} + a(m+r)A_r x^{m+r-1}] = 0. \tag{11}$$

Equating to zero the coefficient of x^{m+r-2},

$$(m+r+1)(m+r-2)A_r + a(m+r-1)A^{r-1} = 0, \tag{12}$$

which, when $r = 0$, gives

$$(m+1)(m-2)A_0 = 0, \tag{13}$$

and when $r > 0$,

$$A_r = -a\frac{m+r-1}{(m+r+1)(m+r-2)}A_{r-1}. \tag{14}$$

The roots of equation (13) are $m = 2$ and $m = -1$; taking $m = 2$, the relation (14) becomes

$$A_r = -a\frac{r+1}{(r+3)r}A_{r-1},$$

whence the first integral is

$$A_0 y_1 = A_0 x^2\left[1 - \frac{2}{4}ax + \frac{3}{4 \cdot 5}a^2x^2 - \frac{4}{4 \cdot 5 \cdot 6}a^3x^3 + \dots\right]. \tag{15}$$

Taking the second value $m = -1$, equation (14) gives

$$B_r = -a\frac{r-2}{r(r-3)}B_{r-1},$$

whence $B_1 = -\frac{a}{2}B_0$, and $B_2 = 0$*; therefore the second integral is the finite expression

$$B_0 y_2 = B_0 x^{-1}\left[1 - \frac{1}{2}ax\right] = B_0\left[\frac{1}{x} - \frac{a}{2}\right]. \tag{16}$$

* B_3 would take the indeterminate form, and if we suppose it to have a finite value, the rest of the series is equivalent to $B_3 y_1$, reproducing the first integral.

When the coefficient of the term of highest degree in the result of substitution, such as equation (11), contains m, it is possible to obtain a solution in descending powers of x. In this case, m occurring only in the first degree, but one such solution can be found; it would be identical with the finite integral (16). In the general case there will be two such solutions, and they will be convergent for values of x greater than unity, while the ascending series will converge for values less than unity.*

When the second member of the equation is a power of x, the particular integral can be determined in the form of a series in a similar manner. For example, suppose the second member of equation (9) to have been $x^{\frac{1}{2}}$. Then, making the substitution as before, we have the same relation between consecutive coefficients; but when $r = 0$, instead of equation (13) we have

$$(m + 1)(m - 2)A_0 x^{m-2} = x^{\frac{1}{2}}$$

to determine the initial term of the series. This gives $m = 2\frac{1}{2}$ and $A_0 = \frac{4}{7}$; hence, putting $m = \frac{5}{2}$ in equation (14), we find for the particular integral †

$$y = \frac{4}{7}x^{\frac{5}{2}}\left[1 - \frac{2 \cdot 5}{9 \cdot 3}ax + \frac{2^2 \cdot 5 \cdot 7}{9 \cdot 11 \cdot 3 \cdot 5}a^2x^2 - \ldots\right].$$

A linear equation remains linear for two important classes of transformations; first, when the independent variable is changed to any function of x, and second, when for y we put $vf(x)$. As an example of the latter, let $y = e^{-ax}v$ be substituted in equation (9) above. After rejecting the factor e^{-ax}, the result is

$$\frac{d^2v}{dx^2} - a\frac{dv}{dx} - \frac{2v}{x^2} = 0.$$

Since this differs from the given equation only in the sign

* When there are two groups of terms, the integrals are expressible in terms of Gauss's "Hypergeometric Series."

† If the second member is a term of the complementary function (for example, in this case, if it is any integral power of x), the particular integral will take the logarithmic form referred to in the foot-note on p. 346.

of a, we infer from equation (16) that it has the finite integral $v = \frac{1}{x} + \frac{a}{2}$. Hence the complete integral of equation (9) can be written in the form

$$xy = c_1(2 - ax) + c_2 e^{-ax}(2 + ax).$$

Prob. 69. Integrate in series the equation $\frac{d^2y}{dx^2} + xy = 0$.

$$\left(\text{Ans. } y = A\left(1 - \frac{1}{3!}x^3 + \frac{1 \cdot 4}{6!}x^6 - \ldots\right) + B\left(x - \frac{2}{4!}x^4 + \frac{2 \cdot 5}{7!}x^7 - \ldots\right).\right)$$

Prob. 70. Integrate in series $x^2\frac{d^2y}{dx^2} + x^2\frac{dy}{dx} + (x - 2)y = 0$.

Prob. 71. Derive for the equation of Prob. 70 the integral $y_3 = e^{-x}(x^{-1} + 1 + \frac{1}{2}x)$, and find its relation to those found above.

ART. 19. SYSTEMS OF DIFFERENTIAL EQUATIONS.

It is shown in Art. 12 that a determinate system of n differential equations of the first order connecting $n + 1$ variables has for its complete solution as many integral equations connecting the variables and also involving n constants of integration. The result of eliminating $n - 1$ variables would be a single relation between the remaining two variables containing in general the n constants. But the elimination may also be effected in the differential system, the result being in general an equation of the nth order of which the equation just mentioned is the complete integral. For example, if there were two equations of the first order connecting the variables x and y with the independent variable t, by differentiating each we should have four equations from which to eliminate one variable, say y, and its two derivatives* with respect to t, leaving a single equation of the second order between x and t.

It is easy to see that the same conclusions hold if some of the given equations are of higher order, except that the order of the result will be correspondingly higher, its index being in

* In general, there would be n^2 equations from which to eliminate $n - 1$ variables and n derivatives of each, that is, $(n - 1)(n + 1) = n^2 - 1$ quantities leaving a single equation of the nth order.

general the sum of the indices of the orders of the given equations. The method is particularly applicable to linear equations with constant coefficients, since we have a general method of solution for the final result. Using the symbolic notation, the differentiations are performed simply by multiplying by the symbol D, and therefore the whole elimination is of exactly the same form as if the equations were algebraic. For example, the system

$$2\frac{d^2y}{dt^2} - \frac{dx}{dt} - 4y = 2t, \qquad 4\frac{dx}{dt} + 2\frac{dy}{dt} - 3x = 0,$$

when written symbolically, is

$$(2D^2 - 4)y - Dx = 2t, \qquad 2Dy + (4D - 3)x = 0;$$

whence, eliminating x,

$$\begin{vmatrix} 2D^2 - 4 & -D \\ 2D & 4D - 3 \end{vmatrix} y = \begin{vmatrix} 2t & -D \\ 0 & 4D - 3 \end{vmatrix},$$

which reduces to

$$(D - 1)^2(2D + 3)y = 2 - \tfrac{3}{2}t.$$

Integrating,

$$y = (A + Bt)e^t + Ce^{-\frac{3}{2}t} - \tfrac{1}{2}t,$$

the particular integral being found by symbolic development, as explained at the end of Art. 16.

The value of x found in like manner is

$$x = (A' + B't)e^t + C'e^{-\frac{3}{2}t} - \tfrac{1}{3}.$$

The complementary function, depending solely upon the determinant of the first members,* is necessarily of the same form as that for y, but involves a new set of constants. The relations between the constants is found by substituting the values of x and y in one of the given equations, and equating to zero in the resulting identity the coefficients of the several terms of the complementary function. In the present example we should thus find the value of x, in terms of A, B, and C, to be

$$x = (6B - 2A - 2Bt)e^t - \tfrac{1}{3}Ce^{-\frac{3}{2}t} - \tfrac{1}{3}.$$

* The index of the degree in D of this determinant is that of the order of the final equation ; it is not necessarily the sum of the indices of the orders of the given equations, but cannot exceed this sum.

In general, the solution of a system of differential equations depends upon our ability to combine them in such a way as to form exact equations. For example, from the dynamical system

$$\frac{d^2x}{dt^2} = X, \qquad \frac{d^2y}{dt^2} = Y, \qquad \frac{d^2z}{dt^2} = Z, \tag{1}$$

where X, Y, Z are functions of x, y, and z, but not of t, we form the equation

$$\frac{dx}{dt}d\frac{dx}{dt} + \frac{dy}{dt}d\frac{dz}{dt} + \frac{dz}{dt}d\frac{dz}{dt} = Xdx + Ydy + Zdz.$$

The first member is an exact differential, and we know that for a conservative field of force the second member is also exact, that is, it is the differential of a function U of x, y, and z. The integral

$$\frac{1}{2}\left[\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2\right] = U + C \tag{2}$$

is that first integral of the system (1) which is known as the equation of energy for the unit mass.

Just as in Art. 13 an equation of the second order was regarded as equivalent to two equations of the first order, so the system (1) in connection with the equation defining the resolved velocities forms a system of six equations of the first order, of which system equation (2) is an "integral" in the sense explained in Art. 12.

Prob. 72. Solve the equations $\dfrac{dx}{-my} = \dfrac{dy}{mx} = dt$ as a system linear in t. (Ans. $x = A\cos mt + B\sin mt$, $y = A\sin mt - B\cos mt$.)

Prob. 73. Solve the system $\dfrac{dz}{dx} + n^2y = e^x$, $\dfrac{dy}{dx} + z = 0$.

(Ans. $y = Ae^{nx} + Be^{-nx} + \dfrac{e^x}{n^2 - 1}$, $z = -nAe^{nx} + nBe^{-nx} - \dfrac{e^x}{n^2 - 1}$.)

Prob. 74. Find for the system $\dfrac{d^2x}{dt^2} = x\phi(x, y)$, $\dfrac{d^2y}{dt^2} = y\phi(x, y)$ a first integral independent of the function ϕ.

(Ans. $x\dfrac{dy}{dt} - y\dfrac{dx}{dt} = C$.)

Prob. 75. The approximate equations for the horizontal motion of a pendulum, when the earth's rotation is taken into account, are

$$\frac{d^2x}{dt^2} - 2r\frac{dy}{dt} + \frac{gx}{l} = 0, \quad \frac{d^2y}{dt} + 2r\frac{dx}{dt} + \frac{gy}{l} = 0;$$

show that both x and y are of the form

$$A \cos n_1t + B \sin n_1t + C \cos n_2t + D \sin n_2t.$$

ART. 20. FIRST ORDER AND DEGREE WITH THREE VARIABLES.

The equation of the first order and degree between three variables x, y and z may be written

$$Pdx + Qdy + Rdz = 0, \tag{1}$$

where P, Q and R are functions of x, y and z. When this equation is exact, P, Q and R are the partial derivatives of some function u, of x, y and z; and we derive, as in Art. 4,

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}; \quad \frac{\partial Q}{\partial z} = \frac{\partial R}{\partial y}, \quad \frac{\partial R}{\partial x} = \frac{\partial P}{\partial z} \tag{2}$$

for the conditions of exactness. In the case of two variables, when the equation is not exact integrating factors always exist; but in this case, there is not always a factor μ such that μP, μQ and μR (put in place of P, Q, and R) will satisfy all three of the conditions (2). It is easily shown, that for this purpose the relation

$$P\left(\frac{\partial Q}{\partial z} - \frac{\partial R}{\partial y}\right) + Q\left(\frac{\partial R}{\partial x} - \frac{\partial P}{\partial z}\right) + R\left(\frac{\partial P}{\partial y} - \frac{\partial Q}{\partial x}\right) = 0 \tag{3}$$

must exist between the given values of P, Q, and R. This is therefore the "condition of integrability" of equation (1).*

When this condition is fulfilled equation (1) may be integrated by first supposing one variable, say z, to be constant. Thus, integrating $Pdx + Qdy = 0$, and supposing the constant of integration C to be a function of z, we obtain the integral, so

* When there are more than three variables such a condition of integrability exists for each group of three variables, but these conditions are not all independent. Thus with four variables there are but three independent conditions.

far as it depends upon x and y. Finally, by comparing the total differential of this result with the given equation we determine dC in terms of z and dz, and thence by integration the value of C.

It may be noticed that when certain terms of an exact equation forms an exact differential, the remaining terms must also be exact. It follows that if one of the variables, say z can be completely separated from the other two (so that in equation (1) R becomes a function of z only and P and Q functions of x and y, but not of z) the terms $Pdx + Qdy$ must be thus rendered exact if the equation is integrable.* For example,

$$zydx - zxdy - y^2dz = 0.$$

is an integrable equation. Accordingly, dividing by y^2z, which we notice separates the variable z from x and y, puts it in the exact form

$$\frac{ydx - xdy}{y^2} - \frac{dz}{z} = 0,$$

of which the integral is $x = y \log cz$.

Regarding x, y and z as coordinates of a moving point, an integrable equation restricts the point to motion upon one of the surfaces belonging to the system of surfaces represented by the integral; in other words, the point (x, y, z) moves in an arbitrary curve drawn on such a surface. Let us now consider in what way equation (1) restricts the motion of a point when it is not integrable. The direction cosines of a moving point are proportional to dx, dy, and dz; hence, denoting them by l, m and n, the direction of motion of the point satisfying equation (1) must satisfy the condition

$$Pl + Qm + Rn = 0. \tag{4}$$

It is convenient to consider in this connection an auxiliary system of lines represented, as explained in Art. 12, by the simultaneous equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R}. \tag{5}$$

* In fact for this case the condition (3) reduces to its last term, which expresses the exactness of $Pdx + Qdy$.

The direction cosines of a point moving in one of the lines of this system are proportional to P, Q and R. Hence, denoting them by λ, μ, ν, equation (4) gives

$$\lambda l + \mu m + \nu n = 0 \qquad (6)$$

for the relation between the directions of two moving points, whose paths intersect, subject respectively to equation (1) and to equations (5). The paths in question therefore intersect at right angles; therefore equation (1) simply restricts a point to move in a path which cuts orthogonally the lines of the auxiliary system.

Now, if there be a system of surfaces which cut the auxiliary lines orthogonally, the restriction just mentioned is completely expressed by the requirement that the line shall lie on one of these surfaces, the line being otherwise entirely arbitrary. This is the case in which equation (1) is integrable.*

On the other hand, when the equation is not integrable, the restriction can only be expressed by two equations involving an arbitrary function. Thus if we assume in advance one such relation, we know from Art. 12 that the given equation (1) together with the first derivative of the assumed relation forms a system admitting of solution in the form of two integrals. Both of these integrals will involve the assumed function. For any particular value of that function we have a system of lines satisfying equation (1), and the arbitrary character of the function makes the solution sufficiently general to include all lines which satisfy the equation.†

Prob. 76. Show that the equation

$$(mz - ny)dx + (nx - lz)dy + (ly - mx)dz = 0$$

is integrable, and infer from the integral the character of the auxil-

* It follows that, with respect to the system of lines represented by equations (5), equation (3) is the condition that the system shall admit of surfaces cutting them orthogonally. The lines of force in any field of conservative forces form such a system, the orthogonal surfaces being the equipotential surfaces.

† So too there is an arbitrary element about the path of a point when the single equation to which it is subject is integrable, but this enters only into *one* of the two equations necessary to define the path.

iary lines. (Compare the illustrative example at the end of Art. 12.)

(Ans. $nx - lz = C(ny - mz)$.)

Prob. 77. Solve $yz^2dx - z^2dy - e^x dz = 0$. (Ans. $yz = e^x(1 + cz)$.)

Prob. 78. Find the equation which in connection with $y = f(x)$ forms the solution of $dz = aydx + bdy$.

Prob. 79. Show that a general solution of

$$ydx = (x - z)(dy - dz)$$

is given by the equations

$$y - z = \phi(x), \qquad y = (x - z)\phi'(x).$$

(This is an example of "Monge's Solution.")

ART. 21. PARTIAL DIFFERENTIAL EQUATIONS OF FIRST ORDER AND DEGREE.

Let x denote an unknown function of the two independent variables x and y, and let

$$p = \frac{\partial z}{\partial x}, \qquad q = \frac{\partial z}{\partial y}$$

denote its partial derivatives: a relation between one or both of these derivatives and the variables is called a "partial differential equation" of the first order. A value of z in terms of x and y which with its derivatives satisfies the equation, or a relation between x, y and z which makes z implicitly such a function, is a "particular integral." The most general equation of this kind is called the "general integral."

If only one of the derivatives, say p, occurs, the equation may be solved as an ordinary differential equation. For if y is considered as a constant, p becomes the ordinary derivative of z with respect to x; therefore, if in the complete integral of the equation thus regarded we replace the constant of integration by an arbitrary function of y, we shall have a relation which includes all particular integrals and has the greatest possible generality. It will be found that, in like manner, when both p and q are present, the general integral involves an arbitrary function.

We proceed to give Lagrange's solution of the equation of the first order and degree, or "linear equation," which may be written in the form

$$Pp + Qq = R, \tag{1}$$

P, Q and R denoting functions of x, y and z. Let $u = a$, in which u is a function of x, y and z, and a, a constant, be an integral of equation (1). Taking derivatives with respect to x and y respectively, we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p = 0, \qquad \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q = 0,$$

and substitution of the values of p and q in equation (1) gives the symmetrical relation

$$P\frac{\partial u}{\partial x} + Q\frac{\partial u}{\partial y} + R\frac{\partial u}{\partial z} = 0. \tag{2}$$

Consider now the system of simultaneous ordinary differential equations

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{R} \tag{3}$$

Let $u = 0$ be one of the integrals (see Art. 12) of this system. Taking its total differential,

$$\frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz = 0;$$

and since by equations (3) dx, dy and dz are proportional to P, Q and R, we obtain by substitution

$$\frac{\partial u}{\partial x}P + \frac{\partial u}{\partial y}Q + \frac{\partial u}{\partial z}R = 0,$$

which is identical with equation (2). It follows that every integral of the system (3) satisfies equation (1), and conversely, so that the general expression for the integrals of (3) will be the general integral of equation (1).

Now let $v = b$ be another integral of equations (3), so that v is also a function which satisfies equation (2). As explained in Art. 12, each of the equations $u = a$, $v = b$ is the equation of a surface passing through a singly infinite system of lines belonging to the doubly infinite system represented by equations (3). What we require is the general expression for any surface passing through lines of the system (and intersecting none of them). It is evident that $f(u, v) = f(a, b) = C$ is such an equation,* and accordingly $f(u, v)$, where f is an arbitrary function, will be found to satisfy equation (2). Therefore, to solve equation (1), we find two independent integrals $u = a$, $v = b$ of the auxiliary system (3), (sometimes called Lagrange's equations,) and then put

$$u = \phi(v), \tag{4}$$

an equation which is evidently equally general with $f(u, v) = 0$.

Conversely, it may be shown that any equation of the form (4), regarded as a primitive, gives rise to a definite partial differential equation of Lagrange's linear form. For, taking partial derivatives with respect to the independent variables x and y, we have

$$\frac{\partial u}{\partial x} + \frac{\partial u}{\partial z}p = \phi'(v)\left[\frac{\partial v}{\partial x} + \frac{\partial v}{\partial z}p\right],$$

$$\frac{\partial u}{\partial y} + \frac{\partial u}{\partial z}q = \phi'(v)\left[\frac{\partial v}{\partial y} + \frac{\partial v}{\partial z}q\right];$$

and eliminating $\phi'(v)$ from these equations, the term containing pq vanishes, giving the result

$$\begin{vmatrix} \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \end{vmatrix} p + \begin{vmatrix} \frac{\partial u}{\partial z} & \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial z} & \frac{\partial v}{\partial x} \end{vmatrix} q = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}, \tag{5}$$

which is of the form $Pp + Qq = R$.†

* Each line of the system is characterized by special values of a and b which we may call its coordinates, and the surface passes through those lines whose coordinates are connected by the perfectly arbitrary relation $f(a, b) = C$.

† These values of P, Q and R are known as the "Jacobians" of the pair of functions u, v with respect to the pairs of variables y, z; z, x; and x, y respectively. Owing to their analogy to the derivatives of a single function they are sometimes denoted thus:

$$P = \frac{\partial(u, v)}{\partial(y, z)}, \quad Q = \frac{\partial(u, v)}{\partial(z, x)}, \quad R = \frac{\partial(u, v)}{\partial(x, y)}.$$

The Jacobian vanishes if the functions u and v are not independent, that is to say, if u can be expressed identically as a function of v. In like manner, As an illustration, let the given partial differential equation be

$$(mz - ny)p + (nx - lz)q = ly - mx, \tag{6}$$

for which Lagrange's Equations are

$$\frac{dx}{mz - ny} = \frac{dy}{nx - lz} = \frac{dz}{ly - mx}. \tag{7}$$

These equations were solved at the end of Art. 12, the two integrals there found being

$$lx + my + nz = a \quad \text{and} \quad x^2 + y^2 + z^2 = b^2. \tag{8}$$

Hence in this case the system of "Lagrangean lines" consists of the entire system of circles having the straight line

$$\frac{x}{l} = \frac{y}{m} = \frac{z}{n} \tag{9}$$

for axis. The general integral of equation (6) is then

$$lx + my + nz = \phi(x^2 + y^2 + z^2), \tag{10}$$

which represents any surface passing through the circles just mentioned, that is, any surface of revolution of which (9) is the axis.*

Lagrange's solution extends to the linear equation containing n independent variables. Thus the equation being

$$P_1\frac{\partial z}{\partial x_1} + P_2\frac{\partial z}{\partial x_2} + \ldots + P_n\frac{\partial z}{\partial x_n} = R,$$

the auxiliary equations are

$$\frac{dx_1}{P_1} = \frac{dx_2}{P_2} = \ldots = \frac{dx_n}{P_n} = \frac{dz}{R},$$

$\dfrac{\partial(\phi, u, v)}{\partial(x, y, z)} = 0$ is the condition that ϕ (a function of x, y and z) is expressible identically as a function of u and v, that is to say, that $\phi = 0$ shall be an integral of $Pp + Qq = R$.

* When the equation $Pdx + Qdy + Rdz = 0$ is integrable (as it is in the above example; see Prob. 76, Art. 20), its integral, which may be put in the form $V = C$, represents a singly infinite system of surfaces which the Lagrangean lines cut orthogonally; therefore, in this case, the general integral may be defined as the general equation of the surfaces which cut orthogonally the system $V = C$. Conversely, starting with a given system $V = C$, $u = f(v)$ is the general equation of the orthogonal surfaces, if $u = a$ and $v = b$ are integrals of

$$dx\Big/\frac{\partial V}{\partial x} = dy\Big/\frac{\partial V}{\partial y} = dz\Big/\frac{\partial V}{\partial z}.$$

and if $u_1 = c_1$, $u_2 = c_2$, ... $u_n = c_n$ are independent integrals, the most general solution is

$$f(u_1, u_2, \ldots u_n) = 0,$$

where f is an arbitrary function.

Prob. 80. Solve $xz\frac{\partial z}{\partial x} + yz\frac{\partial z}{\partial y} = xy$. $\left(\text{Ans. } xy - z^2 = f\left(\frac{x}{y}\right).\right)$

Prob. 81. Solve $(y + z)p + (z + x)q = x + y$.

Prob. 82. Solve $(x + y)(p - q) = z$.

(Ans. $(x + y) \log z - x = f(x + y)$.)

Prob. 83. Solve $x(y - z)p + y(z - x)q = z(x - y)$.

(Ans. $x + y + z = f(xyz)$.)

ART. 22. COMPLETE AND GENERAL INTEGRALS.

We have seen in the preceding article that an equation between three variables containing an arbitrary function gives rise to a partial differential equation of the linear form. It follows that, when the equation is not linear in p and q, the general integral cannot be expressed by a single equation of the form $\phi(u, v) = 0$; it will, however, still be found to depend upon a single arbitrary function.

It therefore becomes necessary to consider an integral having as much generality as can be given by the presence of arbitrary constants. Such an equation is called a "complete integral"; it contains two arbitrary constants (n arbitrary constants in the general case of n independent variables), because this is the number which can be eliminated from such an equation, considered as a primitive, and its two derived equations. For example, if

$$(x - a)^2 + (y - b)^2 + z^2 = k^2,$$

a and b being regarded as arbitrary, be taken as the primitive, the derived equations are

$$x - a + zp = 0, \qquad y - b + zq = 0,$$

and the elimination of a and b gives the differential equation

$$z^2(p^2 + q^2 + 1) = k^2,$$

of which therefore the given equation is a complete integral.

Geometrically, the complete integral represents a doubly infinite system of surfaces; in this case they are spherical surfaces having a given radius and centers in the plane of xy.

In general, a partial differential equation of the first order with two independent variables is of the form

$$F(x, y, z, p, q) = 0, \tag{1}$$

and a complete integral is of the form

$$f(x, y, z, a, b) = 0. \tag{2}$$

In equation (1) suppose x, y and z to have special values, namely, the coordinates of a special point A; the equation becomes a relation between p and q. Now consider any surface passing through A of which the equation is an integral of (1), or, as we may call it, a given "integral surface" passing through A. The tangent plane to this surface at A determines values of p and q which must satisfy the relation just mentioned. Consider also those of the complete integral surfaces [equation (2)] which pass through A. They form a singly infinite system whose tangent planes at A have values of p and q which also satisfy the relation. There is obviously among them one which has the same value of p, and therefore also the same value of q, as the given integral. Thus there is one of the complete integral surfaces which touches at A the given integral surface. It follows that every integral surface (not included in the complete integral) must at every one of its points touch a surface included in the complete integral.*

It is hence evident that every integral surface is the envelope of a singly infinite system selected from the complete integral system. Thus, in the example at the beginning of this article, a right cylinder whose radius is k and whose axis lies in the plane of xy is an integral, because it is the envelope

* Values of x, y, and z, determining a point, together with values of p and q, determining the direction of a surface at that point, are said to constitute an "element of surface." The theorem shows that the complete integral is "complete" in the sense of including all the surface elements which satisfy the differential equation. The method of grouping the "consecutive" elements to form an integral surface is to a certain extent arbitrary.

of those among the spheres represented by the complete integral whose centers are on the axis of the cylinder. If we make the center of the sphere describe an arbitrary curve in the plane of xy we shall have the general integral in this example.

In general, if in equation (2) an arbitrary relation between a and b, such as $b = \phi(a)$, be established, the envelope of the singly infinite system of surfaces thus defined will represent the general integral. By the usual process, the equation of the envelope is the result of eliminating a between the two equations

$$f(x, y, z, a, \phi(a)) = 0, \quad \frac{\partial}{\partial a} f(x, y, z, a, \phi(a)) = 0. \qquad (3)$$

These two equations together determine a line, namely, the "ultimate intersection of two consecutive surfaces." Such lines are called the "characteristics" of the differential equation. They are independent of any particular form of the complete integral, being in fact lines along which all integral surfaces which pass through them touch one another. In the illustrative example above they are equal circles with centers in the plane of xy and planes perpendicular to it.*

The example also furnishes an instance of a "singular solution" analogous to those of ordinary differential equations.

* The characteristics are to be regarded not merely as lines, but as "linear elements of surface," since they determine at each of their points the direction of the surfaces passing through them. Thus, in the illustration, they are circles regarded as great-circle elements of a sphere, or as elements of a right cylinder, and may be likened to narrow hoops. They constitute in all cases a triply infinite system. The surfaces of a complete integral system contain them all, but they are differently grouped in different integral surfaces.

If we arbitrarily select a curve in space there will in general be at each of its points but one characteristic through which the selected curve passes; that is, whose tangent plane contains the tangent to the selected curve. These characteristics (for all points of the curve) form an integral surface passing through the selected curve; and it is the only one which passes through it unless it be itself a characteristic. Integral surfaces of a special kind result when the selected curve is reduced to a point. In the illustration these are the results of rotating the circle about a line parallel to the axis of z.

For the planes $z = \pm k$ envelop the whole system of spheres represented by the complete integral, and indeed all the surfaces included in the general integral. When a singular solution exists it is included in the result of eliminating a and b from equation (2) and its derivatives with respect to a and b, that is, from

$$f = 0, \qquad \frac{\partial f}{\partial a} = 0, \qquad \frac{\partial f}{\partial b} = 0; \tag{4}$$

but, as in the case of ordinary equations, this result may include relations which are not solutions.

Prob. 84. Derive a differential equation from the primitive $lx + my + nz = a$, where l, m, n are connected by the relation $l^2 + m^2 + n^2 = 1$.

Prob. 85. Show that the singular solution of the equation found in Prob. 84 represents a sphere, that the characteristics consist of all the straight lines which touch this sphere, and that the general integral therefore represents all developable surfaces which touch the sphere.

Prob. 86. Find the integral which results from taking in the general integral above $l^2 + m^2 = \cos^2 \theta$ (a constant) for the arbitrary relation between the parameters.

ART. 23. COMPLETE INTEGRAL FOR SPECIAL FORMS.

A complete integral of the partial differential equation

$$F(x, y, z, p, q) = 0 \tag{1}$$

contains two constants, a and b. If a be regarded as fixed and b as an arbitrary parameter, it is the equation of a singly infinite system of surfaces, of which one can be found passing through any given point. The ordinary differential equation of this system, which will be independent of b, may be put in the form

$$dz = pdx + qdy, \tag{2}$$

in which the coefficients p and q are functions of the variables and the constant a. Now the form of equation (2) shows that these quantities are the partial derivatives of z, in an integral of equation (1); therefore they are values of p and q which satisfy equation (1). Conversely, if values of p and q in terms of the variables and a constant a which satisfy equation (1) are such as to make equation (2) the differential equation of a system of surfaces, these surfaces will be integrals. In other words, if we can find values of p and q containing a constant a which satisfy equation (1) and make $dz = pdx + qdy$ integrable, we can obtain by direct integration a complete integral, the integration introducing a second constant.

There are certain forms of equations for which such values of p and q are easily found. In particular there are forms in which p and q admit of constant values, and these obviously make equation (2) integrable. Thus, if the equation contains p and q only, being of the form

$$F(p, q) = 0, \tag{3}$$

we may put $p = a$ and $q = b$, provided

$$F(a, b) = 0. \tag{4}$$

Equation (2) thus becomes

$$dz = adx + bdy,$$

whence we have the complete integral

$$z = ax + by + c, \tag{5}$$

in which a and b are connected by the relation (4) so that a, b and c are equivalent to two arbitrary constants.

In the next place, if the equation is of the form

$$z = px + qy + f(p, q), \tag{6}$$

which is analogous to Clairaut's form, Art. 10, constant values of p and q are again admissible if they satisfy

$$z = ax + by + f(a, b), \tag{7}$$

and this is itself the complete integral. For this equation is of the form $z = ax + by + c$, and expresses in itself the relations between the three constants. Problem 84 of the preceding article is an example of this form.

In the third place, suppose the equation to be of the form

$$F(z, p, q) = 0, \tag{8}$$

in which neither x nor y appears explicitly. If we assume $q = ap$, p will be a function of z determined from

$$F(z, p, ap) = 0, \quad \text{say} \quad p = \phi(z). \tag{9}$$

Then $dz = pdx + qdy = 0$ becomes $dz = \phi(z)(dx + ady)$, which is integrable, giving the complete integral

$$x + ay = \int \frac{dz}{\phi(z)} + b. \tag{10}$$

A fourth case is that in which, while z does not explicitly occur, it is possible to separate x and p from y and q, thus putting the equation in the form

$$f_1(x, p) = f_2(y, q). \tag{11}$$

If we assume each member of this equation equal to a constant a, we may determine p and q in the forms

$$p = \phi_1(x, a), \qquad q = \phi_2(y, a). \tag{12}$$

and $dz = pdx + qdy$ takes an integrable form giving

$$z = \int \phi_1(x, a)dx + \int \phi_2(y, a)dy + b. \tag{13}$$

It is frequently possible to reduce a given equation by transformation of the variables to one of the four forms considered in this article.* For example, the equation $x^2p^2 + y^2 q^2 = z^2$ may be written

$$\left(\frac{xdz}{zdx}\right)^2 + \left(\frac{ydz}{zdy}\right)^2 = 1\,;$$

* The general method, due to Charpit, of finding a proper value of p consists of establishing, by means of the condition of integrability, a linear partial differential equation for p, of which we need only a particular integral. This may be any value of p taken from the auxiliary equations employed in Lagrange's process. See Boole, Differential Equations (London 1865), p. 336; also Forsyth, Differential Equations (London 1885), p. 316, in which the auxiliary equations are deduced in a more general and symmetrical form, involving both p and q. These equations are in fact the equations of the characteristics regarded as in the concluding note to the preceding article. Denoting the partial derivatives of $F(x, y, z, p, q)$ by X, Y, Z, P, Q, they are

$$\frac{dx}{P} = \frac{dy}{Q} = \frac{dz}{Pp+Qq} = -\frac{dp}{X+Zp} = -\frac{dq}{Y+Zq}.$$

See Jordan's Course d'Analyse (Paris, 1887), vol. III, p. 318; Johnson's Differential Equations (New York, 1889), p. 300. Any relation involving one or both the quantities p and q, combined with $F = 0$, will furnish proper values of p

whence, putting $x' = \log x$, $y' = \log y$, $z' = \log z$, it becomes $p'^2 + q'^2 = 1$, which is of the form $F(p', q') = 0$, equation (3). Hence the integral is given by equation (5) when $a^2 + b^2 = 1$; it may therefore be written

$$z' = x' \cos \alpha + y' \sin \alpha + c,$$

and restoring x, y, and z, that of the given equation is

$$z = cx^{\cos \alpha}\, y^{\sin \alpha}.$$

Prob. 87. Find a complete integral for $p^2 - q^2 = 1$.
(Ans. $z = x \sec \alpha + y \tan \alpha + b$.)

Prob. 88. Find the singular solution of $z = px + qy + pq$.
(Ans. $z = -xy$.)

Prob. 89. Solve by transformation $q = 2yp^2$.
(Ans. $z = ax + a^2y^2 + b$.)

Prob. 90. Solve $z(p^2 - q^2) = x - y$.
(Ans. $z^{\frac{3}{2}} = (x + a)^{\frac{3}{2}} + (y + a)^{\frac{3}{2}} + b$.)

Prob. 91. Show that the solution given for the form $F(z, p, q) = 0$ represents cylindrical surfaces, and that $F(z, 0, 0) = 0$ is a singular solution.

Prob. 92. Deduce by the method quoted in the foot-note two complete integrals of $pq = px + qy$.

(Ans. $2z = \left(\frac{x}{\alpha} + \alpha y\right)^2 + \beta$, and $z = xy + y\sqrt{(x^2 + a)} + b$.)

ART. 24. PARTIAL EQUATIONS OF SECOND ORDER.

We have seen in the preceding articles that the general solution of a partial differential equation of the first order depends upon an arbitrary function; although it is only when the equation is linear in p and q that it is expressible by a single equation. But in the case of higher orders no general account can be given of the nature of a solution. Moreover, when we consider the equations derivable from a primitive containing arbitrary functions, there is no correspondence between their number and the order of the equation. For example, if

and q. Sometimes several such relations are readily found; for example, for the equation $z = pq$ we thus obtain the two complete integrals

$$z = (y + a)(x + b) \quad \text{and} \quad 4z = \left(\frac{x}{\alpha} + \alpha y + \beta\right)^2.$$

the primitive with two independent variables contains two arbitrary functions, it is not generally possible to eliminate them and their derivatives from the primitive and its two derived equations of the first and three of the second order.

Instead of a primitive containing two arbitrary functions, let us take an equation of the first order containing a single arbitrary function. This may be put in the form $u = \phi(v)$, u and v now denoting known functions of x, y, z, p, and q. $\phi'(v)$ may now be eliminated from the two derived equations as in Art. 21. Denoting the second derivatives of z by

$$r = \frac{\partial^2 z}{\partial x^2}, \qquad s = \frac{\partial^2 z}{\partial x \partial y}, \qquad t = \frac{\partial^2 z}{\partial^2 y},$$

the result is found to be of the form

$$Rr + Ss + Tt + U(rt - s^2) = V, \tag{1}$$

in which R, S, T, U, and V are functions of x, y, z, p, and q. With reference to the differential equation of the second order the equation $u = \phi(v)$ is called an "intermediate equation of the first order": it is analogous to the first integral of an ordinary equation of the second order. It follows that an intermediate equation cannot exist unless the equation is of the form (1); moreover, there are two other conditions which must exist between the functions R, S, T, and U.

In some simple cases an intermediate equation can be obtained by direct integration. Thus, if the equation contains derivatives with respect to one only of the variables, it may be treated as an ordinary differential equation of the second order, the constants being replaced by arbitrary functions of the other variable. Given, for example, the equation $xr - p = xy$, which may be written

$$xdp - pdx = xy\,dx.$$

This becomes exact with reference to x when divided by x^2, and gives the intermediate equation

$$p = yx \log x + x\phi(y).$$

A second integration (and change in the form of the arbitrary function) gives the general integral

$$z = \tfrac{1}{2} yx^2 \log x + x^2\phi(y) + \psi(y).$$

Again, the equation $p + r + s = 1$ is already exact, and gives the intermediate equation

$$z + p + q = x + \phi(y),$$

which is of Lagrange's form. The auxiliary equations* are

$$dx = dy = \frac{dz}{x - z + \phi(y)},$$

of which the first gives $x - y = a$, and eliminating x from the second, its integral is of the form

$$z = a + \phi(y) + e^{-y}b.$$

Hence, putting $b = \psi(a)$, we have for the final integral

$$z = x + \phi(y) + e^{-y}\psi(x - y),$$

in which a further change is made in the form of the arbitrary function ϕ.

Prob. 93. Solve $t - q = e^x + e^y$.
(Ans. $z = y(e^y - e^x) + \phi(x) + e^y\psi(x)$.)

Prob. 94. Solve $r + p^2 = y^2$.
(Ans. $z = \log[e^{xy}\psi(y) - e^{-xy}] + \psi(y)$.)

Prob. 95. Solve $y^2(s - t) = x$.
(Ans. $z = (x + y)\log y + \phi(x) + \psi(x + y)$.)

Prob. 96. Solve $ps - qr = 0$. (Ans. $x = \phi(y) + \psi(z)$.)

Prob. 97. Show that Monge's equations (see foot-note) give for Prob. 96 the intermediate integral $p = \phi(z)$ and hence derive the solution.

* In Monge's method (for which the reader must be referred to the complete treatises) of finding an intermediate integral of

$$Rr + Ss + Tt = V$$

when one exists, the auxiliary equations

$$Rdy^2 - Sdy\,dx + Tdx^2 = 0, \qquad Rdp\,dy + Tdq\,dx = Vdx\,dy$$

are established. These, in connection with

$$dz = pdx + qdy,$$

form an incomplete system of ordinary differential equations, between the five variables x, y, z, p, and q. But if it is possible to obtain two integrals of the system in the form $u = a$, $v = b$, $u = \phi(v)$ will be the intermediate integral. The first of the auxiliary equations is a quadratic giving two values for the ratio $dy : dx$. If these are distinct, and an intermediate integral can be found for each, the values of p and q determined from them will make $dz = pdx + qdy$ integrable, and give the general integral at once.

Prob. 98. Derive by Monge's method for $q^2r - 2pqs + p^2t = 0$ the intermediate integral $p = q\,\phi(z)$, and thence the general integral. (Ans. $y + x\phi(z) = \psi(z)$.)

ART. 25. LINEAR PARTIAL DIFFERENTIAL EQUATIONS.

Equations which are linear with respect to the dependent variable and its partial derivatives may be treated by a method analogous to that employed in the case of ordinary differential equations. We shall consider only the case of two independent variables x and y, and put

$$D = \frac{\partial}{\partial x}, \quad D' = \frac{\partial}{\partial y},$$

so that the higher derivatives are denoted by the symbols D^2, DD', D'^2, D^3, etc. Supposing further that the coefficients are constants, the equation may be written in the form

$$f(D, D')z = F(x, y), \tag{1}$$

in which f denotes an algebraic function, or polynomial, of which the degree corresponds to the order of the differential equation. Understanding by an "integral" of this equation an explicit value of z in terms of x and y, it is obvious, as in Art. 15, that the sum of a particular integral and the general integral of

$$f(D, D')z = 0 \tag{2}$$

will constitute an equally general solution of equation (1). It is, however, only when $f(D, D')$ is a *homogeneous* function of D and D' that we can obtain a solution of equation (2) containing n arbitrary functions,* which solution is also the "complementary function" for equation (1).

Suppose then the equation to be of the form

$$A_0\frac{d^nz}{dx^n} + A_1\frac{d^nz}{dx^{n-1}dy} + \dots + A_n\frac{d^nz}{dy^n} = 0, \tag{3}$$

and let us assume $\qquad z = \phi(y + mx),$ (4)

* It is assumed that such a solution constitutes the general integral of an equation of the nth order; for a primitive containing more than n independent arbitrary functions cannot give rise by their elimination to an equation of the nth order.

where m is a constant to be determined. From equation (4), $Dz = m\phi'(y + mx)$ and $D'z = \phi'(y + mx)$, so that $Dz = mD'z$, $D^2z = m^2D'^2z$, $DD'z = mD'^2y$, etc. Substituting in equation (3) and rejecting the factor D'^nz or $\phi^{(n)}(y + mx)$, we have

$$A_0m^n + A_1m^{n-1} + \dots + A_n = 0 \tag{5}$$

for the determination of m. If $m_1, m_2, \dots m_n$ are distinct roots of this equation,

$$z = \phi_1(y + m_1x) + \phi_2(y + m_2x) + \dots + \phi_n(y + m_nx) \tag{6}$$

is the general integral of equation (3).

For example, the general integral of $\frac{d^2z}{dx^2} - \frac{d^2z}{dy^2} = 0$ is thus found to be $z = \phi(y + x) + \psi(y - x)$. Any expression of the form $Axy + Bx + Cy + D$ is a particular integral; accordingly it is expressible as the sum of certain functions of $x + y$ and $x - y$ respectively.

The homogeneous equation (3) may now be written symbolically in the form

$$(D - m_1D')(D - m_2D') \dots (D - m_nD')z = 0, \tag{7}$$

in which the several factors correspond to the several terms of the general integral. If two of the roots of equation (5) are equal, say, to m_1, the corresponding terms in equation (6) are equivalent to a single arbitrary function. To form the general integral we need an integral of

$$(D - m_1D')^2z = 0 \tag{8}$$

in addition to $\phi(y + m_1x)$. This will in fact be the solution of

$$(D - m_1D')z = \phi(y + m_1x); \tag{9}$$

for, if we operate with $D - m_1D'$ upon both members of this equation, we obtain equation (8). Writing equation (9) in the form

$$p - m_1q = \phi(y + mx),$$

Lagrange's equations are

$$dx = -\frac{dy}{m_1} = \frac{dz}{\phi(y + m_1x)},$$

giving the integrals $y + m_1x = a$, $z = x\phi(a) + b$. Hence the integral of equation (9) is

$$z = x\phi(y + m_1x) + \psi(y + m_1x), \tag{10}$$

and regarding ϕ also as arbitrary, these are the two independent terms corresponding to the pair of equal roots.

If equation (5) has a pair of imaginary roots $m = \mu \pm i\nu$, the corresponding terms of the integral take the form

$$\phi(y + \mu x + i\nu x) + \psi(y + \mu x - i\nu x),$$

which when ϕ and ψ are real functions contain imaginary terms. If we restrict ourselves to real integrals we cannot now say that there are two radically distinct classes of integrals; but if any real function of $y + \mu x + i\nu x$ be put in the form $X + iY$, either of the real functions X or Y will be an integral of the equation. Given, for example, the equation

$$\frac{d^2z}{dx^2} + \frac{d^2z}{dy^2} = 0,$$

of which the general integral is

$$z = \phi(y + ix) + \psi(y - ix);$$

to obtain a real integral take either the real or the coefficient of the imaginary part of any real form of $\phi(y + ix)$. Thus, if $\phi(t) = e^t$ we find $e^y \cos x$ and $e^y \sin x$, each of which is an integral.

As in the corresponding case of ordinary equations, the particular integral of equation (1) may be made to depend upon the solution of linear equations of the first order. The inverse symbol $\frac{1}{D - mD'}F(x, y)$ in the equation corresponding to equation (14), Art. 16, denotes the value of z in

$$(D - mD')z = F(x, y) \quad \text{or} \quad p - mq = F(x, y). \qquad (11)$$

For this equation Lagrange's auxiliary equations give

$$y + mx = a, \quad z = \int F(x, a - mx)dx + b = F_1(x, a) + b,$$

and the general integral is

$$z = F_1(x, y + mx) + \phi(y + mx). \qquad (12)$$

The first term, which is the particular integral, may therefore be found by subtracting mx from y in $F(x, y)$, integrating with respect to x, and then adding mx to y in the result.*

For certain forms of $F(x, y)$ there exist more expeditious methods, of which we shall here only notice that which applies to the form $F(ax + by)$. Since $DF(ax + by) = aF'(ax + by)$ and $D'F(ax + by) = bF'(ax + by)$, it is readily inferred that, when $f(D, D')$ is a homogeneous function of the nth degree,

$$f(D, D')F(ax + by) = f(a, b)F^{(n)}(ax + by). \qquad (13)$$

That is, if $t = ax + by$, the operation of $f(D, D')$ on $F(t)$ is equivalent to multiplication by $f(a, b)$ and taking the nth derivative, the final result being still a function of t. It follows that, conversely, the operation of the inverse symbol upon a function of t is equivalent to dividing by $f(a, b)$ and integrating n times. Thus,

$$\frac{1}{f(D, D')} F(ax + by) = \frac{1}{f(a, b)} \int\int \dots \int F(t)dt^n. \qquad (14)$$

When $ax + by$ is a multiple of $y + m_1x$, where m_1 is a root of equation (5), this method fails with respect to the corresponding symbolic factor, giving rise to an equation of the form (9), of which the solution is given in equation (10). Given, for example, the equation

$$\frac{d^2z}{dx^2} + \frac{d^2z}{dx\,dy} - 2\frac{d^2z}{dy^2} = \sin(x - y) + \sin(x + y)$$

or $\qquad (D - D')(D + 2D')z = \sin(x - y) + \sin(x + y).$

The complementary function is $\phi(y + x) + \psi(y - 2x)$. The part of the particular integral arising from $\sin(x - y)$, in which $a = 1$, $b = -1$, is $-\frac{1}{2}\int\int \sin t\,dt^2 = \frac{1}{2}\sin(x - y)$. That aris-

* The symbolic form of this theorem is

$$\frac{1}{D - mD'}F(x, y) = e^{mxD'}\int e^{-mxD'}F(x, y)dx$$

corresponding to equation (13), Art. 16. The symbol $e^{mxD'}$ here indicates the addition of mx to y in the operand. Accordingly, using the expanded form of the symbol,

$$e^{mxD'}F(y) = \left(1 + mx\frac{d}{dy} + \frac{m^2x^2}{2!}\frac{d^2}{dy^2} + \dots\right)F(y) = F(y + mx),$$

the symbolic expression of Taylor's Theorem.

ing from sin $(x+y)$ which is of the form of a term in the complementary function is $-\frac{1}{3}\frac{1}{D-D'}\cos(x+y)$, which by equation (10) is $-\frac{1}{3}x\cos(x+y)$. Hence the general integral of the given equation is

$$z=\phi(y+x)+\psi(y-2x)+\tfrac{1}{2}\sin(x-y)-\tfrac{1}{3}x\cos(x+y).$$

If in the equation $f(D, D')z=0$ the symbol $f(D, D')$, though not homogeneous with respect to D and D', can be separated into factors, the integral is still the sum of those corresponding to the several symbolic factors. The integral of a factor of the first degree is found by Lagrange's process; thus that of

$$(D-mD'-a)z=0 \tag{15}$$

is

$$z=e^{ax}\phi(y+mx). \tag{16}$$

But in the general case it is not possible to express the solution in a form involving arbitrary functions. Let us, however, assume

$$z=ce^{hx+ky}, \tag{17}$$

where c, h, and k are constants. Since $De^{hx+ky}=he^{hx+ky}$ and $D'e^{hx+ky}=ke^{hx+ky}$, substitution in $f'(D, D')z=0$ gives $cf(h, k)e^{hx+ky}=0$. Hence we have a solution of the form (17) whenever h and k satisfy the relation

$$f(h, k)=0, \tag{18}$$

c being altogether arbitrary. It is obvious that we may also write the more general solution

$$z=\Sigma ce^{hx+F(h)y}, \tag{19}$$

where $k=F(h)$ is derived from equation (18), and c and h admit of an infinite variety of arbitrary values.

Again, since the difference of any two terms of the form $e^{hx+F(h)y}$ with different values of h is included in expression (19), we infer that the derivative of this expression with respect to h is also an integral, and in like manner the second and higher derivatives are integrals.

For example, if the equation is

$$\frac{d^2z}{dx^2}-\frac{dz}{dy}=0,$$

for which equation (18) is $h^2 - k = 0$, we have classes of integrals of the forms

$$e^{hx + h^2y}, \quad e^{hx + h^2y}(x + 2hy),$$

$$e^{hx + h^2y}[(x + 2hy)^2 + 2y)], \quad e^{hx + h^2y}[(x + 2hy)^3 + 6y(x + 2hy)].$$

. . . .

In particular, putting $h = 0$ we obtain the algebraic integrals c_1x, $c_2(x^2 + 2y)$, $c_3(x^3 + 6xy)$, etc.

The solution of a linear partial differential equation with variable coefficients may sometimes be effected by a change of the independent variables as illustrated in some of the examples below.

Prob. 99. Show that if m_1 is a triple root the corresponding terms of the integral are $x^2\phi(y + m_1x) + x\psi(y + m_1x) + \chi(y + m_1x)$.

Prob. 100. Solve $2\dfrac{\partial^2 z}{\partial x^2} - 3\dfrac{\partial^2 z}{\partial x \partial y} - 2\dfrac{\partial^2 z}{\partial y^2} = 0$.

Prob. 101. Solve $\dfrac{\partial^3 z}{\partial x^2 \partial y} + 2\dfrac{\partial^3 z}{\partial x \partial y^2} + \dfrac{\partial^3 z}{\partial y^3} = \dfrac{1}{x^2}$.

(Ans. $z = \phi(x) + \psi(x + y) + x\chi(x + y) - y \log x$.)

Prob. 102. Solve $(D^2 + 5DD' + 6D'^2)z = (y - 2x)^{-1}$.

(Ans. $z = \phi(y - 2x) + \psi(y - 3x) + x \log (y - 2x)$.)

Prob. 103. Solve $\dfrac{\partial^2 z}{\partial x^2} - \dfrac{\partial^2 z}{\partial x \partial y} + \dfrac{\partial z}{\partial y} - z = 0$.

Prob. 104. Show that for an equation of the form (15) the solution given by equation (19) is equivalent to equation (16).

Prob. 105. Solve $\dfrac{1}{x^2}\dfrac{\partial^2 z}{\partial x^2} - \dfrac{1}{x^3}\dfrac{\partial z}{\partial x} = \dfrac{1}{y^2}\dfrac{\partial^2 z}{\partial y^2} - \dfrac{1}{y^3}\dfrac{\partial z}{\partial y}$ by transposition to the independent variables x^2 and y^2.

Prob. 106. Solve $x^2\dfrac{\partial^2 z}{\partial x^2} + 2xy\dfrac{\partial^2 z}{\partial x \partial y} + y^2\dfrac{\partial^2 z}{\partial y^2} = 0$.