XIV. *On the Calculus of Functions.* *By* W. H. L. Russell, *Esq., A.B.* *Communicated by* Arthur Cayley, *Esq., F.R.S.*

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One of the first efforts towards the formation of the Calculus of Functions is due to Laplace, whose solution of the functional equation of the first order, by means of two equations in finite differences, is well known. Functional equations were afterwards treated systematically by Mr. Babbage; his memoirs were published in the Transactions of this Society, and there is some account of them in Professor Boole's Treatise on the Calculus of Finite Differences. A very important functional equation was solved by Poisson in his Memoirs on Electricity; which suggested to me the investigations I have now the honour to lay before the Society.

I have commenced by discussing the linear functional equation of the first order with constant coefficients, when the subjects of the unknown functions are rational functions of the independent variable, and have shown how the solution of such equations may in a variety of cases be effected by series, and by definite integrals. I have then considered functional equations with constant coefficients of the higher orders, and have proved that they may be solved by methods similar to those used for equations of the first order. I have next proceeded with the solution of functional equations with variable coefficients.

In connexion with functional equations, I have considered equations involving definite integrals, and containing an unknown function under the integral sign; the methods employed for their resolution depend chiefly upon the solution of functional equations, as effected in this paper. The Calculus of Functions has now for a long time engaged the attention of analysts; and I hope that the following investigations will be found to have extended its power and resources.

Let the functional equation be

$$\phi\left\{\frac{n^2+2n-8}{4r}-nx+rx^2\right\}-\alpha\phi(x)=\mathrm{F}(x),$$

where ϕ is an unknown, and F a known function.

Let

$$x=\frac{n}{2r}+\frac{2}{r}\cos z,$$

and the equation becomes

$$\phi\left\{\frac{n}{2r}+\frac{2}{r}\cos 2z\right\}-\alpha\phi\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}=\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\};$$

or if

$$\phi\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}=\phi_1(z),$$

we shall have

$$\varphi_1(2z)-\alpha\varphi_1(z)=\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos z\right\}.$$

Let

$$z=\frac{1}{2^{v+1}},$$

then

$$\varphi_1\left(\frac{1}{2^v}\right)-\alpha\varphi_1\left(\frac{1}{2^{v+1}}\right)=\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos\frac{1}{2^{v+1}}\right\}.$$

Let

$$\mathrm{F}\left\{\frac{n}{2r}+\frac{2}{r}\cos\frac{1}{2^{v+1}}\right\}=\chi\left(\frac{1}{2^{v+1}}\right),$$

then

$$\varphi_1\left(\frac{1}{2^v}\right)-\alpha\varphi_1\left(\frac{1}{2^{v+1}}\right)=\chi\left(\frac{1}{2^{v+1}}\right),$$

and

$$\varphi_1\left(\frac{1}{2^v}\right)=\chi\left(\frac{1}{2^{v+1}}\right)+\alpha\chi\left(\frac{1}{2^{v+2}}\right)+\alpha^2\chi\left(\frac{1}{2^{v+3}}\right)+\ldots+\frac{\mathrm{C}}{\alpha^{v+1}}.$$

Now

$$\int_0^\pi\frac{(\chi(\varepsilon^{i\theta})+\chi(\varepsilon^{-i\theta}))d\theta}{1-2\alpha\cos\theta+\alpha^2}=\frac{2\pi}{1-\alpha^2}\chi(\alpha),$$

whence (α) is less than unity: then if

$$\varepsilon^\omega=2^{v+n+1},$$

and

$$\omega=(v+1)\log_\varepsilon 2+n\log_\varepsilon 2,$$

also

$$f(\theta)=\chi(\varepsilon^{i\theta})+\chi(\varepsilon^{-i\theta}),$$

we shall have

$$\chi\left(\frac{1}{2^{v+n+1}}\right)=\frac{1}{2\pi}\int_0^\pi\frac{\varepsilon^\omega-\varepsilon^{-\omega}}{\varepsilon^\omega-2\cos\theta+\varepsilon^{-\omega}}f(\theta)d\theta$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\pi d\theta d\varrho f(\theta).\frac{\varepsilon^{(\pi-\theta)\varrho}+\varepsilon^{-(\pi-\theta)\varrho}}{\varepsilon^{\pi\varrho}-\varepsilon^{-\pi\varrho}}\sin\{\varrho\log_\varepsilon 2^{v+1}+n\varrho\log_\varepsilon 2\};$$

$$\therefore\quad\chi\left(\frac{1}{2^{v+1}}\right)+\alpha\chi\left(\frac{1}{2^{v+2}}\right)+\alpha^2\chi\left(\frac{1}{2^{v+3}}\right)+\alpha^3\chi\left(\frac{1}{2^{v+4}}\right)+\ldots$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\pi d\theta d\varrho f(\theta).\frac{\varepsilon^{(\pi-\theta)\varrho}+\varepsilon^{-(\pi-\theta)\varrho}}{\varepsilon^{\pi\varrho}-\varepsilon^{-\pi\varrho}}$$

$$\{\sin(\varrho\log_\varepsilon 2^{v+1})+\alpha\sin(\varrho\log_\varepsilon 2^{v+1}+\varrho\log_\varepsilon 2)+\alpha^2\sin(\varrho\log_\varepsilon 2^{v+1}+2\varrho\log_\varepsilon 2)+\&c.\}$$

$$=\frac{1}{\pi}\int_0^\infty\int_0^\pi d\theta d\varrho f(\theta).\frac{\varepsilon^{(\pi-\theta)\varrho}+\varepsilon^{-(\pi-\theta)\varrho}}{\varepsilon^{\pi\varrho}-\varepsilon^{-\pi\varrho}}.\frac{\sin(\varrho\log_\varepsilon 2^{v+1})-\alpha\sin(\varrho\log_\varepsilon 2^v)}{1-2\alpha\cos(\varrho\log_\varepsilon 2^{v+1})+\alpha^2}.$$

Hence

$$\varphi(x)=\frac{1}{\pi}\int_0^\infty\int_0^\pi d\theta d\varrho f(\theta).\frac{\varepsilon^{(\pi-\theta)\varrho}+\varepsilon^{-(\pi-\theta)\varrho}}{\varepsilon^{\pi\varrho}-\varepsilon^{-\pi\varrho}}.\frac{\sin\left\{\varrho\log_\varepsilon\frac{2}{\cos^{-1}\frac{2rx-n}{4}}\right\}-\alpha\sin\left\{\varrho\log_\varepsilon\frac{1}{\cos^{-1}\frac{2rx-n}{4}}\right\}}{1-2\alpha\cos\left\{\varrho\log_\varepsilon\frac{1}{\cos^{-1}\frac{2rx-n}{4}}\right\}+\alpha^2}$$

$$+\frac{\mathrm{C}}{\left(\cos^{-1}\frac{2rx-n}{4}\right)^{\frac{\log_\varepsilon\alpha}{\log_\varepsilon\frac{1}{2}}}},$$

where

$$f(\theta)=\mathrm{F}\left\{\frac{n}{2r}+\frac{1}{r}\cos\cos\theta(\varepsilon^{\theta}+\varepsilon^{-\theta})+\frac{i}{r}\sin\cos\theta(\varepsilon^{\theta}-\varepsilon^{-\theta})\right\}$$

$$+\mathrm{F}\left\{\frac{n}{2r}+\frac{1}{r}\cos\cos\theta(\varepsilon^{\theta}+\varepsilon^{-\theta})-\frac{i}{r}\sin\cos\theta(\varepsilon^{\theta}-\varepsilon^{-\theta})\right\}.$$

Next let the functional equation be

$$\varphi\left\{\left(\frac{n^3}{27r^2}-\frac{4n}{3r}\right)+\left(\frac{n^2}{3r}-3\right)x+nx^2+rx^3\right\}-\alpha\varphi(x)=\mathrm{F}(x).$$

Let

$$x=-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos z,$$

and the equation will be transformed into

$$\varphi\left\{-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos 3z\right\}-\alpha\varphi\left\{-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos z\right\}=\mathrm{F}\left\{-\frac{n}{3r}+\frac{2}{\sqrt{r}}\cos z\right\};$$

this may be written

$$\varphi_1(3z)-\alpha\varphi_1(z)=\chi(z).$$

Let

$$z=\frac{1}{3^{v+1}}$$

$$\varphi_1\left(\frac{1}{3^v}\right)-\alpha\varphi_1\left(\frac{1}{3^{v+1}}\right)=\chi\left(\frac{1}{3^{v+1}}\right),$$

whence

$$\varphi_1\left(\frac{1}{3^v}\right)=\chi\left(\frac{1}{3^{v+1}}\right)+\alpha\chi\left(\frac{1}{3^{v+2}}\right)+\alpha^2\chi\left(\frac{1}{3^{v+3}}\right)+\dots$$

The same method evidently applies, and we thus obtain the value of $\varphi(x)$.

In the same way we may treat the equation

$$\varphi(a_0+a_1x+a_2x^2+a_3x^3+\dots+a_mx^m)-\alpha\varphi(x)=\mathrm{F}(x),$$

where $a_0, a_1, a_2, \dots a_{m-2}$ must be supposed given in terms of a_{m-1}, a_m.

We will now consider the equation

$$\varphi\left\{\frac{(n-r)r^2+2rn^2+2r(rr'-nr'-2nn')x+((n-r)r'^2+2rn'^2)x^2}{(n'-r')r^2+2r'n^2+2r'(rr'-n'r-2nn')x+((n'-r')r'^2+2r'n'^2)x^2}\right\}-\alpha\varphi(x)=\mathrm{F}(x).$$

Let

$$x=\frac{n+r\cos z}{n'+r'\cos z}.$$

Then the equation becomes

$$\varphi\left\{\frac{n+r\cos 2z}{n'+r'\cos 2z}\right\}-\alpha\varphi\left\{\frac{n+r\cos z}{n'+r'\cos z}\right\}=\mathrm{F}\left\{\frac{n+r\cos z}{n'+r'\cos z}\right\},$$

which may be treated as before: as we may write it

$$\varphi_1(2z)-\alpha\varphi_1(z)=\chi(z).$$

Similar methods will evidently apply to the equation

$$\varphi\left\{\frac{a_0+a_1x+a_2x^2+\dots+a_mx^m}{b_0+b_1x+b_2x^2+\dots+b_mx^m}\right\}-\alpha\varphi(x)=\mathrm{F}(x).$$

The general linear functional equation of the nth order with constant coefficients is

$$\phi(\psi^n x)+a\phi(\psi^{n-1}x)+b\phi(\psi^{n-2}x)+\ldots+h\phi\psi(x)+k\phi(x)=\mathrm{F}'(x),$$

where the subject ψx is supposed to be the same as in the preceding equations.

Let $x=\chi(z)$, and $\psi(x)=\psi\chi(z)=\chi(mz)$ suppose; then $\psi^2(x)=\chi m^2 z$, $\psi^3 x=\chi m^3 z$, &c., and the equation becomes

$$\phi\chi(m^n z)+a\phi\chi(m^{n-1}z)+b\phi\chi(m^{n-2}z)+\ldots+h\phi\chi(mz)+k\phi\chi(z)=\mathrm{F}\chi(z).$$

Let $z=\frac{1}{m^{v+n}}$, and the equation becomes

$$\phi\chi\left(\frac{1}{m^v}\right)+a\phi\chi\left(\frac{1}{m^{v+1}}\right)+b\phi\chi\left(\frac{1}{m^{v+2}}\right)+h\phi\chi\left(\frac{1}{m^{v+n-1}}\right)+k\phi\chi\left(\frac{1}{m^{v+n}}\right)=\mathrm{F}\chi\left(\frac{1}{m^{v+n}}\right).$$

This equation may be written

$$\left\{1+a\varepsilon^{\frac{d}{dv}}+b\varepsilon^{2\frac{d}{dv}}+\ldots h\varepsilon^{(n-1)\frac{d}{dv}}+k\varepsilon^{n\frac{d}{dv}}\right\}\phi\chi\left(\frac{1}{m^v}\right)=\mathrm{F}\chi\left(\frac{1}{m^{v+n}}\right),$$

or

$$\left\{\left(1+\alpha_1\varepsilon^{\frac{d}{dv}}\right)\left(1+\alpha_2\varepsilon^{\frac{d}{dv}}\right)\left(1+\alpha_3\varepsilon^{\frac{d}{dv}}\right)\ldots\left(1+\alpha_n\varepsilon^{\frac{d}{dv}}\right)\right\}\phi\chi\left(\frac{1}{m^v}\right)=\mathrm{F}\chi\left(\frac{1}{m^{v+n}}\right),$$

$$\therefore\quad \phi\chi\left(\frac{1}{m^v}\right)$$

$$=\mathrm{A}_1\left\{\mathrm{F}\chi\left(\frac{1}{m^{v+n}}\right)-\alpha_1\mathrm{F}\chi\left(\frac{1}{m^{v+n+1}}\right)+\alpha_1^2\mathrm{F}\chi\left(\frac{1}{m^{v+n+2}}\right)+\&c.\right\}$$

$$+\mathrm{A}_2\left\{\mathrm{F}\chi\left(\frac{1}{m^{v+n}}\right)-\alpha_2\mathrm{F}\chi\left(\frac{1}{m^{v+n+1}}\right)+\alpha_2^2\mathrm{F}\chi\left(\frac{1}{m^{v+n+2}}\right)+\&c.\right\}$$

$$+\mathrm{A}_3\left\{\mathrm{F}\chi\left(\frac{1}{m^{v+n}}\right)-\alpha_3\mathrm{F}\chi\left(\frac{1}{m^{v+n+1}}\right)+\alpha_3^2\mathrm{F}\chi\left(\frac{1}{m^{v+n+2}}\right)+\ldots\right\}+\&c.$$

$$+\frac{\mathrm{C}_1}{(-\alpha_1)^v}+\frac{\mathrm{C}_2}{(-\alpha_2)^v}+\frac{\mathrm{C}_3}{(-\alpha_3)^v}+\&c.,$$

where A_1, A_2, A_3, &c. are certain functions of α_1, α_2, α_3, &c., and C_1, C_2, C_3, &c. are arbitrary constants, from whence we at once obtain the value of $\phi(x)$.

We now proceed to consider functional equations with variable coefficients. And first let the equation be

$$\phi(x)-\chi(x)\phi\left\{\frac{a+bx}{c+ex}\right\}=\mathrm{F}(x),$$

where $\chi(x)$ and $\mathrm{F}(x)$ are known rational functions of (a).

Let

$$x=u_z,\quad \frac{a+bu_z}{c+eu_z}=u_{z+1},$$

then

$$u_{z+1}=\frac{a+bu_z}{c+eu_z}.$$

Suppose a solution of this equation of finite differences to be

$$u_z=\frac{\mathrm{A}+\mathrm{B}z}{\mathrm{C}+\mathrm{E}z},$$

the equation becomes

$$\phi u_z - \chi(u_z)\phi(u_{z+1}) = \mathrm{F}u_z.$$

Let

$$\chi u_z = \alpha \cdot \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\ldots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\ldots},$$

and the equation may then be written

$$\alpha^z \frac{\Gamma(\alpha_1+z)\Gamma(\alpha_2+z)\Gamma(\alpha_3+z)\ldots}{\Gamma(\beta_1+z)\Gamma(\beta_2+z)\Gamma(\beta_3+z)\ldots}\phi(u_z) - \alpha^{z+1}\frac{\Gamma(\alpha_1+z+1)\Gamma(\alpha_2+z+1)\Gamma(\alpha_3+z+1)\ldots}{\Gamma(\beta_1+z+1)\Gamma(\beta_2+z+1)\Gamma(\beta_3+z+1)\ldots}\phi(u_{z+1})$$

$$= \alpha^z \frac{\Gamma(\alpha_1+z)\Gamma(\alpha_2+z)\Gamma(\alpha_3+z)\ldots}{\Gamma(\beta_1+z)\Gamma(\beta_2+z)\Gamma(\beta_3+z)\ldots}\mathrm{F}u_z.$$

Hence

$$\phi(u_z) = \mathrm{F}u_z + \alpha \cdot \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\ldots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\ldots}\mathrm{F}(u_{z+1}) + \alpha^2 \frac{(\alpha_1+z)(\alpha_1+z+1)(\alpha_2+z)(\alpha_2+z+1)\ldots}{(\beta_1+z)(\beta_1+z+1)(\beta_2+z)(\beta_2+z+1)\ldots}\mathrm{F}(u_{z+2})$$

$$+ \ldots + \frac{\mathrm{C}}{\alpha^z} \cdot \frac{\Gamma(\beta_1+z)\Gamma(\beta_2+z)\ldots}{\Gamma(\alpha_1+z)\Gamma(\alpha_2+z)\ldots}.$$

$\mathrm{F}u_z$ is a rational function of (z), and may therefore be decomposed into a series of terms of the form

$$\mathrm{F}u_z = \frac{1}{h_1+k_1 z} + \frac{1}{h_2+k_2 z} + \frac{1}{h_3+k_3 z} + \ldots$$

Hence

$$\phi u_z = \frac{1}{h_1+k_1 z} + \alpha \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\ldots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\ldots} \cdot \frac{1}{h_1+k_1(z+1)} + \text{\&c.}$$

$$+ \frac{1}{h_2+k_2 z} + \alpha \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\ldots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\ldots} \cdot \frac{1}{h_2+k_2(z+1)} + \text{\&c.}$$

$$+ \frac{1}{h_3+k_3 z} + \alpha \frac{(\alpha_1+z)(\alpha_2+z)(\alpha_3+z)\ldots}{(\beta_1+z)(\beta_2+z)(\beta_3+z)\ldots} \cdot \frac{1}{h_3+k_3(z+1)} + \text{\&c.}$$

$$+ \frac{\mathrm{C}}{\alpha^z} \cdot \frac{\Gamma(\beta_1+z)\Gamma(\beta_2+z)\Gamma(\beta_3+z)\ldots}{\Gamma(\alpha_1+z)\Gamma(\alpha_2+z)\Gamma(\alpha_3+z)\ldots}.$$

We may obtain a multiple integral which shall be equivalent to any of the above series, by remembering that

$$\frac{\alpha(\alpha+1)(\alpha+2)\ldots(\alpha+n-1)}{\beta(\beta+1)(\beta+2)\ldots(\beta+n-1)} = \frac{\Gamma\beta}{\Gamma\alpha\Gamma(\beta-\alpha)}\int_0^1 v^{\alpha+\alpha-1}(1-v)^{\beta-x-1}dv$$

$$\frac{1}{\beta(\beta+1)(\beta+2)\ldots(\beta+n-1)} = \frac{\Gamma\beta \, . \, \varepsilon}{2\pi}\int_{-\infty}^{\infty}\frac{\varepsilon^{iv}dv}{(1+iv)^{\beta+n}},$$

also

$$\frac{1}{h+kn} = \int_0^\infty \varepsilon^{-v(h+kn)}dv,$$

and summing accordingly. We may hence immediately deduce the value of $\phi(x)$.

It is evident that the functional equation

$$\phi(a+bx+cx^2) - \alpha\frac{\psi(x)}{\psi(a+cx+cx^2)}\phi(x) = \mathrm{F}(x),$$

when ψ_1 and F_1 are known functions of (x), may be reduced to the form

$$\phi_1(a+bx+cx^2)-\alpha\phi_1(x)=F(x)\psi(a+bx+cx^2)$$

by making

$$\phi_1(x)=\psi(x)\phi(x).$$

The second equation has already been considered; it becomes, therefore, interesting to ascertain what equations are included under the form of the first equation, in other words, to consider what forms the algebraical expression $\frac{\psi(x)}{\psi(a+bx+cx^2)}$ can possibly take. The following are a few of them:—

$$\frac{\sqrt{(x+2\sqrt{ac}-b)}}{\sqrt{a}+\sqrt{c}.x},\quad \frac{x-a}{x(b+cx)},\quad \frac{x^2-a^2}{(2a+bx+cx^2)x(b+cx)},\quad \frac{\sqrt{\{x^2-2bx+b^2-4ac}}{a-cx^2}.$$

Many others may, in like manner, be imagined; and the same methods, *mutatis mutandis*, apply to functional equations of the higher orders with variable coefficients.

I now come to the consideration of equations involving definite integrals, when the equation contains an unknown function under the sign of definite integration.

Let us take the equation

$$\int_0^1 \frac{F_1(\alpha)+F_2(\alpha).x^2}{F_3(\alpha)+F_4(\alpha)x^2+F_5(\alpha).x^4}dx\phi(x)=F(\alpha),$$

where $\phi(x)$ is an unknown function of (x) not containing (α), $F_1(\alpha)$, $F_2(\alpha)\ldots$, $F_5(\alpha)$ rational functions of (α) which is supposed to vary independently of (x), to determine $\phi(x)$.

Suppose the equation can be written in the form

$$\int_0^1 \frac{\lambda(x)dx}{\sqrt{1-x^2}}\left\{\frac{1-(\chi(\alpha))^2}{1-2\chi\alpha(1-2x^2)+(\chi\alpha)^2}+\frac{\mu(1-\alpha^2)}{1-2\alpha(1-2x^2)+\alpha^2}\right\}=F(\alpha),$$

where

$$\lambda(x)=\phi(x)\sqrt{1-x^2}.$$

Let $x=\sin\frac{\theta}{2}$, and the equation reduces to

$$\int_0^\pi d\theta\lambda\left(\sin\frac{\theta}{2}\right)\left\{\frac{1-(\chi\alpha)^2}{1-2\chi\alpha\cos\theta+(\chi\alpha)^2}+\frac{\mu(1-\alpha^2)}{1-2\alpha\cos\theta+\alpha^2}\right\}=2F(\alpha);$$

or if

$$\psi(\alpha)=\int_0^\pi d\theta.\frac{(1-\alpha^2)\lambda\left(\sin\frac{\theta}{2}\right)}{1-2\alpha\cos\theta+\alpha^2},$$

we find

$$\psi\chi(\alpha)+\mu\psi(\alpha)=2F(\alpha).$$

Suppose the solution of this equation be determined by the former investigations to be

$$\psi(\alpha) = f(\alpha),$$

then

$$\int_0^\pi \frac{(1-\alpha^2)d\theta\lambda\left(\sin\frac{\theta}{2}\right)}{1-2\alpha\cos\theta+\alpha^2}=f(\alpha).$$

Assume

$$\lambda\sin\frac{\theta}{2}=a_0+a_1\cos\theta+a_2\cos 2\theta+a_3\cos 3\theta+\ldots,$$

then since

$$\frac{1-\alpha^2}{1-2\alpha\cos\theta+\alpha^2}=1+2\alpha\cos\theta+2\alpha^2\cos 2\theta+\ldots,$$

we have

$$\pi(a_0+a_1\alpha+a_2\alpha^2+\ldots)=f(\alpha);$$

$$\therefore \qquad a_0+a_1\varepsilon^{i\theta}+a_2\varepsilon^{2i\theta}+\ldots=\frac{1}{\pi}f(\varepsilon^{i\theta})$$

$$a_0+a_1\varepsilon^{-i\theta}+a_2\varepsilon^{-2i\theta}+\ldots=\frac{1}{\pi}f(\varepsilon^{-i\theta}).$$

Hence

$$\lambda\left(\sin\frac{\theta}{2}\right)=\frac{1}{2\pi}\{f\varepsilon^{i\theta}+f\varepsilon^{-i\theta}\},$$

whence $\phi(x)$ can be determined.

Similar treatment will of course apply to the equation

$$\int_0^1\frac{F_1\alpha+F_2\alpha x^2+F_3\alpha\,.\,x^4+\ldots}{F_n(\alpha)+F_{n+1}(\alpha)x^2+\ldots}\phi(x)=F(\alpha),$$

but the functional equation employed for its solution (when possible by this method) will be of a higher order.

Let us, lastly, consider the equation

$$\int_0^1 F_1(x)\phi(x\psi(\alpha))=F(\alpha),$$

to find ϕ, where F_1x is a known function of (x) not containing (α), and (α) varies independently of (x).

Let $\psi(\alpha)=\beta$, then $\alpha=\psi^{-1}\beta$, and the equation becomes

$$\int_0^1 F_1(x)\phi(x\beta)=F\psi^{-1}(\beta).$$

Let

$$\phi(x\beta)=A_0+A_1x\beta+A_2x^2\beta^2+\ldots,$$

then we shall have

$$A_0\int_0^1 dxF_1x+A_1\int_0^1 dx\,.\,xF(x)\,.\,\beta+A_2\int_0^1 dxx^2F(x)\,.\,\beta^2+\ldots$$

$$=F\psi^{-1}0+F'\psi^{-1}0\,.\,\beta+F''\psi^{-1}0\,.\,\frac{\beta^2}{1.2}+\ldots;$$

then

$$A_0=\frac{F\psi^{-1}(0)}{\int_0^1 dxF_1x},\quad A_1=\frac{F'\psi^{-1}0}{\int_0^1 dx\,.\,xF(x)}\ldots$$

Hence

$$\phi(\xi)=\frac{F\psi^{-1}0}{\int_0^1 dxF_1x}+\frac{F'\psi^{-1}0}{\int_0^1 dx\,.\,xF_1x}\,.\,\xi+\frac{F''\psi^{-1}0}{\int_0^1 dx\,.\,x^2F_1x}\,.\,\frac{\xi^2}{1.2}+\frac{F'''\psi^{-1}0}{\int_0^1 dx\,.\,x^3\,.\,F_1\,.\,x}\,.\,\frac{\xi^3}{1.2.3}+\ldots,$$

ξ being any variable.

Now suppose

$$\frac{1}{\int_0^1 dx\,.\,x^nF_1x}=\chi(n),$$

and that we are able to express $\chi(n)$ by a definite integral, so that

$$\chi(n)=\int f(v)(\lambda(v))^n dv,$$

the integral being supposed to have certain finite limits, we shall have

$$\varphi(\xi)=\int f(v)\mathrm{F}\psi^{-1}(\xi\lambda(v))\,.\,dv.$$

Thus if

$$\int_0^1(x+1)\varphi(x\psi(\alpha))=\mathrm{F}(\alpha),$$

we have

$$\varphi(\xi)=\frac{1}{2}\xi\frac{d}{d\xi}\mathrm{F}\psi^{-1}(\xi)+\frac{3}{4}\mathrm{F}\psi^{-1}\xi-\frac{1}{8}\int_0^\infty \varepsilon^{-\frac{3v}{4}}dv\mathrm{F}\psi^{-1}(\xi\varepsilon^{-v}).$$