MAXWELL: An Analytical Calculus Vol. II

AN ANALYTICAL CALCULUS

BY

E. A. MAXWELL

VOLUME II

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AN ANALYTICAL CALCULUS

By E. A. Maxwell

Fellow of Queens' College, Cambridge

This is the second of a series of four volumes covering all stages of development of the Calculus, from the last years at school to degree standard. The books are written for students of science and engineering as well as for specialist mathematicians, and are designed to bridge the gap between the works used in schools and more advanced studies, with their emphasis on rigour.

This treatment of algebraic and trigonometric functions is here developed to cover logarithmic, exponential and hyperbolic functions and the expansion of all these functions as power series. There is a chapter on curves and the idea of complex numbers is introduced for the first time. In the two final chapters, the author begins a systematic treatment of methods of integrating functions, introducing principles into what often seems rather a haphazard process.

This volume, like the others, is well endowed with examples.

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FOR SCHOOL AND UNIVERSITY

BY

E. A. MAXWELL

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PREFACE

Appreciation for help received was expressed in the Preface to Volume I, but I would record how much deeper my indebtedness becomes as the work progresses.

E. A. M.

QUEENS' COLLEGE, CAMBRIDGE
June, 1953

CHAPTER VII

THE LOGARITHMIC AND EXPONENTIAL FUNCTIONS

THE particular functions which we have used in the earlier chapters (Volume I) are the powers of x, the ordinary trigonometric functions, and combinations of them such as polynomials. We now introduce an entirely new function, the *logarithm*. The need for it arises, for example, when we seek to evaluate the integral

$$\int x^n \, dx$$

for $n = -1$. The standard formula

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}$$

becomes meaningless; the integral cannot be evaluated in terms of the functions at present at our disposal.

1. The logarithm. Consider the integral

$$\int \frac{dx}{x}.$$

To make the discussion precise, we shall fix the lower limit, giving it the value unity; the effect of this is merely to remove ambiguity about the arbitrary constant. The integral is a function of its upper limit, which we denote by the letter x, replacing the variable in the integration by the letter t. The function is thus

$$f(x) \equiv \int_1^x \frac{dt}{t},$$

where (Vol. I, p. 87) $\quad f'(x) = \dfrac{1}{x}.$

The function defined in this way is called the *logarithm* of x, usually written

$$\log x$$

or

$$\log_e x,$$

the suffix e being inserted for reasons to be given later (p. 27). Thus

$$\log x = \int_1^x \frac{dt}{t}.$$

2. First properties of the logarithm. We now prove some of the basic properties to which the logarithm owes its importance. The reader will note the very close connexion with 'logarithms to the base 10', with which he is presumably familiar.

(i) $$\log 1 = 0.$$

This follows immediately, since (Vol. I, p. 83)

$$\int_1^1 \frac{dt}{t} = 0.$$

(ii) $$\log xy = \log x + \log y.$$

For
$$\log xy = \int_1^{xy} \frac{dt}{t}$$
$$= \int_1^x \frac{dt}{t} + \int_x^{xy} \frac{dt}{t} \quad \text{(Vol. I, p. 83).}$$

Now use the substitution

$$t = xu$$

in the latter integral. We have the relation

$$dt = x\,du$$

(remembering that x is CONSTANT here, the variable of integration being t). Also the values x, xy of t correspond to the values $1, y$ of u. Hence

$$\int_x^{xy} \frac{dt}{t} = \int_1^y \frac{x\,du}{xu} = \int_1^y \frac{du}{u}$$
$$= \log y.$$

We therefore have the required relation

$$\log xy = \log x + \log y.$$

COROLLARY. $\log(x/y) = \log x - \log y.$

For
$$\log x = \log\left\{\left(\frac{x}{y}\right) y\right\}$$
$$= \log(x/y) + \log y.$$

(iii) $$\log(x^n) = n \log x.$$

In the relation
$$\log x = \int_1^x \frac{dt}{t},$$

make the substitution $u = t^n.$

We have the relation $du = nt^{n-1}\,dt.$

Also the values $1, x$ of t correspond to the values $1, x^n$ of u. Therefore, since

$$\frac{1}{t} = \frac{nt^{n-1}}{nt^n},$$

we have
$$\log x = \int_1^x \frac{nt^{n-1}\,dt}{nt^n}$$
$$= \int_1^{x^n} \frac{du}{nu} \quad \text{(applying the substitution)}$$
$$= \frac{1}{n} \log(x^n),$$

so that
$$\log(x^n) = n \log x.$$

Note. n may have any real value, and is not necessarily a positive integer.

(iv) *The value of* $\log x$ *increases indefinitely as* x *does.*

Suppose that N is any large number and m the largest integer such that $2^m < N$. Then

$$\log N = \int_1^2 \frac{dt}{t} + \int_2^4 \frac{dt}{t} + \int_4^8 \frac{dt}{t} + \ldots + \int_{2^{m-1}}^{2^m} \frac{dt}{t} + \int_{2^m}^N \frac{dt}{t}.$$

Now consider
$$\int_{2^{p-1}}^{2^p} \frac{dt}{t}.$$

Throughout the interval $(2^{p-1}, 2^p)$, the variable t is less than 2^p, so that $\frac{1}{t}$ exceeds $\frac{1}{2^p}$. Hence, by the basic definition of an integral (Vol. I, p. 81),

$$\int_{2^{p-1}}^{2^p} \frac{dt}{t} > \int_{2^{p-1}}^{2^p} \frac{dt}{2^p} = \frac{1}{2^p}\Big[t\Big]_{2^{p-1}}^{2^p} = \frac{2^p - 2^{p-1}}{2^p} = \frac{1}{2}.$$

Applying this inequality to the successive integrals in the formula for $\log N$, and noting that $\int_{2^m}^{N} \frac{dt}{t}$ is positive, we obtain the relation

$$\log N > \tfrac{1}{2}+\tfrac{1}{2}+\tfrac{1}{2}+\ldots+\tfrac{1}{2} \quad (m \text{ terms}),$$

so that $\log N$, which exceeds $\frac{1}{2}m$, increases without bound.

Corollary. *The value of* $\log x$ *tends to* minus *infinity as* x *tends to zero.*

For $$\log\left\{\left(\frac{1}{N}\right)N\right\} = \log 1 = 0,$$

so that $$\log\left(\frac{1}{N}\right) + \log N = 0,$$

or $$\log\left(\frac{1}{N}\right) = -\log N.$$

As N tends to infinity, $1/N$ tends to zero, and the result follows.

3. The graph of $\log x$.

If $$y = \log x,$$

then $$\frac{dy}{dx} = \frac{1}{x}.$$

Hence $\frac{dy}{dx}$ is positive for all positive values of x, so that $\log x$ *is a steadily increasing function of* x *for positive* x (Fig. 62). Also the gradient $\frac{1}{x}$ is large and positive when x is small and positive, decreases as x increases, taking the value 1 when $x = 1$, and tends to the value zero as x increases indefinitely. Moreover, as above, y itself tends to 'minus infinity' when x tends to zero, increases with x, taking the value 0 when $x = 1$, and 'tends to infinity' as x increases indefinitely.

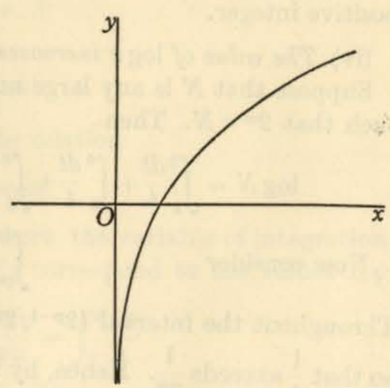


Fig. 62.

The general shape is therefore that shown in the diagram.

The value $x = 0$ imposes a downward barrier on the logarithm, and *the function* $\log x$ *is undefined for negative values of* x.

Note. If $x_1 = x_2$, being positive, then $\log x_1 = \log x_2$; and, what is more important, the converse property holds, that, *if* $\log x_1 = \log x_2$, *then* $x_1 = x_2$. In fact, if $x_1 > x_2$, then $\log x_1 > \log x_2$, since the logarithm is an increasing function; and if $x_1 < x_2$, then $\log x_1 < \log x_2$.

It is also clear from the graph that, *if c is a given number, then the relation*

$$\log x = c$$

defines x uniquely.

Warning. The value of the integral

$$\int_{-2}^{-3} \frac{dt}{t}$$

is fully determinate, but we cannot use the argument:

$$\int_{-2}^{-3} \frac{dt}{t} = \Big[\log t\Big]_{-2}^{-3} = \log(-3) - \log(-2)$$

$$= \log\left(\frac{-3}{-2}\right) = \log(3/2),$$

since $\log(-3)$ and $\log(-2)$ are non-existent. We must proceed as follows:

Substitute $$t = -u,$$

so that $$dt = -du.$$

Then $$\int_{-2}^{-3} \frac{dt}{t} = \int_{2}^{3} \frac{-du}{-u} = \int_{2}^{3} \frac{du}{u}$$

$$= \Big[\log u\Big]_{2}^{3} = \log(3/2).$$

Under no circumstances may we evaluate an integral such as

$$\int_{-2}^{+3} \frac{dt}{t}$$

where the variable of integration t runs through the value zero at which $(1/t)$ has no meaning.

We give two typical examples to show how logarithms arise in physical applications.

ILLUSTRATION 1. *An electric circuit contains a resistance R, a coil of self-inductance L, and a battery of electromotive force E, supposed constant. To find an equation to determine the current t seconds after a switch in the circuit has been closed.*

The equation for the current x is known to be

$$L\frac{dx}{dt}+Rx=E,$$

so that $$L\frac{dx}{dt}=E-Rx.$$

Hence the differentials dt, dx are connected by the relation

$$dt=\frac{L\,dx}{E-Rx},$$

and so $$t=L\int\frac{dx}{E-Rx}.$$

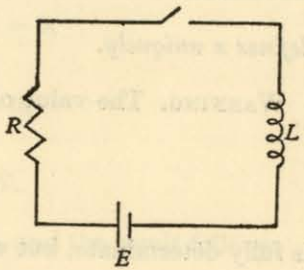


Fig. 63.

[*Note.* We are changing from the conception of x as a function of t to that of t as a function of x.]

The value of the integral may be written down at once, but the beginner may prefer to use the substitution

$$E-Rx=u,$$

so that $$-R\,dx=du,$$

giving $$t=-\frac{L}{R}\int\frac{du}{u}=-\frac{L}{R}\log u+C,$$

assuming that we are dealing with a case in which u is positive.

Hence $$t=-\frac{L}{R}\log(E-Rx)+C,$$

where C is an arbitrary constant. Now $x=0$ when $t=0$, and so

$$0=-\frac{L}{R}\log E+C,$$

or $$C=\frac{L}{R}\log E.$$

Hence $$t=\frac{L}{R}\{\log E-\log(E-Rx)\}$$

$$=\frac{L}{R}\log\frac{E}{E-Rx}.$$

An alternative method for dealing with the relation $L\frac{dx}{dt}+Rx=E$ will be given later (p. 22).

ILLUSTRATION 2. *To find the work done when a given quantity of a perfect gas expands from volume v_1 to volume v_2 at constant absolute temperature T.*

It is known that the volume v and pressure p are connected by the relation

$$pv=RT,$$

where R is constant. Also the work done is known to be

$$W\equiv\int_{v_1}^{v_2}p\,dv.$$

Hence $$W=\int_{v_1}^{v_2}\frac{RT\,dv}{v}$$

$$=RT\Big[\log v\Big]_{v_1}^{v_2}$$

$$=RT(\log v_2-\log v_1)$$

$$=RT\log(v_2/v_1).$$

The following illustrations are typical of integrals involving logarithms.

ILLUSTRATION 3. *To find*

$$I\equiv\int\frac{4x\,dx}{x^4-1}.$$

We have

$$I=\int\left(\frac{2x}{x^2-1}-\frac{2x}{x^2+1}\right)dx=\log(x^2-1)-\log(x^2+1)$$

$$=\log\left(\frac{x^2-1}{x^2+1}\right).$$

ILLUSTRATION 4. *To find*

$$I \equiv \int \tan x\,dx.$$

Write $\qquad u = \cos x,$

so that $\qquad du = -\sin x\,dx.$

Hence
$$I = -\int \frac{du}{u} = -\log u = \log(1/u)$$
$$= \log \sec x.$$

ILLUSTRATION 5. *To find*

$$I \equiv \int x^n \log x\,dx \quad (n \neq -1).$$

On integration by parts (Vol. I, p. 103), we have

$$= \frac{x^{n+1}}{n+1}\log x - \int \frac{x^{n+1}}{n+1}\cdot\frac{1}{x}\,dx$$
$$= \frac{x^{n+1}}{n+1}\log x - \int \frac{x^n}{n+1}\,dx$$
$$= \frac{x^{n+1}}{n+1}\log x - \frac{x^{n+1}}{(n+1)^2}.$$

ILLUSTRATION 6. *To find*

$$I \equiv \int \frac{dx}{\cos x}.$$

We have
$$I = \int \frac{\cos x\,dx}{\cos^2 x} = \int \frac{\cos x\,dx}{1-\sin^2 x}.$$

Let $\qquad u = \sin x,$

so that $\qquad du = \cos x\,dx.$

Then
$$I = \int \frac{du}{1-u^2} = \frac{1}{2}\int \left\{\frac{du}{1-u} + \frac{du}{1+u}\right\}$$
$$= \tfrac{1}{2}\{-\log(1-u) + \log(1+u)\}$$
$$= \tfrac{1}{2}\log\frac{1+u}{1-u}$$
$$= \tfrac{1}{2}\log\frac{1+\sin x}{1-\sin x}.$$

This may also be expressed in the form

$$I = \tfrac{1}{2}\log\frac{(1+\sin x)^2}{1-\sin^2 x}$$
$$= \tfrac{1}{2}\log\left(\frac{1+\sin x}{\cos x}\right)^2 = \log\left(\frac{1+\sin x}{\cos x}\right)$$
$$= \log(\sec x + \tan x).$$

Another form for the answer is

$$I = \log\tan\left(\frac{\pi}{4} + \frac{x}{2}\right).$$

ILLUSTRATION 7.* *To find*

$$I \equiv \int \frac{(4x+7)\,dx}{x^2+4x+13}.$$

Notice that the differential coefficient of the denominator is

$$2x+4,$$

and express the numerator in the form

$$2(2x+4)-1.$$

Then
$$I = 2\int \frac{(2x+4)\,dx}{x^2+4x+13} - \int \frac{dx}{x^2+4x+13}$$
$$= 2\int \frac{(2x+4)\,dx}{x^2+4x+13} - \int \frac{dx}{(x+2)^2+9}$$
$$= 2\log(x^2+4x+13) - \tfrac{1}{3}\tan^{-1}\left(\frac{x+2}{3}\right).$$

ILLUSTRATION 8.* *To find*

$$I \equiv \int \frac{(5x+8)\,dx}{x^2-6x+25}.$$

Notice that the differential coefficient of the denominator is

$$2x-6,$$

and express the numerator in the form

$$\tfrac{5}{2}(2x-6)+23.$$

Then
$$I = \frac{5}{2}\int \frac{(2x-6)\,dx}{x^2-6x+25} + 23\int \frac{dx}{(x-3)^2+16}$$
$$= \frac{5}{2}\log(x^2-6x+25) + \frac{23}{4}\tan^{-1}\left(\frac{x-3}{4}\right).$$

* An important type.

EXAMPLES I

Find the following integrals:

1. $\int \frac{dx}{x+1}$. 2. $\int \frac{dx}{2x+1}$. 3. $\int \frac{dx}{2-3x}$.

4. $\int \frac{x^2+1}{x}\,dx$. 5. $\int \frac{dx}{x^2-1}$. 6. $\int \left(x^2+\frac{1}{x^3}\right)^2 dx$.

Evaluate the following integrals:

7. $\int_2^4 \frac{dx}{x}$. 8. $\int_{-3}^{-5} \frac{dx}{x+1}$. 9. $\int_0^1 \frac{dx}{3x+2}$.

10. $\int_{-2}^{-3} \frac{dx}{4x+7}$. 11. $\int_3^4 \frac{dx}{x^2-4}$. 12. $\int_1^2 \frac{dx}{9x^2-1}$.

Differentiate the following functions with respect to x:

13. $\log(3x+2)$. 14. $\log\tan x$. 15. $\log\operatorname{cosec} x$.

16. $x^2\log x$. 17. $x^n\log x$. 18. $\log(1+x^2)$.

Find the following integrals:

19. $\int \log x\,dx$. 20. $\int \frac{\log x}{x}\,dx$. 21. $\int \cot x\,dx$.

22. $\int x\log x\,dx$. 23. $\int \frac{dx}{\sin x}$. 24. $\int \frac{\cos^2 x}{\sin x}\,dx$.

25. $\int \frac{(2x+5)\,dx}{x^2+5x+12}$. 26. $\int \frac{(2x-3)\,dx}{x^2-3x+7}$. 27. $\int \frac{(2x+5)\,dx}{x^2-2x+17}$.

28. $\int \frac{(2x-6)dx}{x^2+6x+10}$. 29. $\int \frac{(5x+7)\,dx}{x^2-8x+25}$. 30. $\int \frac{(7x-2)\,dx}{x^2+10x+34}$.

4. The use of logarithms in differentiation.

4. The use of logarithms in differentiation. The differentiation of a fraction (in which the numerator and the denominator may themselves be products of factors) is often made easier by the method known as *logarithmic differentiation*, illustrated in the following examples.

ILLUSTRATION 9. *To differentiate the function*

$$y = \frac{x^3(1+x^2)}{(1-x)^4(1+2x)^2}.$$

Take logarithms. Then

$$\log y = 3\log x + \log(1+x^2) - 4\log(1-x) - 2\log(1+2x).$$

Differentiate. Then

$$\frac{1}{y}\frac{dy}{dx} = \frac{3}{x} + \frac{2x}{1+x^2} + \frac{4}{1-x} - \frac{4}{1+2x},$$

and the value of $\frac{dy}{dx}$ follows at once.

With a little practice, the two steps may be taken together:

ILLUSTRATION 10. *To differentiate the function*

$$y = \frac{(1-2x)^2\sin^3 x}{(1+4x^2)^2}.$$

Take logarithms and differentiate. Then

$$\frac{1}{y}\frac{dy}{dx} = \frac{-4}{1-2x} + \frac{3\cos x}{\sin x} - \frac{16x}{1+4x^2}.$$

EXAMPLES II

Use the method of logarithmic differentiation to differentiate the following functions:

1. $\frac{(1+x)^2}{(1-x)^3}$. 2. $\frac{\cos^2 x}{1+x^2}$. 3. $\frac{x\sin^2 x}{1-2x^3}$.

4. $\frac{x^2(1+x)^2}{(1+x^4)^2}$. 5. $\frac{x\sin x}{(1+x)^3(1-x)}$. 6. $\frac{(1+x^2)^2}{2x\cos^2 x}$.

7. $\frac{5x^4(1-x)^3}{\tan^2 2x}$. 8. $\frac{(1+\cos x)^2}{(1+x+x^2)}$. 9. $\frac{(1-x)(1+2x)^2}{(1-3x)^3(1+4x)^4}$.

5. The use of logarithms in integrating simple rational functions. A *rational function of* x is an expression of the form

$$\frac{u(x)}{v(x)},$$

where $u(x), v(x)$ are *polynomials* in x. We shall later (p. 200) give a detailed treatment of the integration of such functions; here we give a preliminary account of the simpler cases.

If the degree of $u(x)$ is higher than that of $v(x)$, we can divide $u(x)$ by $v(x)$, and obtain an expression of the form

$$p(x)+\frac{w(x)}{v(x)},$$

where $p(x)$ is a polynomial, and $w(x)$ *is a polynomial whose degree is less than that of* $v(x)$.

The integration of the polynomial $p(x)$ is immediate. We may therefore confine our attention to the form

$$\frac{w(x)}{v(x)},$$

where $w(x), v(x)$ are polynomials in x, the degree of $w(x)$ being less than that of $v(x)$. The method is to express this quotient in *partial fractions*; details may be found in a text-book on algebra, but, for convenience, a brief account of the calculations involved is inserted for reference.

In order to explain what is required, we consider some typical examples. (Different mathematicians use varying methods. Those which follow have the advantage of giving independent checks of accuracy in some of the more complicated cases.)

(i) $$f(x) \equiv \frac{2x}{(x-2)(x+3)}.$$

The denominator consists of the two linear factors $(x-2), (x+3)$, each occurring to the first degree only. We seek to express $f(x)$ in the form

$$\frac{A}{x-2}+\frac{B}{x+3}.$$

We have $$\frac{A}{x-2}+\frac{B}{x+3} \equiv \frac{2x}{(x-2)(x+3)}.$$

Multiply throughout by $x-2$. Then

$$A+\frac{B(x-2)}{x+3} \equiv \frac{2x}{x+3}.$$

This holds for all values of x; in particular, for $x = 2$. Then

$$A+0 = \frac{2.2}{2+3} = \frac{4}{5}.$$

Hence $$A = \frac{4}{5}.$$

In practice, these steps are usually telescoped, as we now illustrate in finding B. Multiply throughout by $x+3$ and *then* put $x = -3$. Thus

$$B = \frac{2(-3)}{(-3-2)} = \frac{6}{5}$$

Hence $$f(x) \equiv \frac{4}{5(x-2)}+\frac{6}{5(x+3)}.$$

(ii) $$f(x) \equiv \frac{x+1}{(x+2)(x-2)^3}.$$

The denominator consists of two linear factors $(x+2), (x-2)$, of which $(x-2)$ occurs to degree 3. We seek to express $f(x)$ in the form

$$\frac{A}{x+2}+\frac{B}{(x-2)^3}+\frac{C}{(x-2)^2}+\frac{D}{x-2},$$

so that $$\frac{A}{x+2}+\frac{B}{(x-2)^3}+\frac{C}{(x-2)^2}+\frac{D}{x-2} \equiv \frac{x+1}{(x+2)(x-2)^3}.$$

Multiply throughout by $x+2$ and *then* put $x = -2$. Thus

$$A = \frac{-2+1}{(-2-2)^3} = \frac{-1}{-64} = \frac{1}{64}.$$

Multiply throughout by $(x-2)^3$ and *then* put $x = 2$. Thus

$$B = \frac{2+1}{2+2} = \frac{3}{4}.$$

In order to find C, D, we use these values of A, B:

$$\frac{C}{(x-2)^2}+\frac{D}{(x-2)} \equiv \frac{x+1}{(x+2)(x-2)^3}-\frac{1}{64(x+2)}-\frac{3}{4(x-2)^3}$$

$$\equiv \frac{64(x+1)-(x-2)^3-48(x+2)}{64(x+2)(x-2)^3}$$

$$\equiv \frac{64x+64-x^3+6x^2-12x+8-48x-96}{64(x+2)\ (x-2)^3}$$

$$\equiv -\frac{x^3-6x^2-4x+24}{64(x+2)(x-2)^3}.$$

At this point, we are able to check accuracy for the highest common factor of the denominators on the left-hand side is $(x-2)^2$.

Hence $x+2$ and $x-2$ MUST be factors of the numerator on the right. By division, we find that

$$x^3-6x^2-4x+24 \equiv (x+2)(x-2)(x-6).$$

Hence $$\frac{C}{(x-2)^2}+\frac{D}{x-2} \equiv -\frac{x-6}{64(x-2)^2}.$$

Multiply by $(x-2)^2$ and *then* put $x=2$. Thus

$$C = -\frac{(-4)}{64} = \frac{1}{16}.$$

Hence $$\frac{D}{x-2} \equiv -\frac{(x-6)}{64(x-2)^2} - \frac{1}{16(x-2)^2}$$

$$\equiv \frac{-x+2}{64(x-2)^2} \equiv \frac{-1}{64(x-2)},$$

again checking accuracy by the cancelling of $x-2$.

Finally, $$D = -\frac{1}{64}.$$

Hence

$$f(x) \equiv \frac{1}{64(x+2)} + \frac{3}{4(x-2)^3} + \frac{1}{16(x-2)^2} - \frac{1}{64(x-2)}.$$

(iii) $$f(x) \equiv \frac{4x-1}{(x-1)^2(x^2+x+1)}.$$

The denominator consists of the linear factor $(x-1)$, repeated, and the quadratic factor (x^2+x+1). We require to express $f(x)$ in the form

$$\frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1},$$

the numerator above the quadratic factor being of the form $Cx+D$. We thus have

$$\frac{A}{(x-1)^2} + \frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} \equiv \frac{4x-1}{(x-1)^2(x^2+x+1)}.$$

Multiply throughout by $(x-1)^2$, and then put $x=1$. Thus

$$A = \frac{4-1}{1+1+1} = 1.$$

Hence $$\frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} \equiv \frac{4x-1}{(x-1)^2(x^2+x+1)} - \frac{1}{(x-1)^2}$$

$$\equiv \frac{4x-1-(x^2+x+1)}{(x-1)^2(x^2+x+1)}$$

$$\equiv -\frac{x^2-3x+2}{(x-1)^2(x^2+x+1)}.$$

Hence $$\frac{B}{x-1} + \frac{Cx+D}{x^2+x+1} \equiv -\frac{x-2}{(x-1)(x^2+x+1)},$$

the cancelling of the factor $x-1$ providing a check of accuracy.

Multiply throughout by $x-1$, and then put $x=1$. Thus

$$B = -\frac{1-2}{1+1+1} = \frac{1}{3}.$$

Hence $$\frac{Cx+D}{x^2+x+1} \equiv -\frac{x-2}{(x-1)(x^2+x+1)} - \frac{1}{3(x-1)}$$

$$\equiv \frac{-3(x-2)-(x^2+x+1)}{3(x-1)(x^2+x+1)}$$

$$\equiv -\frac{x^2+4x-5}{3(x-1)(x^2+x+1)}$$

$$\equiv -\frac{x+5}{3(x^2+x+1)}$$

on cancelling the factor $x-1$. Hence

$$C \equiv -\tfrac{1}{3}, \quad D \equiv -\tfrac{5}{3},$$

and so $$f(x) \equiv \frac{1}{(x-1)^2} + \frac{1}{3(x-1)} - \frac{x+5}{3(x^2+x+1)}.$$

The following illustrations exhibit some further points about the calculation of partial fractions, and also show how the integration of rational functions is carried out.

ILLUSTRATION 11. *To find*

$$I \equiv \int \frac{x^4\,dx}{(x^2-1)^2}.$$

The numerator is not of less degree than the denominator, so we begin by dividing out:

$$\frac{x^4}{(x^2-1)^2} = 1 + \frac{2x^2-1}{(x^2-1)^2}.$$

Consider, then, the function

$$g(x) \equiv \frac{2x^2-1}{(x^2-1)^2}.$$

Factorize the denominator, so that

$$g(x) \equiv \frac{2x^2-1}{(x-1)^2(x+1)^2}.$$

We therefore have to find constants A, B, C, D such that

$$\frac{A}{(x-1)^2}+\frac{B}{x-1}+\frac{C}{(x+1)^2}+\frac{D}{x+1} \equiv \frac{2x^2-1}{(x-1)^2(x+1)^2}.$$

Multiply throughout by $\begin{cases}(x-1)^2\\(x+1)^2\end{cases}$ and then put $\begin{cases}x=1\\x=-1\end{cases}$. Thus

$$A = \frac{2(1)^2-1}{(1+1)^2} = \frac{1}{4},$$

$$C = \frac{2(-1)^2-1}{(-1-1)^2} = \frac{1}{4}.$$

Hence

$$\frac{B}{x-1}+\frac{D}{x+1} \equiv \frac{2x^2-1}{(x-1)^2(x+1)^2}-\frac{1}{4(x-1)^2}-\frac{1}{4(x+1)^2}$$

$$\equiv \frac{8x^2-4-(x^2+2x+1)-(x^2-2x+1)}{4(x-1)^2(x+1)^2}$$

$$\equiv \frac{6x^2-6}{4(x-1)^2(x+1)^2}$$

$$\equiv \frac{3}{2(x-1)(x+1)}.$$

Hence, by the usual process,

$$B = \tfrac{3}{4}, \quad D = -\tfrac{3}{4}.$$

It follows that

$$I = \int\left\{1+\frac{1}{4(x-1)^2}+\frac{3}{4(x-1)}+\frac{1}{4(x+1)^2}-\frac{3}{4(x+1)}\right\}dx$$

$$= x-\frac{1}{4(x-1)}+\tfrac{3}{4}\log(x-1)-\frac{1}{4(x+1)}-\tfrac{3}{4}\log(x+1).$$

ILLUSTRATION 2. *To find*

$$I \equiv \int \frac{13\,dx}{x^3+x-10}.$$

We must first factorize the denominator. It vanishes when $x = 2$, so that $x-2$ is a factor, and, after division, we find that it is $(x-2)(x^2+2x+5)$. We therefore seek to express

$$f(x) \equiv \frac{13}{x^3+x-10} \equiv \frac{13}{(x-2)(x^2+2x+5)}$$

in the form $\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+2x+5}$,

so that $\dfrac{A}{x-2}+\dfrac{Bx+C}{x^2+2x+5} \equiv \dfrac{13}{(x-2)(x^2+2x+5)}$.

Following a routine which should now be familiar, we have

$$A = \frac{13}{4+4+5} = 1;$$

$$\frac{Bx+C}{x^2+2x+5} \equiv \frac{13}{(x-2)(x^2+2x+5)}-\frac{1}{x-2}$$

$$\equiv \frac{13-x^2-2x-5}{(x-2)(x^2+2x+5)}$$

$$\equiv \frac{8-2x-x^2}{(x-2)(x^2+2x+5)}$$

$$\equiv \frac{-x-4}{x^2+2x+5}.$$

Hence

$$I = \int\left\{\frac{1}{x-2}-\frac{x+4}{x^2+2x+5}\right\}dx$$

$$= \int\left\{\frac{1}{x-2}-\frac{x+1}{x^2+2x+5}-\frac{3}{x^2+2x+5}\right\}dx$$

$$= \log(x-2)-\tfrac{1}{2}\log(x^2+2x+5)-\tfrac{3}{2}\tan^{-1}\left(\frac{x+1}{2}\right).$$

EXAMPLES III

Integrate the following rational functions:

1. $\dfrac{1}{x^2-9}$. 2. $\dfrac{x^2}{(x-1)^2}$. 3. $\dfrac{x}{x^2-3x+2}$.

4. $\dfrac{x^3}{x-1}$. 5. $\dfrac{1}{x^3+x^2+x+1}$. 6. $\dfrac{x^2}{x^2+5x+4}$.

7. $\dfrac{1}{(x-1)^2(x+1)}$. 8. $\dfrac{x}{(x-1)^2(x^2+1)}$. 9. $\dfrac{1}{(x-2)(x^2+4x+9)}$.

10. $\dfrac{x^2}{(x-2)^3}$. 11. $\dfrac{x^3}{(x^2-4)^2}$. 12. $\dfrac{1}{(x-1)^3(x^2+4)}$.

13. $\dfrac{x}{(x-1)(x-2)(x-3)}$. 14. $\dfrac{x^2}{(x-1)(x-2)^2(x-3)}$.

15. $\dfrac{1}{x^2(x^2-6x+13)}$. 16. $\dfrac{1}{x^2(x+4)}$. 17. $\dfrac{x-2}{x^3(x+3)}$.

18. $\dfrac{2x+5}{4x+7}$. 19. $\dfrac{x^2}{x^2+6x+25}$. 20. $\dfrac{(x+1)^2}{x^2+1}$.

21. $\dfrac{(x-1)^2}{x(x^2+4)}$. 22. $\dfrac{2x}{x^3-x^2-8x+12}$.

23. $\dfrac{1}{x^4-2x^3+2x^2-2x+1}$. 24. $\dfrac{x^4}{(x+2)^2(x^2+2x+17)}$.

6. The exponential function. Imagine the graph $y = \log x$ (Fig. 62) to be turned, as it were, through a right angle and viewed through a mirror, and the axes then renamed to give the curve shown in the diagram (Fig. 64). Then y is a certain function of x with the property that

$$x = \log y.$$

Thus y is an 'inverse' function of $\log x$ in a sense similar to that in which $\sin^{-1} x$ is (Vol. I, p. 38) an inverse function of $\sin x$.

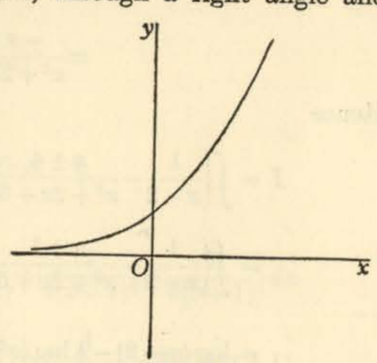


Fig. 64.

We write the relation $\quad x = \log y$

to give y in terms of x, in the form

$$y = \exp x,$$

where $\exp x$, whose properties we now study, is called the *exponential function*. It is defined, as the graph implies, for all values of x, increasing steadily from zero to 'infinity' as x increases from 'minus infinity' to 'infinity'.

The exponential function (of a real variable x) is necessarily POSITIVE.

From the relation $\quad \log y = x,$

we have, by differentiation with respect to x,

$$\frac{1}{y}\frac{dy}{dx} = 1,$$

so that $\quad \dfrac{dy}{dx} = y.$

Hence *the differential coefficient of* $\exp x$ is $\exp x$ itself.

It is convenient to have a name for the value of the function when $x = 1$, and for this we use the letter e. Thus

$$\exp 1 = e,$$

or, in equivalent form, $\quad \log e = 1.$

From the graph, we have the relation

$$e > 1.$$

[The value of e, to four significant figures, is 2·718.]

We now seek to identify the function $\exp x$ in terms of the constant e and the variable x. If

$$y = \exp x,$$

then
$$\log\{\exp x\} = \log y = x.$$

Also the relation $\log(x^n) = n \log x$ leads, on replacing x, n by the letters e, x respectively, to the relation

$$\log(e^x) = x \log e = x,$$

since $\log e = 1$. Hence

$$\log \{\exp x\} = \log (e^x).$$

But we have proved (p. 5) that, if the logarithms of two numbers are equal, then the numbers themselves are equal, and so

$$\exp x = e^x.$$

The exponential function $\exp x$ is therefore identified as the number e raised to the power x.

7. The relations $\dfrac{dy}{dx} = \dfrac{1}{x}, \dfrac{dy}{dx} = y$.

(i) THE LOGARITHM.

It follows from the definition of a logarithm that the relation

$$\frac{dy}{dx} = \frac{1}{x}$$

yields for positive x the result

$$y = \log x + C$$

where C is an arbitrary constant.

If x is negative, say

$$x = -u,$$

where u is positive, then

$$\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = -\frac{dy}{dx};$$

hence

$$\frac{dy}{du} = -\frac{1}{x}$$

$$= \frac{1}{u},$$

so that

$$\begin{aligned} y &= \log u + C \\ &= \log (-x) + C. \end{aligned}$$

We may therefore conclude that, *whether x is positive or negative, the equation*

$$\frac{dy}{dx} = \frac{1}{x}$$

leads to the relation $\qquad y = \log |x| + C,$

where $|x|$ is the numerical value of x.

In practice, it is customary to use the form

$$y = \log x + C$$

with the tacit assumption that x is positive; but this needs care.

The relation $y = \log x + C$ may be put into an alternative form by writing $C = \log a$, where a is also an arbitrary constant. Then

$$\begin{aligned} y &= \log x + \log a \\ &= \log ax \quad (ax \text{ assumed positive}). \end{aligned}$$

By the definition of the exponential function, we then have

$$ax = e^y,$$

or

$$x = b\,e^y,$$

where b is likewise an arbitrary constant, assumed to have the same sign as x.

(ii) THE EXPONENTIAL FUNCTION.

We turn now to the equation

$$\frac{dy}{dx} = y.$$

Writing this in the form $\qquad \dfrac{dx}{dy} = \dfrac{1}{y},$

we see that interchange of x, y in the above relation $x = b\,e^y$ leads to the result

$$y = b\,e^x.$$

Hence *the equation* $\qquad \dfrac{dy}{dx} = y$

leads to the relation $\qquad y = b\,e^x,$

where b is an arbitrary constant, assumed to have the same sign as y.

More generally, *the relation*

$$\frac{dy}{dx} = ky$$

leads to the relation $\qquad y = be^{kx}:$

For the substitution $x = u/k$

gives $$\frac{dy}{du} = \frac{dy}{dx}\frac{dx}{du} = \frac{1}{k}\frac{dy}{dx},$$

so that $$\frac{dy}{du} = \frac{1}{k}(ky) = y.$$

Hence $$y = be^u = be^{kx}.$$

We give two typical examples to show how exponential functions arise in physical applications.

ILLUSTRATION 13. We return to Illustration 1 (p. 6) of a circuit with resistance R, self-inductance L and electromotive force E. The equation for the current x at time t is

$$L\frac{dx}{dt} + Rx = E.$$

Hence $$L\frac{dx}{dt} = E - Rx.$$

Write $$E - Rx = u;$$

then $$\frac{du}{dt} = \frac{du}{dx}\frac{dx}{dt} = (-R)(u/L) = (-R/L)u.$$

Hence $$u = Ae^{-(R/L)t},$$

where A is an arbitrary constant, so that

$$E - Rx = Ae^{-(R/L)t}.$$

If $x = 0$ when $t = 0$, then

$$E = Ae^0 = A,$$

and so $$E - Rx = Ee^{-(R/L)t},$$

or $$x = \frac{E}{R}\{1 - e^{-(R/L)t}\}.$$

ILLUSTRATION 14. *To find the variation of pressure with height in an atmosphere obeying the law*

$$pv = \text{constant},$$

where p, v denote pressure and volume respectively.

Consider a vertical filament of air whose cross-sections have area δA (Fig. 65). Let the pressures at heights $x, x+\delta x$ be $p, p+\delta p$. Then the element of volume (shaded in the diagram) of height δx and base δA is in equilibrium under pressure round its sides, which does not concern us, and also under the following vertical forces:

(i) $p\delta A$ upwards;

(ii) $(p+\delta p)\delta A$ downwards;

(iii) $\rho\delta A\delta x$ downwards,

where ρ is the weight per unit volume at height x. Hence

$$p\delta A - (p+\delta p)\delta A - \rho\delta x\delta A = 0,$$

or $$\delta p + \rho\delta x = 0.$$

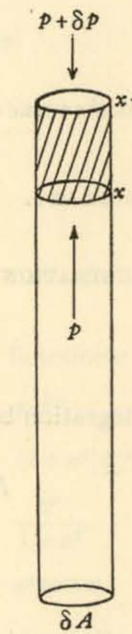


Fig. 65.

Now let p_0, ρ_0, v_0 be the values of p, ρ, v at ground level. Since ρ is the weight per unit volume, the relation

$$pv = p_0 v_0$$

is equivalent to $$\frac{p}{\rho} = \frac{p_0}{\rho_0},$$

and so $$\delta p + \frac{\rho_0}{p_0}p\,\delta x = 0.$$

In the limit, this is $$\frac{dp}{dx} = -\frac{\rho_0}{p_0}p,$$

and so $$p = Ae^{-(\rho_0/p_0)x},$$

where A is an arbitrary constant. But $p = p_0$ when $x = 0$, so that

$$p_0 = Ae^0 = A.$$

Hence the pressure at height x is given by the relation

$$p = p_0 e^{-(\rho_0/p_0)x}.$$

8. The integration of e^x.

To find
$$\int e^x dx,$$
we have merely to note that the relation (p. 19)
$$\frac{d}{dx}(e^x) = e^x$$
leads at once to the result
$$e^x = \int e^x dx,$$
so that *the value of* $\int e^x dx$ *is* e^x *itself.*

COROLLARY. $\int e^{ax} dx = \frac{1}{a} e^{ax}.$

ILLUSTRATION 15. *To find*
$$I \equiv \int e^{ax} \sin bx dx.$$

On integration by parts, we have
$$I = \frac{1}{a} e^{ax} \sin bx - \int \frac{1}{a} e^{ax} . b \cos bx dx$$
$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a} \int e^{ax} \cos bx dx.$$

Integrating again by parts, we have
$$I = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx + \frac{b}{a^2} \int e^{ax}(-b \sin bx) dx$$
$$= \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx - \frac{b^2}{a^2} I.$$

Hence $$\left(1 + \frac{b^2}{a^2}\right) I = \frac{1}{a} e^{ax} \sin bx - \frac{b}{a^2} e^{ax} \cos bx,$$

so that $$I = \frac{e^{ax}}{a^2+b^2}(a \sin bx - b \cos bx).$$

ILLUSTRATION 16. *To prove that, if*
$$y = e^{ax} \sin bx,$$
then $$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0.$$

We have $$\frac{dy}{dx} = a e^{ax} \sin bx + b e^{ax} \cos bx$$
$$= ay + b e^{ax} \cos bx.$$

Hence $$\frac{d^2y}{dx^2} = a\frac{dy}{dx} + ab e^{ax} \cos bx - b^2 e^{ax} \sin bx$$
$$= a\frac{dy}{dx} + a\left\{\frac{dy}{dx} - ay\right\} - b^2 y,$$

so that $$\frac{d^2y}{dx^2} - 2a\frac{dy}{dx} + (a^2+b^2)y = 0.$$

EXAMPLES IV

Find the differential coefficients of the following functions:

1. e^{2x}. 2. e^{x^2}. 3. xe^{5x}.
4. $e^x \cos x$. 5. $e^{\sin x}$. 6. $(1+x^2)e^{-x}$.
7. $\frac{1}{2}(e^x+e^{-x})^2$. 8. $xe^x \sin x$. 9. $\frac{e^x}{1-x^2}$.
10. $(1+e^x)\sin x$. 11. $e^{3x}\cos 4x$. 12. $e^x \tan x$.

Find the following integrals:

13. $\int e^{2x} dx$. 14. $\int e^{-5x} dx$. 15. $\int 2xe^{x^2} dx$.
16. $\int x^2 e^{-x^3} dx$. 17. $\int xe^x dx$. 18. $\int x^2 e^x dx$.
19. $\int e^{\sin x} \cos x dx$. 20. $\int \sin x \cos x e^{\sin^2 x} dx$. 21. $\int \sec^2 x e^{\tan x} dx$.
22. $\int e^x \cos x dx$. 23. $\int e^{3x} \cos 4x dx$. 24. $\int (1+x)e^{2x} dx$.

ILLUSTRATION 17. *To find a formula of reduction for*

$$I_n \equiv \int x^n e^x dx.$$

On integration by parts, we have the relation

$$\begin{aligned} I_n &= e^x . x^n - \int e^x . n x^{n-1} dx \\ &= x^n e^x - n I_{n-1}. \end{aligned}$$

This is the required formula.

ILLUSTRATION 18. *To find a formula of reduction for*

$$I_n = \int e^x \sin^n x\, dx.$$

On integration by parts, we have the relation

$$\begin{aligned} I_n &= e^x . \sin^n x - \int e^x . n \sin^{n-1} x \cos x\, dx \\ &= e^x \sin^n x - n . e^x \sin^{n-1} x \cos x \\ &\qquad + n \int e^x \{(n-1) \sin^{n-2} x \cos^2 x - \sin^n x\} dx \\ &= e^x \sin^n x - n e^x \sin^{n-1} x \cos x \\ &\qquad + n \int e^x \{(n-1) \sin^{n-2} x (1 - \sin^2 x) - \sin^n x\} dx \\ &= e^x \sin^n x - n e^x \sin^{n-1} x \cos x \\ &\qquad + n\{(n-1) I_{n-2} - n I_n\}. \end{aligned}$$

Hence

$$(n^2+1) I_n = e^x \sin^n x - n e^x \sin^{n-1} x \cos x + n(n-1) I_{n-2}.$$

This is the required formula.

EXAMPLES V

Obtain formulæ of reduction for the following integrals:

1. $\int x^n e^{ax} dx.$ 2. $\int e^{ax} \sin^n x\, dx.$ 3. $\int e^{ax} \cos^n bx\, dx.$

Evaluate the following integrals:

4. $\int_0^1 x^5 e^x dx.$ 5. $\int_0^{\frac{1}{2}\pi} e^x \sin^4 x\, dx.$ 6. $\int_0^{\pi} e^x \cos^3 x\, dx.$

9. The reconciliation of $\log_e x$ and $\log_{10} x$. The reader will recall the elementary definition:

The logarithm of a number N to the base a is the index of the power to which a must be raised to give N.

If $\qquad N = a^k,$

then $\qquad \log_a N = k.$

In particular, if $\qquad y = e^x,$

then $\qquad x = \log_e y.$

By this relationship the work which we have just done is reconciled to the more elementary approach, and our use of the word 'logarithm' is justified.

Note. The relation $\frac{d}{dx}(\log x) = \frac{1}{x}$

is true only for the base e.

For other bases we must proceed as follows:

Let $\qquad y = \log_a x.$

Then $\qquad x = a^y.$

Take logarithms of each side to the base e. Then

$$\log_e x = y \log_e a.$$

Differentiate. Then $\qquad \frac{1}{x} = \log_e a \frac{dy}{dx},$

so that $\qquad \frac{dy}{dx} = \frac{1}{x \log_e a}.$

Thus, if $\qquad y = \log_{10} x,$

then $\qquad \frac{dy}{dx} = \frac{1}{x \log_e 10}.$

REVISION EXAMPLES III

'Advanced' Level

1. Differentiate $(1+x)^n \log(1+x)$ (i) with respect to x, (ii) with respect to $(1+x)^n$.

If
$$y = A\sqrt{(1+x)} + B\sqrt{x},$$
prove that
$$4x(1+x)\frac{d^2y}{dx^2} + 2(1+2x)\frac{dy}{dx} - y = 0.$$

2. Prove that, if $y = A\sin^2 x + B\cos^2 x$, then
$$\tan 2x \frac{d^2y}{dx^2} = 2\frac{dy}{dx}.$$

Given that $z = Ae^{2x} + Be^{-2x} + C$, find a differential equation satisfied by z and not containing the constants A, B, C.

3. Find dy/dx in terms of t when
$$x = \frac{1+t}{1-2t}, \quad y = \frac{1+2t}{1-t}.$$

Prove that $\frac{dy}{dx} = 1$ when $t = 0$, and find a second value of t for which $\frac{dy}{dx} = 1$.

Prove that
$$\frac{d^2y}{dx^2} = -\frac{2}{3}\left(\frac{1-2t}{1-t}\right)^3.$$

4. Differentiate with respect to x:
$$\log\sin^2 x, \quad \sqrt{\left(\frac{1-x^2}{1+x^2}\right)}.$$

5. Differentiate with respect to x:
$$\sin^2 x\cos x, \quad e^{ax}(\cos ax + \sin ax),$$
$$\log\{x + \sqrt{(x^2+1)}\}, \quad \tan^{-1}\left(\frac{2}{x}\right).$$

6. Find the differential coefficient of $\sqrt{x}$ from first principles. Differentiate with respect to x:
$$x, \quad \frac{1}{\sqrt{(x^2+1)}}, \quad \cos^{-1}(x^2), \quad \log_{10} x.$$

7. Differentiate the following with respect to x:
$$\frac{1}{x}, \quad \sin 2x, \quad \tan^{-1}\left(\frac{2x}{1-x^2}\right), \quad e^{x^2}.$$

8. Differentiate the following with respect to x:
$$\frac{1}{1-x}, \quad \sin^2 3x, \quad x\log_e x - x, \quad \sin^{-1}\left(\frac{x}{\sqrt{(1+x^2)}}\right).$$

9. Differentiate the following with respect to x, expressing your results as simply as possible:
$$\text{(i)}\ \frac{x}{\sqrt{(1-x^2)}}, \quad \text{(ii)}\ \log_e\tan(\tfrac{1}{2}x + \tfrac{1}{4}\pi), \quad \text{(iii)}\ \sin^{-1}\left(\frac{3+5\cos x}{5+3\cos x}\right).$$

10. Find, from first principles, the differential coefficient of $1/x^2$ with respect to x.

Differentiate the following with respect to x, expressing the results as simply as possible:
$$\frac{x}{(x+1)^2}, \quad \tan^{-1}(x^2), \quad \log_e\tan 2x.$$

11. Differentiate the following expressions with respect to x, giving your results as simply as possible:
$$\frac{x^2+1}{x}, \quad (a^2-x^2)^{\frac{3}{2}}, \quad \sin^{-1}\left(\frac{x}{1+x}\right).$$

12. Differentiate the following functions of x with respect to x:
$$\left(1-\frac{3}{x^2}\right)^2, \quad \sqrt{\left(\frac{1+x}{1-x}\right)}, \quad \frac{\sin x - \cos x}{\sin x + \cos x}.$$

13. Prove from first principles that
$$\frac{d}{dx}\tan x = \sec^2 x$$
and deduce the values of $\frac{d}{dx}\tan^{-1} x$ and $\frac{d}{dx}\cot^{-1} x$.

Differentiate with respect to x:
$$\frac{1+x^2}{1-x^2}, \quad e^{-\frac{1}{2}x}\sin 2x, \quad \log_e(\tan x + \cot x).$$

14. Differentiate the following expressions with respect to x, simplifying your results as much as you can:

$$\frac{x^3}{1+x}, \quad \sin^2 x \cos^3 x, \quad \sin^{-1}(\sqrt{x}).$$

Prove that, if $y = e^{-2x}\cos 4x$, then

$$\frac{d^2y}{dx^2}+4\frac{dy}{dx}+20y = 0.$$

15. Find from first principles the differential coefficient of $1/x^4$ with respect to x.

Differentiate the following expressions with respect to x, simplifying your results as much as you can:

$$\frac{(x^2-1)^{\frac{1}{2}}}{x}, \quad \frac{3+4\tan x}{4+3\tan x}, \quad x\log_e(x^2+1).$$

16. A particle moves along the x-axis so that its displacement x from O at time t is $e^t\cos^2 t$. Find its velocity and acceleration at time $t = \pi$.

Prove that the values of t for which the particle is at rest form two arithmetic progressions, each with common difference π, and that the successive maximum displacements from O form a geometric progression

$$\tfrac{4}{5}e^{\alpha}, \quad \tfrac{4}{5}e^{\alpha+\pi}, \quad \tfrac{4}{5}e^{\alpha+2\pi}, \quad \ldots,$$

where α is an acute angle such that $\tan\alpha = \frac{1}{2}$.

17. Two circles, with centres O and P, radii a ft. and b ft. respectively, intersect at A and B; the chord AB subtends angles 2θ and 2ϕ at O and P respectively; the area common to the two circles is denoted by Δ and you may assume that

$$2\Delta = a^2(2\theta-\sin 2\theta)+b^2(2\phi-\sin 2\phi).$$

Prove that, if P moves towards O with a speed of u ft. per sec., then

$$\frac{d\Delta}{dt} = 2au\sin\theta.$$

18. Find the equations of the tangent and normal at any point on the curve $x = a\cos^3 t, y = a\sin^3 t$, when t is a variable parameter.

Show that the axes intercept a length a on the tangent and a length $2a\cot 2t$ on the normal.

19. A rod AB of length a is hinged at A to a horizontal table and turns about A in a vertical plane with angular velocity ω. A luminous point is situated vertically above A at a height $h(>a)$. Find the length of the shadow when the rod makes an angle θ with the vertical, and prove that the length of the shadow is altering at the rate $ha\omega(h\cos\theta-a)/(h-a\cos\theta)^2$.

20. Prove that, if a, b are positive and $9b>a$, then

$$a\sin x+b\sin 3x$$

will have a maximum value for some value of x between 0 and $\frac{1}{2}\pi$.

Find this maximum value when $a = 3, b = 0{\cdot}5$, proving that it is a maximum and not a minimum.

21. Given that $\quad y = x^5-5x^3+5x^2+1,$

find the stationary values of y. Determine whether these values are maximum or minimum values or neither.

22. The perpendicular from the vertex A to the base BC of a triangular lamina cuts BC at D; $CD = q, DB = p$ (where $q<p$) and $AD = h$. The lamina lies in the quadrant XOY with B on OX, C on OY, and A on the side of BC remote from O. It moves so that B, C slide on OX, OY. Prove that, if $\angle OBC = \theta$, then OA is maximum (not minimum) when

$$\tan 2\theta = \frac{2h}{q-p}.$$

Prove also that $OA = CA$ when

$$\tan\theta = \frac{2h}{q-p}.$$

23. On a fixed diameter of a circle of radius 6 in., and on opposite sides of the centre O, points A, B are taken such that $AO = 3$ in., $OB = 2$ in. The points A, B are joined to any point P on the circle. Prove that, as P moves round the circle, $AP+BP$ takes minimum values 13 in. and 11 in., and takes a maximum value $5\sqrt{7}$ in. twice.

24. Prove that, for real values of x, the function

$$\frac{3\sin x}{2+\cos x}$$

cannot have a value greater than $\sqrt{3}$ or a value less than $-\sqrt{3}$.

Sketch the graph of this function for values of x from $-\pi$ to π.

25. A particle P falls vertically from a point A, its depth below A after t seconds being at^2, where a is constant. B is a fixed point at the same level as A, and at a distance b from A. Prove that the rate of increase of $\angle ABP$ at time t is

$$\frac{2abt}{a^2t^4+b^2},$$

and show that this rate of increase is greatest when

$$\angle ABP = 30^\circ.$$

26. The angle between the bounding radii of a sector of a circle of radius r is θ. Both r and θ vary, but the area of the sector remains constant and equal to c^2. Prove that the perimeter of the sector is a minimum when $r = c$ and $\theta = 2$ radians.

27. If a variable rectangle has a diagonal of constant length 10 inches, prove that its maximum area is 50 square inches.

28. A straight line with variable slope passes through the fixed point (a, b), where a, b are positive, so as to meet the positive part of the x-axis at A and the positive part of the y-axis at B. If O is the origin, prove that the minimum area of the triangle OAB is $2ab$.

Find also the minimum value of the sum of the lengths of OA and OB.

29. Prove that of all isosceles triangles with a given constant perimeter the triangle whose area is greatest is equilateral.

30. A variable line passes through the point $(2, 1)$ and meets the positive axes OX, OY at A, B respectively. If θ denotes the angle OAB $(0<\theta<\frac{1}{2}\pi)$, express the area of the triangle OAB in terms of θ, and prove that the area is a minimum when $\tan\theta = \frac{1}{2}$.

Find also the value of θ if the hypotenuse of the triangle is a minimum.

31. Find the turning points on the graph of the function

$$\frac{2x}{x^2+1},$$

stating (with proof) which is a maximum and which is a minimum.

Sketch the graph and find the equation of the tangent at the point on the curve where $x = \frac{1}{2}$.

32. The slant height of a right circular cone is constant and equal to l. Prove that the volume of the cone is a maximum when the radius of the base is $l\sqrt{(\frac{2}{3})}$.

33. A lighthouse AB of height c ft. stands on the edge of a vertical cliff OA of height b ft. above sea level. From a small boat at a variable distance x from O the angle subtended by AB is θ. Prove that

$$\tan\theta = \frac{cx}{x^2+b(b+c)}.$$

Prove also that, if α is the maximum value of θ, then

$$\tan\alpha = \frac{c}{2(b^2+bc)^{\frac{1}{2}}}.$$

Integrate with respect to x:

34. (i) $\dfrac{2x-7}{2x^2+x-3}$, (ii) $\sin^3 x$, (iii) $x^2\cos x$.

35. (i) $\cos^2 2x$, (ii) $\dfrac{4x}{3x^2-2x-1}$, (iii) $x^3\log_e x$.

36. (i) $\dfrac{\sin^3 x}{\cos^4 x}$, (ii) $x\tan^{-1}x$, (iii) $\dfrac{5}{4x^2+3x-1}$.

37. (i) $\dfrac{10}{2x^2+3x-2}$, (ii) $\sin^2 x\cos^3 x$, (iii) $\dfrac{1}{x^n}\log_e x$ $(n \neq 1)$.

38. (i) $\dfrac{7x}{3x^2-11x+6}$, (ii) $x\sin x$, (iii) $(a^2-x^2)^{\frac{1}{2}}$.

39. Integrate with respect to x:

$$\frac{\cos^3 x}{\sin^2 x}, \quad e^x\sin x.$$

Evaluate
$$\int_2^{12}\frac{2x+5}{(2x-3)(2x+1)}\,dx$$

to three significant figures, given that $\log_{10}e = 0{\cdot}4343$.

40. Interpret by means of a sketch the definite integral

$$\int_0^1\sqrt{(4-x^2)}\,dx,$$

and evaluate this integral.

41. Integrate the following functions with respect to x:

$$\frac{x}{(1-x^2)^{\frac{3}{2}}}, \quad \frac{(x-1)(x-2)}{x+1}.$$

Evaluate $$\int_1^2 x\log_e x\,dx.$$

42. By a suitable change of variable, prove that

$$\int_0^{\frac{1}{2}\pi}\frac{dx}{1+\sin x} = \int_0^{\frac{1}{2}\pi}\frac{dx}{1+\cos x},$$

and by means of the substitution $t=\tan\frac{1}{2}x$, or otherwise, evaluate one of these integrals.

43. Integrate with respect to x:

$$x^2\cos x, \quad \frac{5+3x}{1-9x^2}.$$

44. Integrate with respect to x:

$$x\sqrt{(1+x^2)}, \quad \sqrt{(1+x^2)}, \quad x^2/\sqrt{(1+x^2)}.$$

Evaluate $$\int_0^{\frac{1}{2}\pi}\sin 3x\cos 2x\,dx.$$

45. Explain the method of integration by parts, and employ it to integrate $\sqrt{(1-x^2)}$ with respect to x.

Integrate with respect to x

$$\frac{\sin^{-1}x}{\sqrt{(1-x^2)}}, \quad \frac{1}{(1-x^2)^{\frac{3}{2}}}, \quad e^x(\cos x - \sin x).$$

46. Integrate with respect to x:

$$\frac{1}{x+\sqrt{x}}, \quad \sin^5 x, \quad \frac{x^2+1}{x^2-11x+30}.$$

47. Integrate with respect to x:

$$\frac{1}{\sin^2 x+2\cos^2 x}, \quad \frac{1}{x^3+1}.$$

By integration by parts, or otherwise, integrate $\sin^{-1}x$.

48. Integrate with respect to x:

$$\frac{2-x}{1-x}, \quad (1-\cos^2 x)^2\sin x, \quad xe^x.$$

Prove that $$\int_{\frac{3}{5}}^{\frac{4}{5}}\frac{1-x}{\sqrt{(1-x^2)}}\,dx = \sin^{-1}\tfrac{7}{25}-\tfrac{1}{5}.$$

49. By integration by parts, show that

$$\int_0^\alpha x\sin(\alpha-x)\,dx$$

is equal to $$\alpha-\sin\alpha,$$

and also to $$\frac{\alpha^3}{3!}-\frac{1}{3!}\int_0^\alpha x^3\sin(\alpha-x)\,dx.$$

From graphical or other considerations prove that, if $0<\alpha<\pi$, then

$$\int_0^\alpha x^3\sin(\alpha-x)\,dx < \int_0^\alpha x^3\,dx,$$

and deduce that $$\int_0^\alpha x^3\sin(\alpha-x)\,dx < \tfrac{1}{4}\alpha^4.$$

50. Integrate with respect to x:

$$\frac{1}{x^2(1+x^2)}, \quad \tan^4 x.$$

Evaluate $$\int_0^{\frac{1}{2}\pi}x\sin x\,dx.$$

51. Integrate with respect to x:

$$\frac{x+1}{x+2}, \quad x\sec^2 x.$$

Evaluate $$\int_0^{\frac{1}{2}\pi}\sin 5x\cos 3x\,dx.$$

52. Integrate with respect to x:

$$\frac{4x-1}{2x^2-x-3}, \quad \cos^3 x\sin^2 x.$$

Evaluate $$\int_0^{\frac{1}{2}\pi}x^2\sin x\,dx.$$

53. Integrate $\left(1+\frac{3}{x^2}\right)^2$ with respect to x, and evaluate

$$\int_0^{\frac{1}{2}\pi} \sin^3 x \cos^2 x\,dx, \quad \int_0^1 \frac{(1-x)\,dx}{(x+1)(x^2+1)}.$$

54. Evaluate the definite integrals:

$$\int_0^{\pi} \cos^2 2x\,dx, \quad \int_1^4 x \log x\,dx.$$

If $\quad u_n = \int_0^{\frac{1}{2}\pi} x^n \sin x\,dx,\quad$ and $n>1$, show that

$$u_n = n(\tfrac{1}{2}\pi)^{n-1} - n(n-1)\,u_{n-2},$$

and evaluate u_4.

55. (i) Prove that, if

$$I_n = \int \sec^n x\,dx,$$

then $\quad (n-1)\,I_n = \tan x \sec^{n-2} x + (n-2)\,I_{n-2}.$

Use the formula to evaluate

$$\int_0^{\frac{1}{4}\pi} \sec^5 x\,dx.$$

(ii) Find the positive value of x for which the definite integral

$$\int_0^x \frac{1-t}{\sqrt{(1+t)}}\,dt$$

is greatest, and evaluate the integral for this value of x.

56. Prove that

$$\frac{d}{dx}(\sin^{m+1} x \cos^{n-1} x)$$
$$= (m+n)\sin^m x \cos^n x - (n-1)\sin^m x \cos^{n-2} x\,dx,$$

and deduce a formula connecting

$$\int_0^{\frac{1}{2}\pi} \sin^m x \cos^n x\,dx, \quad \int_0^{\frac{1}{2}\pi} \sin^m x \cos^{n-2} x\,dx.$$

Evaluate

$$\int_0^{\frac{1}{2}\pi} \sin^3 x \cos^5 x\,dx, \quad \int_{-\frac{1}{2}\pi}^{\frac{1}{2}\pi} \sin^3 x \cos^5 x\,dx.$$

57. Prove the rule for integration by parts and use it in finding

$$\int e^{ax} \cos cx\,dx, \quad \int e^{ax} \sin^2 bx\,dx.$$

If I_n denotes $\int_0^a (a^2-x^2)^n\,dx$, prove that, if $n>0$,

$$I_n = \frac{2na^2}{2n+1}\,I_{n-1}.$$

58. (i) Find a reduction formula for

$$\int_0^1 (1+x^2)^{n+\frac{1}{2}}\,dx,$$

and evaluate the integral when $n = 2$.

(ii) By integration by parts, show that, if $0<m<n$, and

$$I = \int_0^1 x^m \frac{d^n}{dx^n}\{x^n(1-x)^n\}\,dx,$$

then $\quad I = -m\int_0^1 x^{m-1} \frac{d^{n-1}}{dx^{n-1}}\{x^n(1-x)^n\}\,dx.$

Deduce that $\quad I = 0.$

CHAPTER VIII

TAYLOR'S SERIES AND ALLIED RESULTS

1. A series giving $\sin x$. By repeated application of the method given in Volume I (p. 53), we may establish that, when x is positive, $\sin x$ lies between the following pairs of functions:

(i) $\quad x \quad$ and $\quad x-\frac{x^3}{3!}$,

(ii) $\quad x-\frac{x^3}{3!} \quad$ and $\quad x-\frac{x^3}{3!}+\frac{x^5}{5!}$,

(iii) $\quad x-\frac{x^3}{3!}+\frac{x^5}{5!} \quad$ and $\quad x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$,

(iv) $\quad x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!} \quad$ and $\quad x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}$,

... ...

A simple inductive step completes the argument. Moreover, the results as stated are equally true when x is negative, though the directions of the inequalities must then be reversed; for example, if x is positive, then

$$x > \sin x > x-\frac{x^3}{3!},$$

whereas, if x is negative,

$$x < \sin x < x-\frac{x^3}{3!}.$$

In either case $\sin x$ is between x and $x-\frac{x^3}{3!}$.

There is, however, a more significant form in which the statements may be cast:

(i) $\sin x$ differs from x by not more than $\frac{|x|^3}{3!}$, where $|x|$ stands for the numerical value of x;

(ii) $\sin x$ differs from $\quad x-\frac{x^3}{3!}$

by not more than $\frac{|x|^5}{5!}$;

(iii) $\sin x$ differs from $\quad x-\frac{x^3}{3!}+\frac{x^5}{5!}$

by not more than $\frac{|x|^7}{7!}$;

(iv) $\sin x$ differs from $\quad x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}$

by not more than $\frac{|x|^9}{9!}$;

and so on.

We are therefore led to a series

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\dots,$$

whose n^{th} term is $(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$, with the property that $\sin x$ *differs from the sum of the first n terms by less than* $\frac{|x|^{2n+1}}{(2n+1)!}$.

Let us examine this 'difference' term $\frac{|x|^{2n+1}}{(2n+1)!}$, writing it in the form

$$\frac{|x|}{1}\cdot\frac{|x|}{2}\cdot\frac{|x|}{3}\cdot\frac{|x|}{4}\cdots\frac{|x|}{2n}\cdot\frac{|x|}{(2n+1)}.$$

Suppose that x has some definite value, positive or negative. If we 'watch' n increase a step at a time, there will come a point when $2n+1$ exceeds $|x|$. Thereafter, the later factors in the product are less than 1; moreover the factor

$$\frac{|x|}{(2n+1)}$$

tends to zero as n continues to increase. By taking n sufficiently large, we may thus ensure that the value of $\sin x$ differs from that of the sum of the first n terms of the series

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\frac{x^9}{9!}-\dots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}+\dots$$

by as little as we please. In that case, $\sin x$ is called the 'sum to infinity' of the series, and we describe the series as an *expansion* of $\sin x$ in ascending powers of x.

EXAMPLES I

1. Complete the inductive step in the argument to prove that $\sin x$ lies between

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^{n-1}\frac{x^{2n-1}}{(2n-1)!}$$

and
$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\ldots+(-1)^{n}\frac{x^{2n+1}}{(2n+1)!}.$$

2. Obtain the expansion for $\cos x$ as a series of ascending powers of x in the form

$$1-\frac{x^2}{2!}+\frac{x^4}{4!}-\frac{x^6}{6!}+\ldots+(-)^{n-1}\frac{x^{2n-2}}{(2n-2)!}+\ldots.$$

2. A series giving $1/(1+x)$. Consider the sum

$$S_n \equiv 1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1},$$

consisting of n terms. By direct multiplication, we have

$$xS_n = \quad x-x^2+x^3-\ldots+(-1)^{n-2}x^{n-1}+(-1)^{n-1}x^n,$$

so that
$$(1+x)S_n = \quad 1+(-1)^{n-1}x^n,$$

or
$$S_n=\frac{1}{1+x}+\frac{(-1)^{n-1}x^n}{1+x}.$$

Hence
$$\frac{1}{1+x}=1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1}+\frac{(-1)^n x^n}{1+x},$$

as is probably familiar.

We have therefore obtained a series

$$1-x+x^2-x^3+\ldots,$$

whose n^{th} term is $(-1)^{n-1}x^{n-1}$, with the property that $1/(1+x)$ *differs from the sum of the first n terms in this series by precisely the amount* $|x^n/(1+x)|$.

Consider, then, the 'difference' term $|x^n/(1+x)|$. When x lies between $-1, 1$, so that $-1<x<1$, this difference can be made as small as we please by taking n sufficiently large, and so we may ensure that, when $|x|<1$, the value of $1/(1+x)$ differs from that of the sum of the n terms

$$1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1}$$

by as little as we please.

On the other hand, when $|x|>1$, the difference $|x^n/(1+x)|$ becomes larger and larger with increasing n, and the value of

$$1-x+x^2-x^3+\ldots+(-1)^{n-1}x^{n-1},$$

so far from approximating to $1/(1+x)$, oscillates wildly as the number of terms increases when x is positive, and increases beyond all bounds when x is negative.

We have therefore obtained a series

$$1-x+x^2-x^3+\ldots,$$

which represents the function $1/(1+x)$ for a certain range of values of x (namely $-1<x<1$), but whose sum has no value when $|x|>1$; in contrast to the series

$$x-\frac{x^3}{3!}+\frac{x^5}{5!}-\frac{x^7}{7!}+\ldots$$

which was shown (p. 39) to be an expansion for $\sin x$ for all values of x.

The intermediate values $x=+1$, $x=-1$ require separate consideration:

When $x=+1$, the series is

$$1-1+1-1+\ldots,$$

whose sum to n terms is oscillating, being 1 when n is odd and 0 when n is even.

When $x=-1$, the series is

$$1+1+1+1+\ldots,$$

and the sum of the first n terms increases indefinitely as n increases.

In neither of these cases can we assign a meaning to the sum 'to infinity'.

3. Expansion in series. The two examples given in §§ 1, 2 illustrate the way in which a function $f(x)$ can be expanded as a series of ascending powers of x in the form

$$a_0+a_1x+a_2x^2+a_3x^3+\ldots+a_nx^n+\ldots,$$

possibly for a restricted range of values of x. They suffer, however, by referring to very particular functions, and the treatment given for $\sin x$ and $1/(1+x)$ leaves us with no idea of how to proceed in more general cases.

In the next paragraph we shall give a formula for the coefficients $a_0, a_1, a_2, \ldots$ in terms of $f(x)$ and its differential coefficients and later we proceed to a more detailed discussion. First, however, we must say a few words about the meaning of the 'sum to infinity' in general; for a fuller treatment, a text-book on analysis should be consulted.

Suppose that we have a series whose successive terms are, say, $u_1, u_2, u_3, \ldots, u_n, \ldots$. It may happen that the sum of the first n terms

$$S_n \equiv u_1 + u_2 + \ldots + u_n$$

tends to a limit S as n tends to infinity.

(For example, if $x \neq 1$,

$$1 + x + x^2 + \ldots + x^{n-1} = \frac{1}{1-x} - \frac{x^n}{1-x},$$

so that, for this series, $S_n = \dfrac{1}{1-x} - \dfrac{x^n}{1-x}$,

and, if $|x| < 1$, $\qquad S = \lim_{n\to\infty} S_n = \dfrac{1}{1-x}$.)

In this case we call S the *sum to infinity* of the series. We write

$$S = u_1 + u_2 + \ldots + u_n + \ldots,$$

and say that the series *converges* to S.

On the other hand, if S_n does not tend to a limit as n tends to infinity, the series has no 'sum to infinity', and the expression

$$u_1 + u_2 + \ldots + u_n + \ldots$$

has no arithmetical meaning.

If the terms $u_1, u_2, \ldots$ of the series depend on x (as in the particular example just quoted) so also does the sum S_n of the first n terms, and the sum to infinity when there is one. We shall be concerned exclusively with series of the form

$$a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$$

where the coefficients $a_0, a_1, \ldots$ are constants, and we have seen (§§ 1, 2) that such a series may converge for all values of x or for some values only.

Note. It is important to realize that the word 'converge' implies definite tending to a limit. A series such as

$$1 - 1 + 1 - 1 + 1 - 1 + \ldots,$$

for which $S_n = 1$ when n is odd and 0 when n is even, has a finite sum for all values of n, but does not converge. The series is said to *oscillate boundedly*.

4. The coefficients in an infinite series. We first assume that expansion in an infinite series is possible, and seek a formula to determine the coefficients:

To prove that, *if $f(x)$ is a given function which* CAN *be expanded in the form*

$$f(x) \equiv a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n + \ldots,$$

then

$$a_n = \frac{f^{(n)}(0)}{n!},$$

where $f^{(n)}(0)$ is the value of $f^{(n)}(x)$ when $x = 0$.

We assume without proof that (in normal cases) we can differentiate the sum of an infinite series by differentiating the terms separately and adding the results, as we should for a finite number of terms. Then

$$f'(x) = a_1 + 2a_2 x + 3a_3 x^2 + \ldots + n a_n x^{n-1} + \ldots,$$
$$f''(x) = 2a_2 + 3.2 a_3 x + 4.3 a_4 x^2 + \ldots + n(n-1) a_n x^{n-2} + \ldots,$$
$$f'''(x) = 3.2 a_3 + 4.3.2 a_4 x + \ldots + n(n-1)(n-2) a_n x^{n-3} + \ldots,$$

and so on. Putting $x = 0$, we have successively

$$f(0) = a_0,$$
$$f'(0) = a_1,$$
$$f''(0) = 2a_2,$$
$$f'''(0) = 3.2 a_3,$$
$$f^{(\mathrm{iv})}(0) = 4.3.2.a_4,$$

and, generally,

$$f^{(n)}(0) = n(n-1)\ldots 3.2 a_n = n!\, a_n.$$

Hence

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \ldots + \frac{x^n}{n!} f^{(n)}(0) + \ldots.$$

Note. This work gives no help about whether the function CAN be expanded; for that we must go to the next paragraph.

ILLUSTRATION 1. *A particle falls from rest under gravity in a medium whose resistance is proportional to the speed. To find an expression for the distance fallen in time t.*

If x is the distance fallen, then the acceleration is $\ddot{x}$ downwards; the forces acting per unit mass are (i) gravity, of magnitude g downwards, (ii) resistance of magnitude $k\dot{x}$ upwards. Hence

$$\ddot{x} = g - k\dot{x}.$$

The formula just given, when adapted to this notation, is

$$x = x_0 + t\dot{x}_0 + \frac{t^2}{2!}\ddot{x}_0 + \frac{t^3}{3!}\dddot{x}_0 + \ldots,$$

where $x_0, \dot{x}_0, \ddot{x}_0, \ldots$ are the values of $x, \dot{x}, \ddot{x}, \ldots$ when $t = 0$.

From the initial conditions,

$$x_0 = 0, \quad \dot{x}_0 = 0,$$

so that

$$\ddot{x}_0 = g - k\dot{x}_0 = g.$$

By successive differentiation of the equation of motion, we have

$$\dddot{x}_0 = -k\ddot{x}_0 = -kg,$$

$$\ddddot{x}_0 = -k\dddot{x}_0 = k^2g,$$

and so on. Hence

$$x = \frac{t^2}{2!}g + \frac{t^3}{3!}(-kg) + \frac{t^4}{4!}(k^2g) + \frac{t^5}{5!}(-k^3g) + \ldots$$

$$= gt^2\left\{\frac{1}{2!} - \frac{(kt)}{3!} + \frac{(kt)^2}{4!} - \frac{(kt)^3}{5!} + \ldots\right\}.$$

The series converges for all values of t.

EXAMPLES II

Use the formula of § 4 to find expansions for the following functions:

1. $\sin x$. 2. $\cos x$. 3. $1/(1+x)$.

5. Taylor's theorem. We come to a somewhat difficult theorem on which the validity of expansion in series can be based.

Suppose that $f(x)$ is a given function of x, possessing as many differential coefficients as are required in the subsequent work.

To prove that, if a, b are two given values of x, then there exists a number ξ between a, b such that, for given n,

$$f(b) = f(a) + (b-a)f'(a) + \frac{(b-a)^2}{2!}f''(a) + \ldots$$
$$+ \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) + \frac{(b-a)^n}{n!}f^{(n)}(\xi).$$

Write

$$F(x) \equiv f(b) - f(x) - (b-x)f'(x) - \frac{(b-x)^2}{2!}f''(x) - \ldots$$
$$- \frac{(b-x)^{n-1}}{(n-1)!}f^{(n-1)}(x) - \frac{(b-x)^k}{(b-a)^k}R_n,$$

where R_n is a number whose properties will be described as required, and k is a positive integer to be specified later. We propose to use Rolle's theorem (Vol. I, p. 60) for $F(x)$ exactly as we did (Vol. I, p. 61) for the mean value theorem, of which this is, indeed, a generalization; we therefore want the relations $F(a) = F(b) = 0$.

By direct substitution of b for x in the expression for $F(x)$, we have

$$F(b) = 0.$$

In order to obtain the relation $F(a) = 0$, we substitute a for x on the right-hand side and equate the result to zero; thus

$$0 = f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!}f''(a) - \ldots$$
$$- \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a) - R_n.$$

We must therefore give to R_n the value

$$R_n \equiv f(b) - f(a) - (b-a)f'(a) - \frac{(b-a)^2}{2!}f''(a) - \ldots$$
$$- \frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a).$$

We have now ensured the relations

$$F(b) = F(a) = 0,$$

and so, by Rolle's theorem, there exists a value ξ of x between a, b at which $F'(x) = 0$.

The next step is to evaluate $F'(x)$. This will involve a number of terms of which
$$\frac{(b-x)^p}{p!}f^{(p)}(x)$$
is typical, and the differential coefficient of this term is
$$\frac{(b-x)^p}{p!}f^{(p+1)}(x)+\frac{p(b-x)^{p-1}(-1)}{p!}f^{(p)}(x)$$
$$=\frac{(b-x)^p}{p!}f^{(p+1)}(x)-\frac{(b-x)^{p-1}}{(p-1)!}f^{(p)}(x).$$

Hence, remembering that R_n is a constant, we have
$$F'(x)=-f'(x)-\{(b-x)f''(x)-f'(x)\}$$
$$-\left\{\frac{(b-x)^2}{2!}f'''(x)-(b-x)f''(x)\right\}-\dots$$
$$-\left\{\frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x)-\frac{(b-x)^{n-2}}{(n-2)!}f^{(n-1)}(x)\right\}+\frac{k(b-x)^{k-1}}{(b-a)^k}R_n$$
$$=-\frac{(b-x)^{n-1}}{(n-1)!}f^{(n)}(x)+\frac{k(b-x)^{k-1}}{(b-a)^k}R_n,$$
after cancelling like terms of opposite signs. But there is a number ξ between a, b for which $F'(\xi)=0$, and so
$$R_n=\frac{(b-a)^k(b-\xi)^{n-k}}{k.(n-1)!}f^{(n)}(\xi).$$

Equating this to the value of R_n obtained above, we have
$$f(b)=f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\dots$$
$$+\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a)+\frac{(b-a)^k(b-\xi)^{n-k}}{k.(n-1)!}f^{(n)}(\xi).$$

The value of k is still at our disposal. If we put $k=n$ (as we might have done from the start, of course, had we so desired) we have
$$R_n=\frac{(b-a)^n}{n!}f^{(n)}(\xi),$$

and the result enunciated at the head of the paragraph follows at once. This gives us the most usual form for Taylor's theorem, and we have adopted it in the formal statement; but there are advantages in keeping k more general, as we see below.

The theorem is therefore established.

The expression
$$R_n\equiv\frac{(b-a)^k(b-\xi)^{n-k}}{k.(n-1)!}f^{(n)}(\xi)$$
is called *the remainder after n terms*. It is, in the first instance, what it says, namely a remainder, the difference between
$$f(b)$$
and $$f(a)+(b-a)f'(a)+\frac{(b-a)^2}{2!}f''(a)+\dots\frac{(b-a)^{n-1}}{(n-1)!}f^{(n-1)}(a).$$

The theorem just proved enables us, however, to express this remainder in the suggestive form (with $k=n$)
$$\frac{(b-a)^n}{n!}f^{(n)}(\xi)$$
by choosing ξ suitably. This expression is known as *Lagrange's form* of the remainder.

By giving k other values, we obtain various forms for the remainder. In particular, when $k=1$, we have
$$R_n\equiv\frac{(b-a)(b-\xi)^{n-1}}{(n-1)!}f^{(n)}(\xi).$$

This may be expressed alternatively by writing ξ, which lies between a, b, as $a+\theta(b-a)$, where θ lies between 0 and 1. Then
$$R_n\equiv\frac{(b-a)\{b-a-\theta(b-a)\}^{n-1}}{(n-1)!}f^{(n)}\{a+\theta(b-a)\}$$
$$=\frac{(b-a)^n(1-\theta)^{n-1}}{(n-1)!}f^{(n)}\{a+\theta(b-a)\}.$$

This is called *Cauchy's form* of the remainder. Though less 'in sequence' than Lagrange's form, it enables us to deal with some series for which Lagrange's form does not work.

ALTERNATIVE TREATMENT. There is an alternative treatment of Taylor's theorem, based on integration by parts, which leads to yet another form of the remainder. The method is essentially a continued application of the formula

$$\int_0^b \frac{(b-t)^{k-1}}{(k-1)!} f^{(k)}(a+t)\,dt$$

$$= \left[-\frac{(b-t)^k}{k!} f^{(k)}(a+t)\right]_0^b + \int_0^b \frac{(b-t)^k}{k!} f^{(k+1)}(a+t)\,dt$$

$$= \frac{b^k}{k!} f^{(k)}(a) + \int_0^b \frac{(b-t)^k}{k!} f^{(k+1)}(a+t)\,dt,$$

the term $\frac{(b-t)^k}{k!} f^{(k)}(a+t)$ vanishing for $t=b$ when $k>0$.

This formula may be expressed more concisely by writing

$$u_k \equiv \int_0^b \frac{(b-t)^{k-1}}{(k-1)!} f^{(k)}(a+t)\,dt \quad (k \geqslant 1),$$

so that

$$u_k = \frac{b^k}{k!} f^{(k)}(a) + u_{k+1}.$$

Hence

$$u_1 = bf'(a) + u_2$$

$$= bf'(a) + \frac{b^2}{2!} f''(a) + u_3$$

$$= bf'(a) + \frac{b^2}{2!} f''(a) + \frac{b^3}{3!} f'''(a) + u_4$$

$$= bf'(a) + \frac{b^2}{2!} f''(a) + \ldots + \frac{b^{n-1}}{(n-1)!} f^{(n-1)}(a) + u_n.$$

Moreover,

$$u_1 = \int_0^b f'(a+t)\,dt = f(a+b) - f(a).$$

Equating the two values of u_1, we obtain the 'Taylor' relation (with our original 'b' now replaced by '$a+b$')

$$f(a+b) = f(a) + bf'(a) + \frac{b^2}{2!} f''(a) + \ldots + \frac{b^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n,$$

where

$$R_n = \int_0^b \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(a+t)\,dt.$$

EXAMPLE III

1. Prove that, if $R_n(x)$ is the function of x defined by the relation

$$R_n(x) = \left\{f(x) + (b-x)f'(x) + \ldots + \frac{(b-x)^{n-1}}{(n-1)!} f^{(n-1)}(x)\right\} - f(b),$$

then

$$R_n'(x) = \frac{(b-x)^{n-1}}{(n-1)!} f^{(n)}(x),$$

and deduce that

$$R_n(x) = \int_b^x \frac{(b-t)^{n-1}}{(n-1)!} f^{(n)}(t)\,dt.$$

6. Maclaurin's theorem. If we write $b = a+h$, Taylor's theorem (with the Lagrange remainder) becomes

$$f(a+h) = f(a) + hf'(a) + \frac{h^2}{2!} f''(a) + \ldots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + \frac{h^n}{n!} f^{(n)}(\xi),$$

where ξ is a certain number between $a, a+h$.

A convenient form is found by putting $a = 0$ and then renaming h to be the current variable x:

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n}{n!} f^{(n)}(\xi),$$

where ξ is a certain number between $0, x$. This important result is known as Maclaurin's theorem. Compare p. 43.

With the Cauchy form of remainder, the corresponding result is

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x),$$

where θ is a certain number between $0, 1$.

7. Maclaurin's series. The remainder R_n in Maclaurin's theorem appears in the form

$$\frac{x^n}{n!} f^{(n)}(\xi) \quad \text{or} \quad \frac{x^n(1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x),$$

where ξ lies between $0, x$ and θ between $0, 1$. Then

$$f(x) \equiv f(0) + xf'(0) + \ldots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n.$$

It may be possible to prove that, as n becomes larger and larger (x having a definite value for a particular problem) the remainder R_n tends to the limit zero. When this happens, the sum of the first n terms of the series

$$f(0) + xf'(0) + \frac{x^2}{2!} f''(0) + \ldots$$

tends to the limit $f(x)$, and so the sum to infinity of the series exists, and is $f(x)$.

The condition for the remainder to tend to zero may involve x, so that it is fulfilled for some values of x but not for others.

It is on this basis that the possibility of obtaining an expansion rests. The succeeding paragraphs give the details for a number of important functions.

8. The series for $\sin x$ and $\cos x$. Let

$$f(x) \equiv \sin x.$$

Then

$$f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x, \quad \ldots,$$

so that

$$f(0) = 0, \quad f'(0) = 1, \quad f''(0) = 0, \quad f'''(0) = -1, \quad \ldots.$$

Hence the Maclaurin series is

$$0 + x.1 + \frac{x^2}{2!}.0 + \frac{x^3}{3!}(-1) + \ldots,$$

or

$$x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots.$$

To see whether the series converges to $\sin x$, we consider R_n, the remainder after n terms, where

$$R_n \equiv \frac{x^n}{n!} f^{(n)}(\xi).$$

The numerical value $|f^{(n)}(\xi)|$ is certainly not greater than 1, since $f^{(n)}(\xi)$ is a sine or a cosine. Hence

$$|R_n| \leqslant \frac{|x^n|}{n!} = \frac{|x|^n}{n!}.$$

But (see p. 39) $\quad \lim_{n \to \infty} \frac{|x|^n}{n!} = 0,$

for all values of x, and so the series converges to $\sin x$ for all values of x.

By similar argument, we obtain the expansion

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots,$$

convergent for all values of x.

9. The binomial series. Let

$$f(x) \equiv (1+x)^p,$$

where p may be positive or negative, and not necessarily an integer. Then

$$f'(x) = p(1+x)^{p-1},$$
$$f''(x) = p(p-1)(1+x)^{p-2},$$
$$f'''(x) = p(p-1)(p-2)(1+x)^{p-3},$$

and, generally,

$$f^{(n)}(x) = p(p-1)\ldots(p-n+1)(1+x)^{p-n}.$$

Hence

$$f(0) = 1,$$
$$f'(0) = p,$$
$$f''(0) = p(p-1),$$
$$f'''(0) = p(p-1)(p-2),$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$f^{(n)}(0) = p(p-1)\ldots(p-n+1).$$

The Maclaurin series is thus

$$1 + px + \frac{p(p-1)}{2!} x^2 + \frac{p(p-1)(p-2)}{3!} x^3 + \ldots$$
$$+ \frac{p(p-1)\ldots(p-n+1)}{n!} x^n + \ldots.$$

If p is a positive integer, the series terminates, giving the expansion, familiar from any text-book on algebra,

$$(1+x)^p = 1 + c_1 x + c_2 x^2 + \ldots + c_p x^p,$$

where

$$c_n = \frac{p(p-1)\ldots(p-n+1)}{n!} = \frac{p!}{n!(p-n)!}.$$

If p is NOT *a positive integer*, we obtain an infinite series, and the conditions for convergence become important. We prove that *the expansion*

$$(1+x)^p = 1+px+\frac{p(p-1)}{2!}x^2+\ldots+\frac{p(p-1)\ldots(p-n+1)}{n!}x^n+\ldots,$$

where p is not a positive integer, is valid for values of x in the interval

$$-1<x<1.$$

The Cauchy form of remainder* gives

$$R_n = \frac{x^n(1-\theta)^{n-1}}{(n-1)!}p(p-1)\ldots(p-n+1)(1+\theta x)^{p-n}$$

$$= x^n\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}\frac{p(p-1)\ldots(p-n+1)}{(n-1)!}(1+\theta x)^{p-1}.$$

Now
$$1-\theta<1+\theta x$$
whether x is positive or negative, since $-1<x<1, 0<\theta<1$.

Hence
$$\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}<1.$$

Also, if $p>1$, $\quad (1+\theta x)^{p-1}<(1+|x|)^{p-1}$,

and, if $p<1$, $\quad (1+\theta x)^{p-1}=\frac{1}{(1+\theta x)^{1-p}}$

$$<\frac{1}{(1-|x|)^{1-p}}.$$

For any given x in the interval $-1<x<1$, the product

$$\left(\frac{1-\theta}{1+\theta x}\right)^{n-1}(1+\theta x)^{p-1}$$

is therefore less than an ascertainable positive number A.

Consider next the product

$$\frac{p(p-1)\ldots(p-n+1)}{(n-1)!}x^n,$$

which we denote by the symbol u_n. Then

$$\frac{u_{n+1}}{u_n}=\frac{p(p-1)\ldots(p-n+1)(p-n)}{n!}x^{n+1}\cdot\frac{(n-1)!}{p(p-1)\ldots(p-n+1)}\cdot\frac{1}{x^n}$$

$$=\frac{p-n}{n}x.$$

* The proof which follows may be postponed, if desired.

Take n so large as to be greater than p. Then

$$\left|\frac{u_{n+1}}{u_n}\right|=\frac{n-p}{n}|x|.$$

Write $y=\frac{1}{2}(1+|x|)$, so that $0<y<1$. Let n_0 be the first positive integer such that $(n-p)\,|x|/n<y$, that is, such that

$$n(1-|x|)>-2p|x|.$$

Then
$$\left|\frac{u_{n+1}}{u_n}\right|\cdot\left|\frac{u_n}{u_{n-1}}\right|\cdot\left|\frac{u_{n-1}}{u_{n-2}}\right|\ldots\left|\frac{u_{n_0+2}}{u_{n_0+1}}\right|\cdot\left|\frac{u_{n_0+1}}{u_{n_0}}\right|$$

$$<y.y.y.\ldots.y.y,$$

so that
$$|u_{n+1}|<|u_{n_0}|y^{n-n_0+1}\quad(n>n_0).$$

Since u_{n_0} is a definite ascertainable number, and since $|y|<1$, it follows that

$$|u_n|\to 0$$

as $n\to\infty$.

But
$$|R_n|<A|u_n|,$$
when A is the positive number already defined. Hence

$$|R_n|\to 0$$

as $n\to\infty$, and the validity of the expansion is established.

10. The logarithmic series.

10. The logarithmic series. Let

$$f(x)\equiv\log(1+x).$$

Then
$$f'(x)=\frac{1}{1+x},$$

$$f''(x)=-\frac{1}{(1+x)^2},$$

$$f'''(x)=\frac{2}{(1+x)^3},$$

$$f^{(\mathrm{iv})}(x)=-\frac{3.2}{(1+x)^4},$$

and generally, $\quad f^{(n)}(x)=(-1)^{n-1}\frac{(n-1)!}{(1+x)^n}.$

Hence $$f(0) = \log 1 = 0,$$

$$f^{(n)}(0) = (-1)^{n-1}(n-1)!$$

so that the Maclaurin series is

$$0+x.1+\frac{x^2}{2!}(-1)+\frac{x^3}{3!}(2!)+\frac{x^4}{4!}(-3!)+\dots,$$

or $$x-\tfrac{1}{2}x^2+\tfrac{1}{3}x^3-\tfrac{1}{4}x^4+\dots+(-1)^{n-1}\frac{1}{n}x^n+\dots.$$

This expansion for $\log(1+x)$ *is valid for all values of* x *in the range*

$$-1<x\leqslant 1.$$

When $x=1$, we have the result

$$\log 2 = 1-\tfrac{1}{2}+\tfrac{1}{3}-\tfrac{1}{4}+\dots.$$

The case $x=-1$ is reflected in the graph (Fig. 62, p. 4) where $y\to-\infty$ as $x\to 0$.

EXAMPLE IV

1. Use the method given for the binomial series to prove the validity of the logarithmic series when $-1<x<1$.

11. The exponential series.

Let

$$f(x) = e^x.$$

Then $$f'(x) = e^x,$$

$$f''(x) = e^x,$$

and, generally, $$f^{(n)}(x) = e^x.$$

Hence $$f(0) = f'(0) = f''(0) = \dots = 1.$$

We therefore have the Maclaurin series

$$1+x+\frac{x^2}{2!}+\frac{x^3}{3!}+\dots+\frac{x^n}{n!}+\dots.$$

It may be proved that *this expansion for* e^x *is valid for all values of* x. [See Examples V (1) below.]

Corollary. The work of this paragraph enables us to fill a gap in the discussion of the exponential function (pp. 18–20) by obtaining an expression for the number e. Putting $x=1$, we have

$$e = 1+1+\frac{1}{2!}+\frac{1}{3!}+\dots+\frac{1}{n!}+\dots.$$

EXAMPLES V

1. Use the method given for the sine series to prove the validity of the exponential series for all values of x.

By direct calculation of the differential coefficients and substitution in Maclaurin's formula, obtain expansions for the following functions as series of ascending powers of x:

2. e^{3x}.	3. $\log(1+2x)$.	4. $\sin 2x$.
5. $1/(1+x)$.	6. $1/(1-x)^2$.	7. $\cos 4x$.
8. e^{-2x}.	9. $\sqrt{(1+2x)}$.	10. $\log(1-3x)$.

12. Approximations.

If the successive terms in the expansion

$$f(x) \equiv a_0+a_1x+a_2x^2+\dots$$

become rapidly smaller, a good approximation to the value of the function $f(x)$ may be found by taking the first few terms.

Illustration 2. *To estimate* $\sqrt{(3{\cdot}98)}$.

Writing the expression in the form

$$(4-{\cdot}02)^{\frac{1}{2}} = 2(1-{\cdot}005)^{\frac{1}{2}},$$

we may expand by the binomial series (p. 51) to obtain

$$2\left\{1+\tfrac{1}{2}(-{\cdot}005)+\frac{\frac{1}{2}(-\frac{1}{2})}{2!}(-{\cdot}005)^2+\frac{\frac{1}{2}(-\frac{1}{2})(-\frac{3}{2})}{3!}(-{\cdot}005)^3+\dots\right\}$$

$$= 2\{1-{\cdot}0025-\tfrac{1}{8}({\cdot}000025)+\tfrac{1}{16}({\cdot}000000125)\dots\},$$

and a good approximation is

$$2(1-{\cdot}0025)$$

$$= 1{\cdot}9950.$$

A limit may be set to the error by means of Taylor's theorem, as follows:

Since, for a general function $f(x)$,

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(\theta x) \quad (0 < \theta < 1),$$

the error cannot exceed the greatest value of

$$\left|\frac{x^2}{2!}f''(\theta x)\right|,$$

or, here, $$2 \times \frac{(\cdot 005)^2}{2!}\cdot\frac{1}{4(1-\cdot 005\theta)^{\frac{3}{2}}} \equiv \frac{\cdot 00000625}{(1-\cdot 005\theta)^{\frac{3}{2}}}.$$

In the most unfavourable case, with $\theta = 1$, this gives a value less than $\cdot 0000063$ for the error.

EXAMPLES VI

Estimate the values of the following expressions:

1. $\sqrt{(4\cdot 02)}$.
2. $\sqrt{(8\cdot 97)}$.
3. $\sqrt[3]{(8\cdot 02)}$.
4. $\sqrt[3]{(26\cdot 98)}$.
5. $\sqrt[5]{(31\cdot 97)}$.
6. $1/\sqrt{(9\cdot 03)}$.

13. Newton's approximation to a root of an equation.

Suppose that we are given an equation

$$f(x) = 0$$

and know that there is a root somewhere near the value $x = a$. Newton's method, which we now describe, shows that, *under suitable circumstances, a better approximation to the root is*

$$a - \frac{f(a)}{f'(a)}.$$

If the correct root being sought is $\xi \equiv a + h$, then

$$f(a+h) = f(\xi) = 0,$$

so that, by Taylor's theorem for $n = 2$, with Lagrange's form of the remainder,

$$f(a) + hf'(a) + \frac{h^2}{2!}f''(\eta) = 0$$

for a value of η between a and $a+h$. If h is reasonably small, then the term involving h^2 may be regarded as negligible for practical purposes. We therefore have

$$f(a) + hf'(a) = 0,$$

or $$h = -\frac{f(a)}{f'(a)},$$

so that the root is approximately

$$a - \frac{f(a)}{f'(a)}.$$

This crude statement, however, should be supplemented by more careful analysis, and the following graphical treatment shows the precautions which ought to be taken.

We begin with an examination of the curve

$$y = f(x)$$

near the point ξ, taking first the case where the gradient is positive and the concavity 'upwards' near that point, so that $f'(x), f''(x)$ are both positive.

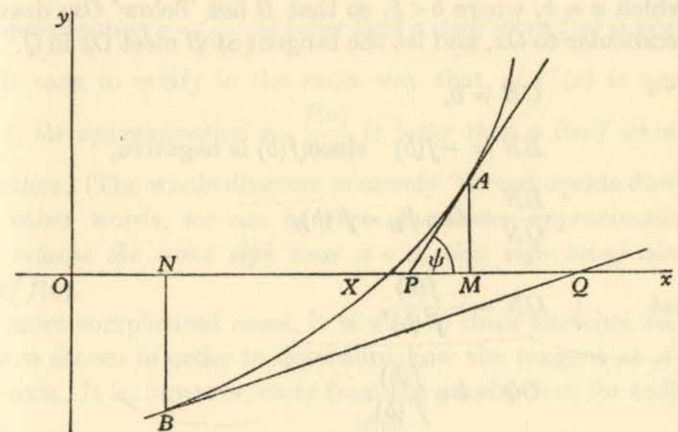


Fig. 66.

(i) Let X be the point $(\xi, 0)$ on the curve, and let A be the point for which $x = a$, where $a > \xi$; draw AM perpendicular to Ox, and let the tangent at A meet Ox in P.

Then $\qquad OM = a,$

$$AM = f(a),$$

$$\frac{AM}{PM} = \tan\psi_A = f'(a),$$

so that
$$PM = \frac{f(a)}{f'(a)},$$

and
$$OP = a - \frac{f(a)}{f'(a)}.$$

But under our assumptions that $f'(x), f''(x)$ are positive near X (so that the gradient is positive and the concavity upwards) the tangent AP lies between AM and the curve, so that P lies between X and M. Hence, under these conditions, *OP is a better approximation than OM to OX*. That is, $a - \dfrac{f(a)}{f'(a)}$ is a better approximation than a to the root.

It is assumed that $f'(a)$ is not zero, and, indeed, that $f'(x)$ is not zero near the required root.

(ii) Suppose next that, with the same diagram, B is the point for which $x = b$, where $b < \xi$, so that B lies 'below' Ox; draw BN perpendicular to Ox, and let the tangent at B meet Ox in Q. Then

$$ON = b,$$

$$BN = -f(b) \quad \text{since } f(b) \text{ is negative},$$

$$\frac{BN}{QN} = \tan\psi_B = f'(b),$$

so that
$$QN = -\frac{f(b)}{f'(b)},$$

and
$$OQ = b - \frac{f(b)}{f'(b)}.$$

But now we cannot be sure that Q is nearer to X than N, for Q may be anywhere to the right of X according to the shape of the curve. Hence *we cannot be sure whether $b - \dfrac{f(b)}{f'(b)}$ is, or is not, a better approximation than b to the root.*

Similar argument applied to the accompanying diagram (Fig. 67) shows that the results (i), (ii) are also true if (the concavity still being 'upwards') the gradient is negative near X; P is closer than M to X, whereas Q may or may not be closer than N to X.

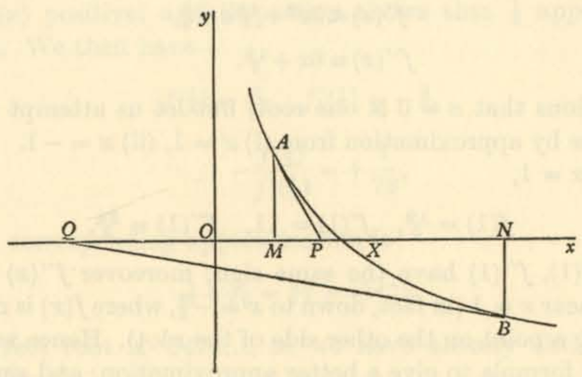


Fig. 67.

We have therefore proved, so far, that, *if $f''(x)$ is positive near ξ, the approximation $a - \dfrac{f(a)}{f'(a)}$ is better than a itself when $f(a)$ is positive.*

It is easy to verify in the same way that, *if $f''(x)$ is negative near ξ, the approximation $a - \dfrac{f(a)}{f'(a)}$ is better than a itself when $f(a)$ is negative.* (The whole diagram is merely 'turned upside down'.)

In other words, *we can be sure of a better approximation if $f''(x)$ retains the same sign near $x = a$, that sign being also the sign of $f(a)$.*

In more complicated cases, it is wise to draw sketches such as we have shown in order to determine how the tangent at A cuts the x-axis. It is, however, clear from the graphs that, for ordinary functions, the approximation $a - \dfrac{f(a)}{f'(a)}$ is ALWAYS better than a itself once we come sufficiently close to the correct answer. In other words, once it is ascertained that an approximation is reasonably good, it may confidently be expected that Newton's approximation will make it better still.

ILLUSTRATION 3. An example where the answer is apparent from the outset may help to make the principle clear. Consider the equation

$$x^3+\tfrac{8}{3}x^2+\tfrac{8}{3}x=0,$$

so that

$$f(x)\equiv x^3+\tfrac{8}{3}x^2+\tfrac{8}{3}x,$$

$$f'(x)\equiv 3x^2+\tfrac{16}{3}x+\tfrac{8}{3},$$

$$f''(x)\equiv 6x+\tfrac{16}{3}.$$

It is obvious that $x=0$ is one root, but let us attempt to reach that value by approximation from (i) $x=1$, (ii) $x=-1$.

When $x=1$,

$$f(1)=\tfrac{19}{3},\quad f'(1)=11,\quad f''(1)=\tfrac{34}{3},$$

so that $f(1)$, $f''(1)$ have the same sign; moreover $f''(x)$ remains positive near $x=1$ (in fact, down to $x=-\frac{8}{9}$, where $f(x)$ is negative, indicating a point on the other side of the root). Hence we expect Newton's formula to give a better approximation; and since

$$1-\frac{f(1)}{f'(1)}=1-\frac{19}{33}=\frac{14}{33},$$

this is actually the case.

On the other hand, when $x=-1$,

$$f(-1)=-1,\quad f'(-1)=\tfrac{1}{3},\quad f''(-1)=-\tfrac{2}{3}.$$

Here, again, $f(-1)$, $f''(-1)$ have the same sign; but $f''(x)$ changes sign from negative to positive at $x=-\frac{8}{9}$, which is near -1; we are therefore in a doubtful region; and since

$$-1-\frac{f(-1)}{f'(-1)}=-1+3=2,$$

the approximation is actually worse.

ILLUSTRATION 4. *To find an approximation to that root of the equation*

$$x^3-3x+1=0$$

which lies between 0, 1.

We have

$$f(x)\equiv x^3-3x+1,$$

$$f'(x)\equiv 3x^2-3,$$

$$f''(x)=6x.$$

Since $f''(x)$ is positive when x lies in the given interval 0, 1, it is advisable to begin with an approximation which makes $f(x)$ also positive.

Now

$$f(0)=1,\quad f(\tfrac{1}{4})=\tfrac{17}{64},\quad f(\tfrac{1}{2})=-\tfrac{3}{8}.$$

We should therefore like an approximation between $\frac{1}{4}$ and $\frac{1}{2}$ which keeps $f(x)$ positive, and inspection shows that $\frac{1}{3}$ appears very suitable. We then have

$$f(\tfrac{1}{3})=\tfrac{1}{27},\quad f'(\tfrac{1}{3})=-\tfrac{8}{3},$$

so that

$$-\frac{f(\frac{1}{3})}{f'(\frac{1}{3})}=+\frac{1}{72},$$

and the corresponding approximation is

$$\tfrac{1}{3}+\tfrac{1}{72}=\tfrac{25}{72}=\cdot 3472.$$

The correct root is ·3472..., so we have already obtained four correct figures.

ILLUSTRATION 5. *If η is a small positive number, to find an approximation to that root of the equation*

$$\sin x=\eta x$$

which lies near to $x=\pi$.

(The intersection of the graphs $y=\sin x$, $y=\eta x$ shows that there *is* a root near to π.)

Since $\sin\pi=0$ and η is small, the approximation $x=\pi$ is reasonably good.

Write

$$f(x)\equiv\sin x-\eta x,$$

so that

$$f'(x)=\cos x-\eta,$$

$$f''(x)=-\sin x.$$

Then

$$f(\pi)=-\eta\pi,\quad f'(\pi)=-1-\eta,\quad f''(\pi)=0.$$

Although $f''(\pi)$ is actually zero, so that the curve (Fig. 68) $y=\sin x-\eta x$ has an inflexion at $x=\pi$, the concavity is 'downwards' in the interval $0,\pi$ and $f(\pi)$ is negative; moreover the gradient $f'(\pi)$ is also negative. The accompanying sketch shows that the tangent lies between the ordinate $x=\pi$ and the curve, so that Newton's method will improve the approximation.

The corresponding solution is

$$\pi - \frac{f(\pi)}{f'(\pi)}$$

$$= \pi - \frac{(-\eta\pi)}{(-1-\eta)} = \pi - \frac{\eta\pi}{1+\eta} = \frac{\pi}{1+\eta}.$$

To the first order in η this is, on expansion of $(1+\eta)^{-1}$,

$$\pi(1-\eta).$$

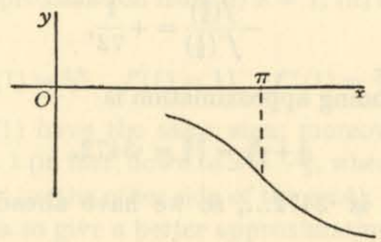


Fig. 68.

Note. This solution is less than π, as we expected from the diagram.

14. Leibniz's theorem. The theorem which follows is useful in calculating the higher differential coefficients necessary for a Maclaurin expansion.

To prove that, *if $f(x)$ is the product of two functions u, v, so that*

$$f(x) = uv,$$

then

$$f^{(n)}(x) = u^{(n)}v + {}_nc_1 u^{(n-1)}v' + {}_nc_2 u^{(n-2)}v'' + \dots$$
$$+ {}_nc_p u^{(n-p)}v^{(p)} + \dots + {}_nc_{n-1}u'v^{(n-1)} + {}_nc_n uv^{(n)},$$

where ${}_nc_p$ is the binomial coefficient

$${}_nc_p \equiv \frac{n!}{p!(n-p)!}.$$

We use the method of mathematical induction, assuming the result to be true for a certain integer N, so that

$$f^{(N)}(x) = u^{(N)}v + \dots + {}_Nc_p u^{(N-p)}v^{(p)} + \dots + {}_Nc_N uv^{(N)}.$$

Now differentiate this expression to obtain $f^{(N+1)}(x)$. The differential coefficient of a product such as $u^{(N-p)}v^{(p)}$ is

$$u^{(N-p+1)}v^{(p)} + u^{(N-p)}v^{(p+1)}.$$

As we write down these terms for the series on the right, we put the answer in two lines, the top line consisting of terms such as $u^{(N-p+1)}v^{(p)}$ and the lower of $u^{(N-p)}v^{(p+1)}$; also we displace the lower line one place to the right, thus:

$$u^{(N+1)}v + \dots\dots + {}_Nc_{p-1}u^{(N-p+2)}v^{(p-1)} + {}_Nc_p\, u^{(N-p+1)}v^{(p)} + \dots\dots\dots + u'v^{(N)}$$
$$+ u^{(N)}v' + \dots\dots\dots + {}_Nc_{p-1}u^{(N-p+1)}v^{(p)} + {}_Nc_p u^{(N-p)}v^{(p+1)} + \dots\dots + uv^{(N+1)}$$

Now the coefficient of $u^{(N-p+1)}v^{(p)}$ is

$${}_Nc_p + {}_Nc_{p-1}$$

$$= \frac{N!}{p!(N-p)!} + \frac{N!}{(p-1)!(N-p+1)!}$$

$$= \frac{N!}{p!(N-p+1)!}\{(N-p+1)+p\}$$

$$= \frac{(N+1)!}{p!(N+1-p)!}$$

$$= {}_{N+1}c_p.$$

Hence

$$f^{(N+1)}(x) = u^{(N+1)}v + \dots + {}_{N+1}c_p u^{(N+1-p)}v^{(p)} + \dots + uv^{(N+1)}.$$

It follows that, if the theorem is true for any particular value N, then it is true for $N+1, N+2$, and all subsequent values. But it is easily established when $N = 1$, being merely the result

$$f'(x) = u'v + uv'.$$

It is therefore true generally.

Note. The expression is symmetrical when regarded from the two ends, and will equally well be written in the form

$$f^{(n)}(x) = uv^{(n)} + \dots + {}_nc_p u^{(p)}v^{(n-p)} + \dots + u^{(n)}v.$$

EXAMPLES VII

Use Leibniz's theorem to find the following differential coefficients:

1. $\dfrac{d^4}{dx^4}(x^8 \sin x)$.
2. $\dfrac{d^4}{dx^4}(x^2 \sin x)$.
3. $\dfrac{d^5}{dx^5}(e^{2x} \cos 3x)$.
4. $\dfrac{d^6}{dx^6}(x^3 e^{3x})$.
5. $\dfrac{d^n}{dx^n}(x^3 \cos x) \quad (n > 3)$.
6. $\dfrac{d^n}{dx^n}(x^3 e^{2x}) \quad (n > 3)$.
7. $\dfrac{d^5}{dx^5}\{x^2(1-2x)^{10}\}$.
8. $\dfrac{d^8}{dx^8}\{x^2(3x+1)^{12}\}$.

ILLUSTRATION 6. *To apply Leibniz's theorem in finding a Maclaurin expansion for*

$$f(x) \equiv \log\{x + \sqrt{(x^2+1)}\}.$$

We have

$$f'(x) = \frac{1}{x+\sqrt{(x^2+1)}} \cdot \left\{1 + \tfrac{1}{2} \cdot \frac{2x}{\sqrt{(x^2+1)}}\right\}$$

$$= \frac{1}{\sqrt{(x^2+1)}}$$

on simplification. Hence

$$(x^2+1)\{f'(x)\}^2 = 1.$$

Differentiate. Then

$$(x^2+1) \cdot 2f'(x)f''(x) + 2x\{f'(x)\}^2 = 0,$$

or

$$(x^2+1)f''(x) + xf'(x) = 0.$$

Differentiate n times, using Leibniz's theorem. Write the expansion from $(x^2+1)f''(x)$ on the first line, and from $xf'(x)$ on the second; note that the expansions terminate since $(x^2+1)''' = 0$, $x'' = 0$. We obtain

$$(x^2+1)f^{(n+2)}(x) + n \cdot 2x \cdot f^{(n+1)}(x) + \frac{n(n-1)}{2!} \cdot 2 \cdot f^{(n)}(x)$$

$$+ \quad x \cdot f^{(n+1)}(x) + \quad n \quad \cdot 1 \cdot f^{(n)}(x) = 0.$$

We require values when $x = 0$, so that

$$f^{(n+2)}(0) + \left\{\frac{2n(n-1)}{2!} + n\right\} f^{(n)}(0) = 0,$$

or

$$f^{(n+2)}(0) = -n^2 f^{(n)}(0).$$

But $\qquad f(0) = \log 1 = 0,$

and so $\qquad f''(0) = f^{(\text{iv})}(0) = \ldots = f^{(2n)}(0) = 0.$

Also $\qquad f'(0) = 1,$

so that $\qquad f'''(0) = -1^2 \cdot 1,$

$$f^{(\text{v})}(0) = +3^2 \cdot 1,$$

$$f^{(\text{vii})}(0) = -5^2 \cdot 3^2 \cdot 1,$$

and so on. Hence the Maclaurin expansion is

$$x \cdot 1 + \frac{x^3}{3!}(-1) + \frac{x^5}{5!}(3^2 \cdot 1) + \frac{x^7}{7!}(-5^2 \cdot 3^2 \cdot 1) + \ldots,$$

or

$$x - \frac{1}{2}\frac{x^3}{3} + \frac{1 \cdot 3}{2 \cdot 4}\frac{x^5}{5} - \frac{1 \cdot 3 \cdot 5}{2 \cdot 4 \cdot 6}\frac{x^7}{7} + \ldots,$$

converging to $\log\{x + \sqrt{(x^2+1)}\}$ for such values of x (not considered here) as make the series convergent.

EXAMPLES VIII

Use the method of Illustration 6 to obtain the following expansions:

1. $f(x) \equiv \sin^{-1} x$.
2. $f(x) \equiv \tan^{-1} x$.
3. Prove that, if $\quad f(x) \equiv \cos(m \sin^{-1} x)$, then

$$(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) + (m^2-n^2)f^{(n)}(x) = 0.$$

4. Prove that, if $f(x) \equiv (\sin^{-1} x)^2$, then

$$(1-x^2)f^{(n+2)}(x) - (2n+1)xf^{(n+1)}(x) - n^2 f^{(n)}(x) = 0.$$

REVISION EXAMPLES IV

'Advanced' Level

1. Differentiate

(i) $e^{ax^2 \log bx}$, (ii) $\sin^{-1}\left\{\dfrac{ax}{\sqrt{(1+a^2x^2)}}\right\}$.

If $y = e^{a \sin bx}$, and y', y'' are the first and second differential coefficients of y with respect to x show that

$$(yy''-y'^2)^2 = b^2y^2(a^2b^2y^2-y'^2).$$

Calculate the values of y', y'' when $x = 0$.

2. Differentiate

$$\tan^{-1}\frac{1-2x}{2+x}, \quad \sin^{-1}\sqrt{\left(\frac{1-x}{1+x}\right)},$$

$$\log(\sec x+\tan x), \quad \frac{2x+1}{x^2-8x-2}.$$

Find the values of x for which

$$\frac{2x+1}{x^2-8x-2}$$

is a maximum or minimum, distinguishing the maximum from the minimum.

3. If $y = e^x \tan x$, prove that

$$\frac{d^2y}{dx^2}-2(1+\tan x)\frac{dy}{dx}+(1+2\tan x)\,y = 0.$$

Prove that $y = ue^x \tan x$ also satisfies this equation when u is a function of x such that

$$\tan x\frac{d^2u}{dx^2}+2\frac{du}{dx} = 0.$$

Verify that $u = \cot x$ is such a function.

4. Differentiate $y = \sec^m x \tan^n x$.

Deduce that the k^{th} differential coefficient of $\sec^m x$ can be expressed in the form $\sec^m x\, P_k(\tan x)$, where $P_k(\tan x)$ is a polynomial in $\tan x$ of degree k.

Evaluate $P_4(\tan x)$ when both m and k are taken equal to 4.

5. Differentiate with respect to x:

$$\frac{1}{x-2}+2\log\frac{x-3}{x-2}, \quad x-\tan x+\tfrac{1}{3}\tan^3 x,$$

simplifying your answers as much as you can.

Prove that the first function has a maximum at $x = 1$ and that the second function (whose differential coefficient vanishes at $x = 0$) has neither a maximum nor a minimum at $x = 0$.

6. Find the first four differential coefficients of $\sin^4 x$.

Show that the function

$$x^4(x-\tfrac{1}{2}\pi)^2+\sin^4 x$$

has turning values at $x = 0$ and at $x = \frac{1}{2}\pi$, and determine whether they are maxima or minima.

7. Prove that, if $y = \sin^2(x^2)$, then

$$x\frac{d^2y}{dx^2}-\frac{dy}{dx}+16x^3y = 8x^3.$$

Prove that the result of changing the independent variable in this equation from x to ξ, where $\xi = x^2$, is

$$\frac{d^2y}{d\xi^2}+ay = b,$$

where a, b are constants to be determined.

Prove also that, if A, B are any constants, the function

$$y = \tfrac{1}{2}+A\cos(2x^2)+B\sin(2x^2)$$

satisfies the first differential equation.

8. Differentiate

$$\left(x+\frac{1}{x}\right)^n, \quad (\cos x+\sec x)^n$$

with respect to x, and find the nth differential coefficient of

$$\left(x+\frac{1}{x}\right)^2$$

for all positive integral values of n, distinguishing the cases $n \leqslant 2$ and $n > 2$.

9. Show that the polynomial

$$f(x) \equiv x - \frac{4}{\pi^3}(2\pi - 5)x^3 + \frac{16}{\pi^5}(\pi - 3)x^5$$

and its differential coefficient $f'(x)$ have the same values as $\sin x$ and its differential coefficient respectively, at the values $x = 0$, $x = \pm\frac{1}{2}\pi$.

Show that the error in using $\int_0^{\frac{1}{2}\pi} f(x)dx$ as an approximation for $\int_0^{\frac{1}{2}\pi} \sin x\,dx$ is less than 0·1 per cent.

10. (i) Given that $\dfrac{du}{dx} = e^x$, $\dfrac{dv}{dx} = \sin x$,

and that $u = 1, v = 0$ when $x = 0$, show that

$$\frac{d^2(uv)}{dx^2} = 1$$

when $x = 0$.

(ii) Given that $y = ax^n + bx^{1-n}$, prove that

$$x\frac{dy}{dx} + (n-1)y = (2n-1)ax^n,$$

and form an equation in $x, y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ which does not contain a or b.

11. Differentiate with respect to x:

$$\sqrt{\left(\frac{x-1}{x+1}\right)}, \quad \log \sec x.$$

Prove that $$\frac{d^n}{dx^n}\left(\frac{1}{1-x}\right) = \frac{n!}{(1-x)^{n+1}},$$

and find the nth differential coefficient of

$$\frac{x}{(1-x)(1-2x)}.$$

12. Find the nth derivatives with respect to x of

(i) $\dfrac{1}{x}$ and $\dfrac{1}{x^2-1}$;

(ii) $\sin x$ and $x \sin x$.

13. (i) Find the derivative of $y = \sin^2 x$ with respect to $z = \cos x$ by evaluating the limit of $\delta y/\delta z$.

(ii) Differentiate with respect to x:

$$\frac{\sin x}{1+\tan x}, \quad (1-x)\sqrt{(1+x^2)}, \quad \log_e(e^x + e^{-x}).$$

(iii) Prove that $$\frac{d^2x}{dy^2} = -\frac{d^2y}{dx^2}\bigg/\left(\frac{dy}{dx}\right)^3.$$

14. Find from first principles the differential coefficient of $1/x^3$ with respect to x.

What is the differential coefficient of $1/x^3$ with respect to x^2?

Differentiate with respect to x:

$$\log_e \cos x, \quad x\sqrt{(1-x^2)}, \quad \frac{e^x}{(1+x^2)^2}.$$

15. Find the nth differential coefficient of y with respect to x in each of the following cases:

(i) $y = 1/x^2$, (ii) $y = \sin 2x$, (iii) $y = e^{2x}\sin 2x$.

There are three values of k for which the function $y = x^k e^{-x}$ satisfies the equation

$$\frac{d^3y}{dx^3} + 3\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + y = 0.$$

Find these values.

16. Differentiate with respect to x:

$$\frac{1-2x}{(1+3x)^2}, \quad e^{\tan^2 x}, \quad \sin^{-1}\frac{1}{\sqrt{(1+x^2)}}.$$

Prove that, when $y = at^2 + 2bt + c$, $t = ax^2 + 2bx + c$, and a, b, c are constants,

$$\frac{d^3y}{dx^3} = 24a^2(ax+b).$$

17. Differentiate $\quad e^{\tan x}, \quad \sqrt{\left(\dfrac{x+1}{x-2}\right)}.$

Show that, if $y = \sin\theta$, $x = \cos\theta$, then

$$\frac{d}{dx}(y^3) = -\frac{3}{2}\sin 2\theta, \quad \frac{d^2}{dx^2}(y^5) = 5\sin 3\theta.$$

18. A particle moves along the axis of x so that its distance from the origin t seconds after starting is given by the formula

$$x = a\cos kt + \tfrac{1}{2}akt,$$

where a, k are positive constants. Find expressions for the velocity and acceleration in terms of t, and prove that the particle is not always moving in the same direction along the axis.

Find the positions of the particle at the two times $t = \pi/6k$ and $t = 13\pi/6k$; also find the position of the particle at the instant between these times at which it is momentarily at rest, and deduce the *total* distance travelled between the two times.

19. A particle moves in a plane so that its coordinates at time t are given by $x = e^t \cos t, y = e^t \sin t$. Find the magnitudes of the velocity and of the acceleration* at time t and prove that the acceleration is always at right angles to the radius vector.

Draw a rough sketch of the path of the particle, from time $t = 0$ to $t = \pi$, and indicate the direction of motion at times 0, $\frac{1}{4}\pi, \frac{1}{2}\pi, \frac{3}{4}\pi, \pi$.

20. A particle moves in a plane so that its position at time t is given by $x = a\cos pt, y = b\sin pt$. Find expressions for (i) the magnitude v of the velocity at time t; (ii) the magnitude f of the acceleration at time t; and prove that there is no value of t for which $f = dv/dt$.

Prove also that the resultant acceleration makes an angle θ with the normal to the path, where

$$2ab\tan\theta = (a^2 - b^2)\sin 2pt.$$

21. A point moves in a straight line so that its distance at time t from a given point O of the line is x, where

$$x = t^2 \sin t + 6t\cos t - 12\sin t.$$

Find its velocity at time t, and prove that the acceleration is then

$$-t^2\sin t - 2t\cos t + 2\sin t.$$

Determine the times $(t > 0)$ at which the acceleration has a turning value, distinguishing between maxima and minima.

* If v, f are the velocity and acceleration respectively, then $v^2 = \dot{x}^2 + \dot{y}^2$, $f^2 = \ddot{x}^2 + \ddot{y}^2$.

22. A particle moves along the axis of x so that its distance from the origin t seconds after starting is given by the formula $x = a\cos pt$. Prove that the velocity of the particle changes direction once, and only once, between the times $t = 0$, $t = 2\pi/p$ and that the change of direction occurs at the point $x = -a$.

The distance from the origin of a second particle is given by the formula

$$x = a\cos pt + \tfrac{1}{2}a\cos 2pt.$$

Write down expressions for its velocity and acceleration at time t. Show that between $t = 0$ and $t = 2\pi/p$ the velocity of the particle changes direction three times, and find the values of x at which these changes occur.

23. Prove that the equation of the tangent to the curve given by

$$x = 3t^2 + 1, \quad y = 2t^3 - 1$$

at the point where $t = \tan\alpha$ is

$$y - x\tan\alpha + \tan^3\alpha + \tan\alpha + 1 = 0.$$

Show that the curve lies on the positive side of the line $x = 1$ and is symmetrical about the line $y = -1$, and prove that the area bounded by the line $x = 4$ and the curve is $\frac{24}{5}$.

24. Prove that there are two distinct tangents to the curve

$$y = x^4 - x + 3$$

which pass through the origin. Find their equations, and their points of contact with the curve.

Give a rough sketch of the curve.

25. A curve is defined by the parametric equations

$$x = a(1-t^2), \quad y = a(2-t)(1-t),$$

where a is a positive constant. Prove that

(i) the curve passes through the points A, O, B whose coordinates are $(0, 6a), (0, 0), (-3a, 0)$.

(ii) the point t is in the first quadrant when $-1 < t < 1$ and in the third quadrant when $1 < t < 2$.

Make a rough sketch showing the part of the curve corresponding to values of t between -1 and $+2$.

Find the equation of the tangent to the curve at O, and prove that the area bounded by the arc AB and the chord AB is $27a^2/2$.

26. Prove that the slope of the curve whose equation is

$$y = 1 + x + \tfrac{1}{2}x^2 + \tfrac{1}{6}x^3$$

is always positive.

Show that the curve has a point of inflexion where $x = -1$, the slope there being $\frac{1}{2}$.

Prove also that the tangent at the point $(0, 1)$ meets the curve again at the point $(-3, -2)$.

Sketch the curve, indicating clearly the point of inflexion and the tangents at the points $(-1, \frac{1}{3})$ and $(0, 1)$.

27. A, O, B are three fixed points in order on a straight line, and $AO = p$, $OB = q$. A fixed circle has centre O and radius a greater than p or q, and P is a point on this circle. Show that the perimeter of the triangle APB is a maximum when OP bisects the angle APB, and find the corresponding magnitude of the angle POA.

28. Prove that the maximum and minimum values of the function $y = x\cos 3x$ occur when $3\tan 3x = 1/x$, and discuss the behaviour of the function when $x = 0$.

By considering the curves $y = \tan 3x$, $y = 1/x$, show that maximum values occur near the values $x = \frac{2}{3}k\pi$, and minimum values near $x = \frac{1}{3}(2k+1)\pi$, the approximation becoming more exact as x becomes larger.

29. Given that $a^2x^4 + b^2y^4 = c^6$, where a, b, c are constants, show that xy has a stationary value $c^3/\sqrt{(2ab)}$. Is this value a maximum or a minimum?

30. (i) A right circular cylinder is inscribed in a given right circular cone. Prove that its volume is a maximum when its altitude is one-third that of the cone.

(ii) A right circular cone of height h stands on a base of radius $h\tan\alpha$. A cylinder of height $h - x$ is inscribed in the cone. Prove that S, the total surface of the cylinder, is equal to

$$2\pi\{x^2(\tan^2\alpha - \tan\alpha) + xh\tan\alpha\},$$

and prove that, when $\tan\alpha > \frac{1}{2}$, S increases steadily as x increases.

31. P is a variable point in the circumference of a fixed circle of which AB is a fixed diameter and O is the centre. Prove that, when OP is perpendicular to AB, then the function $AP + PB$ has a maximum value and the function $AP^3 + PB^3$ has a minimum value.

32. If $f(x)$ is a function of x and $f'(x)$ is its differential coefficient, show that, when h is small, $f(a+h)$ is approximately equal to $f(a) + hf'(a)$.

Without using trigonometrical tables, find to three significant figures (i) the value of $\cos 31°$, and (ii) the positive acute angle whose sine is $0{\cdot}503$. Give your answer to (ii) in degrees and tenths of a degree.

33. Calculate

(i) $\sqrt[3]{8{\cdot}05}$ to 4 significant figures,

(ii) $\cos 59°$ to 3 significant figures.

[Take π to be $3{\cdot}142$.]

34. Determine to two places of decimals that root of the equation

$$\frac{x^4}{x+2} = 3{\cdot}2104$$

whose value is nearly equal to 8.

35. Determine to 3 places of decimals the value of that root of the equation

$$x^3 - 3x + 1 = 0$$

which lies between $1{\cdot}5$ and $1{\cdot}6$.

36. Prove, graphically or otherwise, that, if n is a large positive integer, there is a root of the equation $x\sin x = 1$ nearly equal to $2n\pi$. Show that a better approximation is $2n\pi + (1/2n\pi)$.

37. Prove that, if η is small, the equation

$$\theta + \sin\theta\cos\theta = 2\eta\cos\theta$$

has a small root approximately equal to

$$\eta - \tfrac{1}{6}\eta^3.$$

[You may assume the power-series expansions for $\sin\theta$ and $\cos\theta$.]

38. Prove that the result of differentiating the equation

$$(1+x^2)\frac{dy}{dx}-2x = 0$$

$n+1$ times $(n \geqslant 1)$ with respect to x is

$$(1+x^2)\frac{d^{n+2}y}{dx^{n+2}}+2(n+1)\,x\frac{d^{n+1}y}{dx^{n+1}}+n(n+1)\frac{d^n y}{dx^n} = 0.$$

Hence verify that, if k is a positive integer and

$$z = \frac{d^k}{dx^k}\log(1+x^2),$$

then z is a solution of the equation

$$(1+x^2)\frac{d^2z}{dx^2}+2(k+1)x\frac{dz}{dx}+k(k+1)z = 0.$$

39. By using Maclaurin's theorem, or otherwise, obtain the expansion of $\log(1+\sin x)$ in ascending powers of x as far as the term in x^4.

40. Prove that, if $y = \log_e \cos x$, then

$$\frac{d^3y}{dx^3}+2\frac{d^2y}{dx^2}\frac{dy}{dx} = 0.$$

Hence, or otherwise, obtain the Maclaurin expansion of $\log_e \cos x$ as far as the term in x^4.

Deduce the approximate relation

$$\log_e 2 = \frac{\pi^2}{16}\left(1+\frac{\pi^2}{96}\right).$$

41. Prove that, if $y = e^{\tan x}$, then

$$\frac{dy}{dx} = y(1+t^2), \quad \frac{d^2y}{dx^2} = \frac{dy}{dx}(1+t)^2,$$

where $t \equiv \tan x$.

Prove that the expansion of y as far as the term in x^3 is

$$y = 1+x+\tfrac{1}{2}x^2+\tfrac{1}{2}x^3.$$

42. Prove that, if $y = \sin(\log x)$, where $x > 0$, then

$$x^2\frac{d^2y}{dx^2}+x\frac{dy}{dx}+y = 0.$$

By induction, or otherwise, prove that

$$x^2\frac{d^{n+2}y}{dx^{n+2}}+(2n+1)\,x\frac{d^{n+1}y}{dx^{n+1}}+(n^2+1)\frac{d^n y}{dx^n} = 0,$$

where n is a positive integer.

43. If $y = e^x \log x$, show that

$$x\frac{d^2y}{dx^2}-(2x-1)\frac{dy}{dx}+(x-1)\,y = 0.$$

Find the equation obtained by differentiating this equation n times.

44. Prove that, when $y = e^{ax}\sin bx$,

$$\frac{d^2y}{dx^2}-2a\frac{dy}{dx}+(a^2+b^2)\,y = 0,$$

and that $\qquad \dfrac{dy}{dx} = y(a+b\cot bx).$

Prove that, if $\qquad e^{ax}\sin bx = \sum_{n=1}^{\infty}\frac{c_n}{n!}x^n,$

then $\qquad c_{n+2}-2ac_{n+1}+(a^2+b^2)c_n = 0,$

and find the values of c_1, c_2, c_3.

45. If $\cos y = \cos\alpha\cos x$, where x, y, α lie between 0 and $\frac{1}{2}\pi$ radians and α is constant, find the values of $y, \dfrac{dy}{dx}, \dfrac{d^2y}{dx^2}$ when $x = 0$.

Taking x to be so small that x^3 and higher powers of x are negligible, use Maclaurin's theorem to show that

$$y = \alpha+\tfrac{1}{2}x^2\cot\alpha.$$

Hence calculate y in degrees, correct to $0{\cdot}001°$, if $\alpha = 45°$ and $x = 1°\,48'$. [Take $\pi = 3{\cdot}142$.]

46. By using Taylor's theorem obtain the expansion of

$$\tan(x+\tfrac{1}{4}\pi)$$

in powers of x up to the term in x^3.

Hence calculate the value of tan 44° 48′ correct to four places of decimals. [Take $\pi = 3{\cdot}142$.]

47. Prove that, if $y = \log_e\left(\frac{1+\sin x}{1-\sin x}\right)$, then

$$\text{(i)}\quad \frac{dy}{dx} = 2\sec x,$$

and (ii) the expansion of y in a series of powers of x as far as the term in x^4 is $2x+\frac{1}{3}x^3$.

Find, correct to four significant figures, the value of y when $x = 1°\ 48'$, taking $\pi = 3{\cdot}142$.

48. Prove that, if $y = (\sin^{-1}x)^2$, then

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 = 4y,$$

and

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 2.$$

The Maclaurin expansion of y in powers of x is taken to be

$$y = a_0 + a_1x + \frac{a_2}{2!}x^2 + \ldots + \frac{a_n}{n!}x^n + \ldots.$$

Given that $y = 0$ when $x = 0$, prove that $a_0 = 0$ and $a_1 = 0$. Prove also that $a_{n+2} = n^2a_n$ when $n > 0$, and hence show that

$$y = x^2 + \sum_{n=2}^{\infty}\frac{2^{2n-1}(n-1)!}{n(n+1)\ldots(2n)}x^{2n}.$$

49. If $y = \int_1^x u^u\,du$, prove that

$$\frac{d^2y}{dx^2} = (1+\log x)\frac{dy}{dx}.$$

Find the values of the first four differential coefficients of y when $x = 1$, and, by using Taylor's expansion in the form

$$f(1+h) = f(1) + hf'(1) + \tfrac{1}{2}h^2f''(1) + \ldots,$$

deduce that the value of $\int_1^{1{\cdot}1} u^u\,du$ is approximately $0{\cdot}1053$.

50. If $y = \tan x$, prove that

$$\frac{dy}{dx} = 1+y^2,\quad \frac{d^2y}{dx^2} = 2y+2y^3,$$

and find the third, fourth, and fifth derivatives of y.

Hence find the expansion of $\tan x$ in a series of powers of x up to x^5.

51. Prove that, if $y = \sin(m\sin^{-1}x)$, then

$$(1-x^2)\left(\frac{dy}{dx}\right)^2 + m^2y^2 = m^2,$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + m^2y = 0.$$

Show that the first two terms in the expansion of the principal value of y in ascending powers of x are

$$mx + \tfrac{1}{6}m(1-m^2)x^3.$$

52. Find the indefinite integrals:

$$\int xe^{-x^2}dx,\quad \int\frac{\tan^{-1}x}{1+x^2}dx,\quad \int x^2e^x\,dx.$$

If u, v are functions of x, and dashes denote differentiation with respect to x, show that

$$\int(uv''' + u'''v)\,dx = uv'' - u'v' + u''v + \text{constant}.$$

53. Show that $\quad \int_0^a f(x)dx = \int_0^a f(a-x)dx.$

Deduce that $\int_0^{\frac{1}{2}\pi}\left(\frac{1-\sin 2x}{1+\sin 2x}\right)dx = \int_0^{\frac{1}{2}\pi}\tan^2 x\,dx,$

and evaluate the integral.

54. Find the indefinite integrals:

$$\int x\log x\,dx,\quad \int\frac{dx}{\sin x},\quad \int\frac{x^2}{1-x^4}dx.$$

Prove that, when the expression $e^x\sin x$ is integrated n times, the result is

$$2^{-\frac{1}{2}n}e^x\sin(x-\tfrac{1}{4}n\pi) + P_{n-1},$$

where P_{n-1} is a polynomial of degree $n-1$ in x.

55. Find the indefinite integrals of

$$\cos^3 x, \quad \frac{1}{x^3(x+1)}, \quad (\log x)^3, \quad \frac{1}{3-2\cos x}.$$

56. Find the indefinite integrals of

$$32\cos^4 x, \quad 27x^2(\log x)^2, \quad \frac{2(1+x^4)}{1-x^4}, \quad a^x.$$

57. Integrate with respect to x:

$$\frac{4}{x(1+x)(1+x^2)}.$$

Evaluate the definite integrals:

$$\int_1^2 \log x\,dx, \quad \int_0^{\frac{1}{2}\pi} \frac{dx}{1+3\cos^2 x}.$$

58. Find the indefinite integrals:

$$\int \frac{x^2\,dx}{1-x^6}, \quad \int \frac{dx}{\sqrt{\{x(2-x)\}}}, \quad \int x\sec^2 x\,dx, \quad \int \tan^4 x\,dx.$$

59. Find the indefinite integrals:

$$\int \frac{dx}{(1-3x)^3}, \quad \int \tan^2 2x\,dx, \quad \int 2x\tan^{-1} x\,dx, \quad \int \frac{\cos x\,dx}{2-\cos^2 x}.$$

60. Find the values of

$$\int_0^{\frac{1}{2}\pi} \sin^5 x\,dx, \quad \int_0^{\pi} \cos^4 x\,dx, \quad \int_0^{2\pi} e^x \sin^2 x\,dx.$$

61. (i) Find the indefinite integrals of

$$\frac{1}{x(x+1)^2}, \quad \frac{2x+3}{x^2+2x+2}.$$

(ii) By substitution, or otherwise, prove that

$$\int_0^{1/\sqrt{2}} x\sin^{-1} x\,dx = \tfrac{1}{8}, \quad \int_0^1 x^3\sqrt{(x^2+1)}\,dx = \frac{2(1+\sqrt{2})}{15}.$$

62. Find the indefinite integrals of

$$\cos^3 3x, \quad \frac{1}{3-2\cos x}, \quad x^3\sin x.$$

Use the method of integration by parts to integrate

$$(1-x^2)\frac{d^2y}{dx^2} - 4x\frac{dy}{dx} - 2y - 8$$

twice with respect to x.

63. By means of the substitution $x = \alpha\cos^2\theta + \beta\sin^2\theta$, or otherwise, prove that

$$\int_\alpha^\beta \frac{dx}{\sqrt{\{(x-\alpha)(\beta-x)\}}} = \pi.$$

Prove also that

$$\int_\alpha^\beta \{(x-\alpha)(\beta-x)\}^{n-\frac{1}{2}}\,dx = \left(\frac{\beta-\alpha}{2}\right)^{2n}\int_0^\pi \sin^{2n}\psi\,d\psi.$$

64. Prove that

$$\int_0^\pi \cos^2\theta\,d\theta = \tfrac{1}{2}\pi, \quad \int_0^\pi \cos^4\theta\,d\theta = \tfrac{3}{8}\pi.$$

Find the area bounded by the curve $r = a(1+\cos\theta)$ and determine the position of the centre of gravity of the area.

65. Find the equation of the normal at the point $(\xi, \sin\xi)$ to the curve whose equation is $y = \sin x$.

Prove that, if ξ lies between 0, π, the normal at P divides the area bounded by the x-axis and that arc of the curve for which $0 \leqslant x \leqslant \pi$ in the ratio

$$(2-\cos\xi-\cos^3\xi) : (2+\cos\xi+\cos^3\xi).$$

66. The ellipse $$\frac{x^2}{a^2}+\frac{y^2}{b^2} = 1$$

is rotated through two right angles about the x-axis. Prove that the volume generated is $\frac{4}{3}\pi ab^2$.

(i) Prove that, if a and b are varied subject to the condition $a+b = \frac{1}{2}$, then the greatest volume generated is $2\pi/81$.

(ii) The volume is cut in two by the plane generated by the rotation of the y-axis. Prove that the centre of gravity of either part of the volume is at a distance $\frac{3}{8}a$ from the plane of separation.

67. The complete curve $x^2/a^2 + y^2/b^2 = 1$ is rotated round the y-axis through two right angles. Find the volume generated by the area enclosed by the curve.

The semi-axes a and b of the curve are each increased by ϵ. Prove that, if ϵ is small, the increase in volume is approximately $\frac{4}{3}\pi a(a+2b)\epsilon$.

68. OA is a straight rod of length a in which the density at a point distant x from O is $b+cx$, where b and c are constants. Find the distance of the centre of gravity of the rod from O.

69. The gradient at any point (x, y) of a curve is given by

$$\frac{dy}{dx} = -3x^2+3,$$

and the curve passes through the point $(2, 0)$. Find its equation and sketch the graph, indicating the turning points.

Find the distance from the y-axis of the centre of gravity of a uniform lamina bounded by the curve and the positive halves of the x and y axes.

70. A lamina in the shape of the parabola $y^2 = 4ax$ bounded by the chord $x = a$ is rotated (i) about the axis of y, (ii) about the line $x = a$. Prove that the volumes generated in the two cases are $\frac{16}{5}\pi a^3, \frac{32}{15}\pi a^3$.

71. Integrate with respect to x:

$$\sin^2 x, \quad \sin^3 x, \quad x^2\cos x.$$

The portion of the curve $y = \sin x$ from $x = 0$ to $x = \frac{1}{2}\pi$ revolves round the axis of y. Prove that the volume contained between the surface so formed and the plane $y = 1$ is $\frac{1}{4}\pi(\pi^2-8)$.

72. The coordinates of a point on a curve referred to rectangular axes are $(at^2, 2at)$, where t is a variable parameter which lies between 0 and 1. Make a rough graph of the curve.

Calculate (i) the area enclosed by the curve and the lines $x = a, y = 0$; and (ii) the area of the surface obtained by revolving this part of the curve about the x-axis.

73. Find the area contained between the x-axis and that part of the curve $x = 2t^2+1, y = t^2-2t$ which corresponds to values of t lying between 0 and 2.

Find also the coordinates of the centroid of this area.

74. Prove that the tangent to the curve

$$\sqrt{x}+\sqrt{y} = \sqrt{a}$$

at the point (x, y) makes intercepts $\sqrt{(ax)}, \sqrt{(ay)}$ on the axes.

A solid is generated by rotating about the x-axis the area whose complete boundary is formed by (i) the arc of this curve joining the points $(a, 0), (0, a)$, and (ii) the straight lines joining the origin to these two points. Prove that, when this solid is of uniform density ρ, its mass M is $\frac{1}{15}\pi\rho a^3$.

Prove also that the moment of inertia of the solid about Ox is $\frac{1}{6}Ma^2$.

75. Find the area bounded by the curve $x = 2a(t^3-1), y = 3at^2$ and the straight lines $x = 0, y = 0$.

Prove that the tangent to the curve at the point $t = 1$ meets the curve again at the point $t = -\frac{1}{2}$, and find the area bounded by the parts of the tangent and of the curve that lie between these points.

76. Prove that the parabola $y^2 = 2ax$ divides the area of the ellipse $4x^2+3y^2 = 4a^2$ into two parts whose areas are in the ratio $4\pi+\sqrt{3} : 8\pi-\sqrt{3}$.

77. The portion of the curve $y^2 = 4ax$ from $(a, 2a)$ to $(4a, 4a)$ revolves round the tangent at the origin. Prove that the volume bounded by the curved surface so formed and plane ends perpendicular to the axis of revolution is $\frac{62}{5}\pi a^3$, and find the square of the radius of gyration of this volume about the axis of revolution.

78. Find the coordinates of the centre of gravity of the area enclosed by the loop of the curve whose equation is $r = a\cos 2\theta$, which lies in the sector bounded by the lines $\theta = \pm\frac{1}{4}\pi$.

Find also the volume obtained by rotating this loop about the line $\theta = \frac{1}{2}\pi$.

79. Find the coordinates of the centre of gravity of the loop of the curve traced out by the point $x = 1-t^2, y = t-t^3$.

Find also the volume obtained by rotating this loop about the line $x = y$.

80. A plane uniform lamina is bounded by the curve $y^2 = 4ax$ and the straight lines $y = 0, x = a$. Find the area and the centre of gravity of the lamina.

The lamina is rotated about the axis Ox to form a (uniform) solid of revolution. Find the centre of gravity of the solid and, assuming the density of the solid to be ρ, find its moment of inertia about the axis Ox.

81. A lamina in the shape of the parabola $y^2 = 4ax$, bounded by the chord $x = a$, is rotated (i) about the axis of y, (ii) about the line $x = a$. Prove that the two volumes thus generated are in the ratio 3 : 2.

82. Prove by integration that the moment of inertia of a uniform circular disc, of mass m and radius a, about a line through its centre perpendicular to its plane is $\frac{1}{2}ma^2$.

The mass of a uniform solid right circular cone is M, and the radius of its base is a. Prove that its moment of inertia about its axis is $\frac{3}{10}Ma^2$.

83. Evaluate $\int_0^{\frac{1}{2}\pi} \cos^n\theta\, d\theta$ when $n = 1, 2, 3, 4$.

The area bounded by the axis of x, the line $x = a$, and the curve

$$x = a\sin\theta, \quad y = a(1-\cos\theta),$$

from $\theta = 0$ to $\theta = \frac{1}{2}\pi$, revolves round the axis of x. Prove that the volume generated is $\frac{1}{6}\pi a^3(10-3\pi)$.

84. The portion of the curve $x^2 = 4a(a-y)$ from $x = -2a$ to $x = 2a$ revolves round the axis of x. Prove that the volume contained by the surface so formed is $\frac{32}{15}\pi a^3$, and find its radius of gyration about the axis of revolution.

85. Sketch the curve $r = a(1+\cos\theta)$, and find the area it encloses and the volume of the surface formed by revolving it about the line $\theta = 0$.

86. Find the three pairs of consecutive integers (positive, negative, or zero) between which the roots of the equation

$$x^3 - 3x^2 + 1 = 0$$

lie, and evaluate the largest root correct to two places of decimals.

87. Show that the equation

$$x^3 - 3x - 7 = 0$$

has one real root, and find it correct to three places of decimals.

88. Show that the equation

$$x^3 + 2x^2 + 3x + 5 = 0$$

has one real root, and find it correct to three places of decimals.

89. For a function f having an Nth differential coefficient, Taylor's theorem expresses $f(a+x)$ as a polynomial of degree $N-1$ in x, together with a remainder. State the form taken by this polynomial, and one form of the remainder.

Prove by induction, or otherwise, that

$$\frac{d^n}{dx^n}(e^x \sin x\sqrt{3}) = 2^n e^x \sin(x\sqrt{3} + \tfrac{1}{3}n\pi).$$

Hence find the coefficients a_n in the Maclaurin series $\Sigma a_n x^n$ of the function $e^x \sin x\sqrt{3}$.

By means of Taylor's theorem, show that when $x > 0$ the difference between $e^x \sin x\sqrt{3}$ and $\sum_{n=0}^{N-1} a_n x^n$ is not greater than

$$\frac{(2x)^N e^x}{N!}.$$

[A proof showing that the difference is not greater than

$$\frac{(2x)^N e^x}{(N-1)!}$$

is acceptable if the form of remainder which you have quoted leads to the result.]

90. If $$y = (1+x)^{-1}\log(1+x),$$

show that $$(1+x)^2\frac{dy}{dx} + (1+x)y = 1.$$

Deduce the first four terms in the Maclaurin series for y in powers of x.

91. Show that $$y = \{x + \sqrt{(1+x^2)}\}^k$$

satisfies the relations $$y'\sqrt{(1+x^2)} = ky,$$

$$y''(1+x^2) + xy' = k^2y.$$

Deduce the expansion

$$y = 1 + kx + \frac{k^2}{2!}x^2 + \frac{k(k^2-1)}{3!}x^3 + \frac{k^2(k^2-2^2)}{4!}x^4 + \ldots.$$

Verify that this agrees with the series derived from the binomial series when $k = 1$.

CHAPTER IX

THE HYPERBOLIC FUNCTIONS

1. The hyperbolic cosine and sine. There are two functions with properties closely analogous to those of $\cos x$ and $\sin x$. They are called the *hyperbolic cosine of* x, and the *hyperbolic sine of* x, and are written as $\cosh x$ and $\sinh x$. We define them in terms of the exponential function as follows:

$$\cosh x = \tfrac{1}{2}(e^x + e^{-x}),$$

$$\sinh x = \tfrac{1}{2}(e^x - e^{-x}).$$

We establish a succession of properties similar to those of the cosine and sine:

(i) *To prove that* $\cosh^2 x - \sinh^2 x = 1.$

The left-hand side is

$$\begin{aligned}&\tfrac{1}{4}\{(e^x+e^{-x})^2-(e^x-e^{-x})^2\}\\ &= \tfrac{1}{4}\{(e^{2x}+2+e^{-2x})-(e^{2x}-2+e^{-2x})\}\\ &= \tfrac{1}{4}(4)\\ &= 1.\end{aligned}$$

(ii) *To prove that*

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y.$$

It is easier to start with the right-hand side:

$$\begin{aligned}&\tfrac{1}{4}\{(e^x+e^{-x})(e^y+e^{-y})+(e^x-e^{-x})(e^y-e^{-y})\}\\ &= \tfrac{1}{4}\{(e^{x+y}+e^{x-y}+e^{-x+y}+e^{-x-y})+(e^{x+y}-e^{x-y}-e^{-x+y}+e^{-x-y})\}\\ &= \tfrac{1}{2}(e^{x+y}+e^{-x-y})\\ &= \cosh(x+y).\end{aligned}$$

(iii) *Similarly*

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y.$$

(iv) *As particular cases of* (ii), (iii),

$$\cosh 2x = \cosh^2 x + \sinh^2 x,$$

and

$$\sinh 2x = 2\sinh x \cosh x.$$

COROLLARY (i). $\cosh^2 x = \tfrac{1}{2}(\cosh 2x + 1),$

$$\sinh^2 x = \tfrac{1}{2}(\cosh 2x - 1).$$

COROLLARY (ii). Since

$$\cosh x - 1 = 2\sinh^2\frac{x}{2}$$

and $\sinh^2\frac{x}{2}$ is positive, it follows that $\cosh x$ *is greater than unity for all (real) values of* x, i.e. $\cosh x \geqslant 1$.

Note. $\cosh 0 = 1,$

$$\sinh 0 = 0.$$

(v) *To prove that* $\cosh x = \cosh(-x)$.

The right-hand side is

$$\begin{aligned}&\tfrac{1}{2}(e^{-x}+e^{-(-x)}) = \tfrac{1}{2}(e^{-x}+e^x)\\ &= \cosh x.\end{aligned}$$

(vi) *To prove that* $\sinh x = -\sinh(-x)$.

For $$\begin{aligned}\sinh(-x) &= \tfrac{1}{2}(e^{-x}-e^{-(-x)}) = \tfrac{1}{2}(e^{-x}-e^x)\\ &= -\sinh x.\end{aligned}$$

Note. $\sinh x$ is positive when x is positive, and negative when x is negative. For example, if x is positive, then e^x is greater than e^{-x}, since e is greater than 1. Hence $\tfrac{1}{2}(e^x - e^{-x})$ is positive.

(vii) *To prove that*

$$\frac{d}{dx}(\cosh x) = \sinh x,$$

$$\frac{d}{dx}(\sinh x) = \cosh x.$$

We have $$\frac{d}{dx}(\cosh x) = \frac{1}{2}\frac{d}{dx}(e^x + e^{-x})$$
$$= \tfrac{1}{2}(e^x - e^{-x})$$
$$= \sinh x;$$

and $$\frac{d}{dx}(\sinh x) = \frac{1}{2}\frac{d}{dx}(e^x - e^{-x})$$
$$= \tfrac{1}{2}(e^x + e^{-x})$$
$$= \cosh x.$$

(viii) *To prove that*

$$\int \cosh dx = \sinh x,$$

$$\int \sinh x dx = \cosh x.$$

These results follow at once from (vii).

(ix) *By the formulæ of* (vii), we have the relations

$$\frac{d^2}{dx^2}(\cosh x) = \cosh x,$$

$$\frac{d^2}{dx^2}(\sinh x) = \sinh x.$$

Thus $\cosh x$, $\sinh x$ both satisfy the relation

$$\frac{d^2 y}{dx^2} = y.$$

(x) The following *expansions in power series* are immediate consequences of Maclaurin's theorem:

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \dots,$$

$$\sinh x = x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \dots.$$

The series converge to the functions for all values of x.

2. Other hyperbolic functions. The following functions are defined by analogy with the corresponding functions of elementary trigonometry:

$$\tanh x = \frac{\sinh x}{\cosh x},$$

$$\coth x = \frac{\cosh x}{\sinh x},$$

$$\operatorname{cosech} x = \frac{1}{\sinh x},$$

$$\operatorname{sech} x = \frac{1}{\cosh x}.$$

The relations $$\operatorname{sech}^2 x + \tanh^2 x = 1,$$
$$\coth^2 x - \operatorname{cosech}^2 x = 1,$$

are found by dividing the equation

$$\cosh^2 x - \sinh^2 x = 1$$

by $\cosh^2 x$, $\sinh^2 x$ respectively.

Note the implications

$$\operatorname{sech}^2 x < 1, \quad \tanh^2 x < 1.$$

The differential coefficients are easily obtained from the definitions:

(i) If
$$y = \tanh x = (\sinh x)(\cosh x)^{-1},$$
then $$\frac{dy}{dx} = (\cosh x)(\cosh x)^{-1} - (\sinh x)(\cosh x)^{-2}\sinh x$$
$$= 1 - \tanh^2 x$$
$$= \operatorname{sech}^2 x.$$

(ii) If
$$y = \coth x = (\cosh x)(\sinh x)^{-1},$$
then $$\frac{dy}{dx} = (\sinh x)(\sinh x)^{-1} - (\cosh x)(\sinh x)^{-2}\cosh x$$
$$= 1 - \coth^2 x$$
$$= -\operatorname{cosech}^2 x.$$

(iii) If $$y = \operatorname{cosech} x = (\sinh x)^{-1},$$

then $$\frac{dy}{dx} = -(\sinh x)^{-2}\cosh x$$
$$= -\operatorname{cosech} x \coth x.$$

(iv) If $$y = \operatorname{sech} x = (\cosh x)^{-1},$$

then $$\frac{dy}{dx} = -(\cosh x)^{-2}\sinh x$$
$$= -\operatorname{sech} x \tanh x.$$

ILLUSTRATION 1. *A body of mass m falls from rest under gravity in a medium whose resistance to motion is gv^2/k^2 per unit mass when the speed is v. To prove that the speed after t seconds is $k\tanh(gt/k)$.*

Let x be the distance dropped in time t. Then the acceleration downward is $\ddot{x}$ and the forces are

(i) mg downwards due to gravity,

(ii) $mg\dot{x}^2/k^2$ upwards due to the resistance.

Hence $$m\ddot{x} = mg - mg\dot{x}^2/k^2.$$

Write $\dot{x} = v$. Then $$\frac{dv}{dt} = g - gv^2/k^2,$$

or $$k^2\frac{dv}{dt} = g(k^2 - v^2).$$

Substitute $$v = k\tanh\theta.$$

(This substitution is possible so long as v is less than k, since (p. 87) $\tanh^2\theta < 1$.)

Then $$k^2 . k\operatorname{sech}^2\theta\frac{d\theta}{dt} = gk^2(1-\tanh^2\theta)$$
$$= gk^2\operatorname{sech}^2\theta.$$

Hence $$\frac{d\theta}{dt} = \frac{g}{k},$$

so that $$\theta = \frac{gt}{k} + C,$$

where C is an arbitrary constant. It follows that
$$v = k\tanh\left(\frac{gt}{k} + C\right).$$

Now we are given that $v = 0$ when $t = 0$, so that
$$0 = \tanh C.$$

Hence $$C = 0,$$

and so $$v = k\tanh(gt/k).$$

Note that $$v = k\tanh(gt/k)$$
$$= k\frac{e^{gt/k} - e^{-gt/k}}{e^{gt/k} + e^{-gt/k}}$$
$$= k\frac{1 - e^{-2gt/k}}{1 + e^{-2gt/k}}.$$

Now as t increases, $e^{2gt/k}$ tends to infinity (see, for example, the diagram (Fig. 64) for e^x on p. 18), so that $e^{-2gt/k} \to 0$. Hence
$$v \to k$$
as t increases. In other words, v tends to a *terminal value* k as the time of falling increases.

The example which follows deserves close attention. It brings out very clearly the points of similarity between the trigonometric and the hyperbolic functions.*

ILLUSTRATION 2. Suppose that a particle P (Fig. 69), of mass m, is free to move on a fixed smooth circular wire, of radius a, whose plane is vertical. A light string, of natural length a and modulus of elasticity $2kmg$ joins P to the highest point B of the wire. We wish to examine what happens if P receives a slight displacement from the lowest point of the wire.

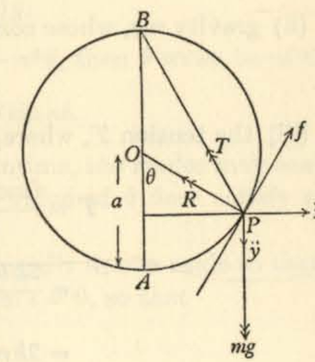


Fig. 69.

Let AB be the diameter through B, and O the centre of the circle. Denote by θ the angle $\angle AOP$.

* It may be postponed or omitted by a reader who finds the mechanics difficult.

If x denotes the horizontal distance of P to the right of the vertical diameter AB, and if y denotes the depth of P below O, then

$$x = a \sin\theta,$$

$$y = a \cos\theta.$$

Hence, differentiating with respect to time,

$$\dot{x} = a\dot{\theta}\cos\theta,$$

$$\dot{y} = -a\dot{\theta}\sin\theta,$$

and
$$\ddot{x} = a\ddot{\theta}\cos\theta - a\dot{\theta}^2\sin\theta,$$

$$\ddot{y} = -a\ddot{\theta}\sin\theta - a\dot{\theta}^2\cos\theta.$$

The acceleration f of P in the direction of the (upward) tangent is thus

$$f = \ddot{x}\cos\theta - \ddot{y}\sin\theta$$
$$= a\ddot{\theta}.$$

[This is a standard formula of applied mathematics.]

Now the forces on the particle are

(i) the reaction R along PO, which has no component along the tangent;

(ii) gravity mg, whose component along the (upward) tangent is

$$-mg\sin\theta;$$

(iii) the tension T, where, by definition of modulus,

$$T \equiv \frac{\text{(modulus) (extension)}}{\text{natural length}}$$

$$= \frac{2kmg\{2a\cos\frac{1}{2}\theta - a\}}{a}$$

$$= 2kmg(2\cos\tfrac{1}{2}\theta - 1);$$

the component of T along the (upward) tangent is thus

$$T\sin\tfrac{1}{2}\theta$$

$$= 2kmg(2\cos\tfrac{1}{2}\theta - 1)\sin\tfrac{1}{2}\theta.$$

Equating the component of the acceleration to the component of the forces, we have the equation of motion

$$a\ddot{\theta} = -g\sin\theta + 2kg(2\cos\tfrac{1}{2}\theta - 1)\sin\tfrac{1}{2}\theta.$$

Now suppose that θ is small. By the work given on p. 38, the value of $\sin\theta$ is nearly θ itself, while $\cos\theta$ is nearly equal to 1. Hence the equation is approximately

$$a\ddot{\theta} = -g\theta + 2kg\{2(1) - 1\}(\tfrac{1}{2}\theta)$$
$$= -g\theta + kg\theta.$$

Hence we reach the equation

$$a\ddot{\theta} = g(k-1)\theta,$$

valid during the time while θ is small.

The argument now divides, according as k is less than or greater than unity; that is, according as the string is 'fairly slack' or 'fairly tight'.

(i) *Suppose that* $k < 1$.

Then
$$\ddot{\theta} = -\frac{(1-k)g}{a}\theta$$
$$= -n^2\theta,$$

where
$$an^2 = (1-k)g.$$

It may be proved that, when $\ddot{\theta} = -n^2\theta$, then θ MUST be of the form

$$\theta = A\cos nt + B\sin nt,$$

where A, B are constants. In the meantime, the reader may easily verify the converse result, that this value of θ does satisfy the relation.

If we suppose that the particle is initially drawn aside so that θ has the small value α, then $\theta = \alpha$ when $t = 0$, so that

$$A = \alpha.$$

If also the particle is released from rest, then $\dot{\theta} = 0$ when $t = 0$, so that

$$B = 0.$$

Hence
$$\theta = \alpha\cos nt.$$

Thus if θ is initially small, it remains small, and its value oscillates between $\pm\alpha$. The equilibrium is stable at the lowest point.

(ii) *Suppose that* $k > 1$.

Then
$$\ddot{\theta} = \frac{(k-1)g}{a}\theta$$
$$= p^2\theta,$$
where
$$ap^2 = (k-1)g.$$

This equation is not satisfied by sines and cosines, but we can express the relation between θ and t in the form
$$\theta = Ae^{pt} + Be^{-pt},$$
where, again, the reader may verify the converse result that this value of θ does satisfy the equation.

With the same initial conditions as before, we have
$$\alpha = A + B,$$
$$0 = Ap - Bp,$$
so that
$$A = B = \tfrac{1}{2}\alpha.$$
Hence
$$\theta = \tfrac{1}{2}\alpha(e^{pt} + e^{-pt})$$
$$= \alpha \cosh pt.$$

As t increases, $\cosh pt$ increases steadily to 'infinity' (since e^{pt} does), so that θ ceases to be small. The equilibrium is unstable at the lowest point.

The differential equation, in fact, ceases to be accurate once θ ceases to be small.

It is instructive to consider the same problem under the alternative initial conditions that the particle is projected from the lowest point with speed v. Thus $\theta = 0, a\dot{\theta} = v$ when $t = 0$. We take the two cases in succession:

(i)
$$\theta = A\cos nt + B\sin nt,$$
where
$$0 = A,$$
$$v/a = nB.$$
Hence
$$\theta = (v/an)\sin nt.$$

(ii)
$$\theta = Ae^{pt} + Be^{-pt},$$
where
$$0 = A + B,$$
$$v/a = pA - pB,$$
so that
$$A = \frac{v}{2ap}, \quad B = -\frac{v}{2ap}.$$
Hence
$$\theta = \frac{v}{2ap}(e^{pt} - e^{-pt})$$
$$= (v/ap)\sinh pt.$$

The analogy between the pairs of solutions
$$\alpha\cos nt, \quad \alpha\cosh pt$$
and
$$(v/an)\sin nt, \ (v/ap)\sinh pt$$
affords striking confirmation of the analogy between the two classes of functions.

EXAMPLES I

Differentiate the following functions:

1. $\sinh 3x$. 2. $\cosh^2 2x$. 3. $x\tanh x$.
4. $\sinh^2(2x+1)$. 5. $\sinh x\cos x$. 6. $\operatorname{sech} x\sin^2 x$.
7. $(1+x)^3\cosh^3 3x$. 8. $x^2\tanh^2 4x$. 9. $\log\sinh x$.
10. $\log(\sinh x + \cosh x)$. 11. $e^{\sinh x}$. 12. $xe^{-\tanh x}$.

Find the following integrals:

13. $\int \sinh 4x\,dx$. 14. $\int \sinh^2 x\,dx$. 15. $\int \cosh^2 x\,dx$.
16. $\int x\sinh x\,dx$. 17. $\int e^x\cosh x\,dx$. 18. $\int \sinh 3x\cosh x\,dx$.
19. $\int x\sinh^2 x\,dx$. 20. $\int \cosh^3 x\,dx$. 21. $\int \tanh^2 x\,dx$.
22. $\int x^2\cosh x\,dx$. 23. $\int e^{2x}\sinh 5x\,dx$. 24. $\int \tanh x\operatorname{sech}^2 x\,dx$.

Establish the following formulæ:

25. $\sinh A + \sinh B = 2 \sinh \dfrac{A+B}{2} \cosh \dfrac{A-B}{2}$.

26. $\sinh A - \sinh B = 2 \cosh \dfrac{A+B}{2} \sinh \dfrac{A-B}{2}$.

27. $\cosh A + \cosh B = 2 \cosh \dfrac{A+B}{2} \cosh \dfrac{A-B}{2}$.

28. $\cosh A - \cosh B = 2 \sinh \dfrac{A+B}{2} \sinh \dfrac{A-B}{2}$.

29. Prove that, for all values of u, the point $(a \cosh u, b \sinh u)$ lies on the hyperbola whose equation is

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1,$$

and that the tangent at that point is

$$\frac{x}{a} \cosh u - \frac{y}{b} \sinh u = 1.$$

(But note that that point is restricted to the part of the hyperbola for which x is positive.)

3. The graph $y = \cosh x$. The two relations

$$y = \cosh x,$$

$$\frac{dy}{dx} = \sinh x$$

give us sufficient information to indicate the general shape of the curve:

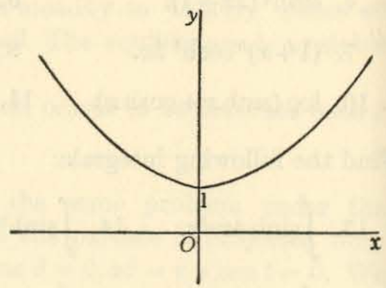


Fig. 70.

Since

$$\cosh(-x) = \cosh(x),$$

the curve is symmetrical about the y-axis; and since

$$\cosh x \geqslant 1,$$

the curve lies entirely above the line $y = 1$. The value of y increases rapidly with x.

Taking x to be positive (when it is negative the corresponding part of the curve is obtained simply by reflexion in the y-axis) we have $\sinh x$ positive, so that the gradient is positive. Moreover,

$$\frac{d^2y}{dx^2} = \cosh x,$$

which is positive, and so (Vol. I, p. 54) the concavity is 'upwards'.

The general shape of the curve is therefore that indicated in the diagram (Fig. 70).

4. The graph $y = \sinh x$. We have the relations

$$y = \sinh x,$$

$$\frac{dy}{dx} = \cosh x.$$

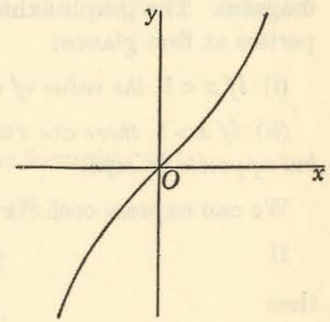


Fig. 71.

Since $\dfrac{dy}{dx}$ is positive, the gradient is always positive, and y is an increasing function of x, running from $-\infty$ to $+\infty$ as x increases from $-\infty$ to $+\infty$.

At the origin, $\dfrac{dy}{dx} = 1$, so that the curve crosses the x-axis there at an angle of $\frac{1}{4}\pi$. Also

$$\frac{d^2y}{dx^2} = \sinh x,$$

which is positive for positive x and negative for negative. Hence the curve lies entirely in the first and third quadrants, with (Vol. I, p. 54) concavity 'upwards' in the first and 'downwards' in the third. At the origin, $\dfrac{d^2y}{dx^2} = 0$, so that (Vol. I, p. 55) the curve has an inflexion there.

Since

$$\sinh(-x) = -\sinh(x),$$

the curve is symmetrical about the origin.

The general shape of the curve is therefore that indicated in the diagram (Fig. 71).

5. The inverse hyperbolic cosine. The problem arises in practice to determine a function whose hyperbolic cosine has a given value x. If the function is y, then

$$x = \cosh y,$$

and we use the notation

$$y = \cosh^{-1} x$$

to denote the *inverse hyperbolic cosine.*

The graph (Fig. 72) $y = \cosh^{-1} x$ is found by 'turning the graph $y = \cosh x$ through a right angle' and then renaming the axes, as shown in the diagram. The graph exhibits two properties at first glance:

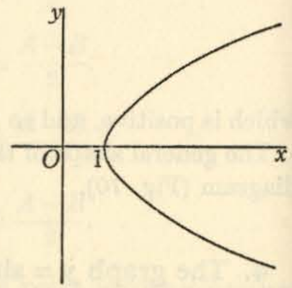


Fig. 72.

(i) *If* $x < 1$, *the value of* $\cosh^{-1} x$ *does not exist;*

(ii) *If* $x > 1$, *there are* TWO *values of* $\cosh^{-1} x$, *equal in magnitude but opposite in sign.*

We can express $\cosh^{-1} x$ in terms of logarithms as follows:

If $$y = \cosh^{-1} x,$$

then $$x = \cosh y$$

$$= \tfrac{1}{2}(e^y + e^{-y}),$$

so that $$e^{2y} - 2xe^y + 1 = 0.$$

Solving this equation as a quadratic in e^y, we have

$$e^y = x \pm \sqrt{(x^2 - 1)},$$

and so, by definition of the exponential function,

$$y = \log\{x \pm \sqrt{(x^2 - 1)}\}.$$

These are the two values of y. Moreover their sum is

$$\begin{aligned} &\log\{x + \sqrt{(x^2-1)}\} + \log\{x - \sqrt{(x^2-1)}\} \\ &= \log[\{x + \sqrt{(x^2-1)}\}\{x - \sqrt{(x^2-1)}\}] \\ &= \log[x^2 - (x^2 - 1)] = \log 1 \\ &= 0. \end{aligned}$$

Hence the two roots are equal and opposite. The positive root is $\log\{x + \sqrt{(x^2-1)}\}$ and the negative $\log\{x - \sqrt{(x^2-1)}\}$.

To find the differential coefficient of $\cosh^{-1} x$, we differentiate the relation

$$\cosh y = x$$

with respect to x. Then

$$\sinh y \frac{dy}{dx} = 1,$$

or $$\frac{dy}{dx} = \frac{1}{\sinh y} = \frac{1}{\pm\sqrt{(\cosh^2 y - 1)}}$$

$$= \pm \frac{1}{\sqrt{(x^2-1)}}.$$

The gradient is positive when y is positive and negative when y is negative. In particular, if we take the POSITIVE value of $\cosh^{-1} x$, then

$$\frac{dy}{dx} = \frac{1}{\sqrt{(x^2-1)}}.$$

We can also obtain the result from the formula

$$y = \log\{x + \sqrt{(x^2-1)}\},$$

taking the positive value. For if

$$u \equiv x + (x^2 - 1)^{\frac{1}{2}},$$

then $$\frac{du}{dx} = 1 + x(x^2-1)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{(x^2-1)}}\{\sqrt{(x^2-1)} + x\} = \frac{u}{\sqrt{(x^2-1)}}.$$

Hence $$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \frac{1}{u}\frac{u}{\sqrt{(x^2-1)}}$$

$$= \frac{1}{\sqrt{(x^2-1)}}.$$

Note the corresponding integral

$$\int \frac{dx}{\sqrt{(x^2-1)}} = \cosh^{-1} x \quad \text{(positive value)}$$

$$= \log\{x + \sqrt{(x^2-1)}\}.$$

6. The inverse hyperbolic sine. On 'turning the graph of $\sinh x$ through a right angle and taking a mirror image', and then renaming the axes, we obtain the graph (Fig. 73)

$$y = \sinh^{-1} x$$

of the *inverse hyperbolic sine* of x, that is, of the number whose hyperbolic sine is x. The function $\sinh^{-1} x$ is a SINGLE-VALUED function, uniquely determined for all values of x.

To express $\sinh^{-1} x$ in terms of logarithms, we write

$$\sinh^{-1} x = y,$$

so that

$$x = \sinh y = \tfrac{1}{2}(e^y - e^{-y}),$$

giving $e^{2y} - 2xe^y - 1 = 0.$

Solving this equation as a quadratic in e^y, we have

$$e^y = x \pm \sqrt{(x^2+1)}.$$

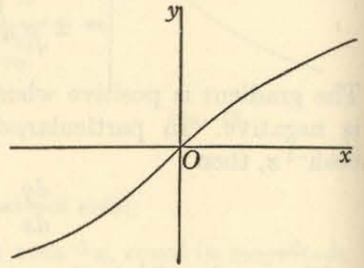


Fig. 73.

But (p. 19) the exponential function is positive, so that we must take the positive sign for the square root. Hence

$$e^y = x + \sqrt{(x^2+1)},$$

or

$$y = \log\{x + \sqrt{(x^2+1)}\}$$

without ambiguity.

To find the differential coefficient of $\sinh^{-1} x$, we differentiate the relation

$$\sinh y = x$$

with respect to x. Then

$$\cosh y \frac{dy}{dx} = 1,$$

or

$$\frac{dy}{dx} = \frac{1}{\cosh y} = \frac{1}{\pm\sqrt{(\sinh^2 y + 1)}}$$

$$= \pm \frac{1}{\sqrt{(x^2+1)}}.$$

But, from the graph (Fig. 73), the gradient is always positive, and so

$$\frac{dy}{dx} = + \frac{1}{\sqrt{(x^2+1)}}.$$

We can also obtain this result from the formula

$$y = \log\{x + \sqrt{(x^2+1)}\}.$$

For if

$$u = x + (x^2+1)^{\frac{1}{2}},$$

then

$$\frac{du}{dx} = 1 + x(x^2+1)^{-\frac{1}{2}}$$

$$= \frac{1}{\sqrt{(x^2+1)}}\{\sqrt{(x^2+1)} + x\} = \frac{u}{\sqrt{(x^2+1)}}.$$

Hence

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}$$

$$= \frac{1}{u}\frac{u}{\sqrt{(x^2+1)}}$$

$$= \frac{1}{\sqrt{(x^2+1)}}.$$

Note the corresponding integral

$$\int \frac{dx}{\sqrt{(x^2+1)}} = \sinh^{-1} x$$

$$= \log\{x + \sqrt{(x^2+1)}\}.$$

EXAMPLES II

Find the differential coefficients of the following functions:

1. $x\cosh^{-1} x$.
2. $\sinh^{-1}(1+x^2)$.
3. $\tanh^{-1} x$.
4. $\operatorname{sech}^{-1} x$.
5. $\operatorname{cosech}^{-1} x$.
6. $\log(\cosh^{-1} x)$.
7. $x\cosh^{-1}(x^2+1)$.
8. $(\cosh^{-1} x)^2$.
9. $1/(\sinh^{-1} x)$.

Find the following integrals:

10. $\int \frac{dx}{\sqrt{(x^2-4)}}$.
11. $\int \frac{dx}{\sqrt{(9x^2+1)}}$.
12. $\int \frac{dx}{\sqrt{(4x^2-9)}}$.
13. $\int \frac{dx}{\sqrt{\{(x+1)^2-4\}}}$.
14. $\int \frac{dx}{\sqrt{(x^2+2x+5)}}$.
15. $\int \frac{dx}{\sqrt{(4x^2-4x-15)}}$.

CHAPTER X

CURVES

1. Parametric representation. Hitherto we have regarded a curve as defined by an equation of the form

$$y = f(x).$$

For many purposes it is more convenient to adopt a *parametric representation* whereby the coordinates x, y of a point T of the curve are expressed as functions of a *parameter* t in the form

$$x = f(t), \quad y = g(t).$$

(Of course, there is no reason why the parameter should not be x itself.) For economy of notation, however, we often write

$$x = x(t), \quad y = y(t).$$

Familiar examples from elementary coordinate geometry are the representations

$$x = at^2, \quad y = 2at$$

for the parabola $y^2 = 4ax$, and

$$x = ct, \quad y = c/t$$

for the rectangular hyperbola $xy = c^2$.

We confine ourselves to the simplest case, in which $x(t), y(t)$ are single-valued functions of t with as many continuous differential coefficients as the argument may require.

The 'dot' notation

$$\dot{x}, \dot{y}, \ddot{x}, \ddot{y}, \ldots$$

will be used to denote the differential coefficients

$$\frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}, \ldots.$$

The tangent at the point 't' of the curve

$$x = f(t), \quad y = g(t)$$

may easily be found; it is the line through that point with gradient

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}}$$

$$= \frac{g'(t)}{f'(t)}.$$

For the reader familiar with determinants, an alternative form of equation may be given:

The equation of any straight line is

$$lx + my + n = 0.$$

If this line passes through the point $x = f(t), y = g(t)$, then

$$lf(t) + mg(t) + n = 0;$$

if it passes through the point $x = f(t+\delta t), y = g(t+\delta t)$, then

$$lf(t+\delta t) + mg(t+\delta t) + n = 0.$$

Subtracting, we have the relation

$$l\{f(t+\delta t) - f(t)\} + m\{g(t+\delta t) - g(t)\} = 0,$$

or, on division by δt,

$$l\frac{f(t+\delta t) - f(t)}{\delta t} + m\frac{g(t+\delta t) - g(t)}{\delta t} = 0.$$

For the tangent, we must take the limiting form of this relation as $\delta t \to 0$, namely

$$lf'(t) + mg'(t) = 0.$$

Hence, on eliminating the ratios $l : m : n$, we obtain the equation of the tangent in the form

$$\begin{vmatrix} x & y & 1 \\ f(t) & g(t) & 1 \\ f'(t) & g'(t) & 0 \end{vmatrix} = 0.$$

2. The sense of description of a curve. We regard that sense of description of a curve as *positive* which is followed by a variable point for *increasing* values of the defining parameter. For example, if the points A, P, Q (Fig. 74) correspond to the values a, p, q respectively, and if $a < p < q$, then the sense is $\overrightarrow{APQ}$.

It is important to realize that sense is not an inherent property; it is a man-made convention. Thus different parametric representations may give rise to different senses along the curve.

For example, the positive quadrant of the circle $x^2+y^2=1$ may be expressed with x as parameter in the form

$$x = x, \quad y = +\sqrt{(1-x^2)}.$$

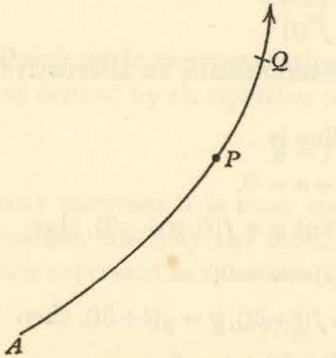


Fig. 74.

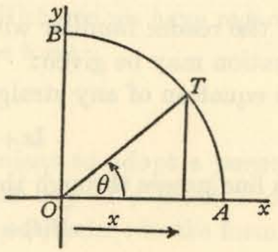


Fig. 75.

As x increases from 0 to 1, the arc is described in the sense $\overrightarrow{BA}$ of the diagram (Fig. 75). On the other hand, if the polar angle θ is taken as parameter, we have

$$x = \cos\theta, \quad y = \sin\theta$$

As θ increases from 0 to $\frac{1}{2}\pi$, the arc is described in the sense $\overrightarrow{AB}$.

3. The 'length' postulate. If we confine our attention to the simplest case, where the curve has a continuously turning tangent, as in the diagram (Fig. 76), then our instinctive ideas will be satisfied if we ensure that the length of a curved arc PQ is nearly the same as that of the chord PQ when the points P, Q are very close together. For this purpose, we shall base our treatment of length on the postulate

$$\lim_{Q\to P} \frac{\text{arc } QP}{\text{chord } QP} = 1.$$

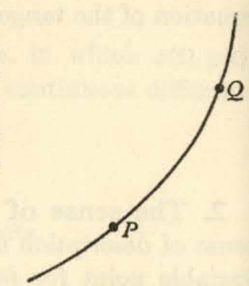


Fig. 76.

Our aim is to let the derivation of all the standard formulæ of the geometry of curves rest on this single assumption, together with the normal manipulations of algebra, trigonometry and the calculus.

The warning ought perhaps to be added that in a more advanced treatment it would be necessary to examine whether 'length' exists at all and to proceed somewhat differently. The present treatment suffices when $dx/dt, dy/dt$ are continuous; this is true for 'ordinary' cases such as we shall be considering.

4. The length of a curve. Suppose that U P, P' are three points on the curve (Fig. 77)

$$x = x(t), \quad y = y(t)$$

given by the values $u, p, p+\delta p$ of the parameter. For convenience, we assume that

$$u < p < p + \delta p,$$

so that the curve is described in the sense $\overrightarrow{UPP'}$.

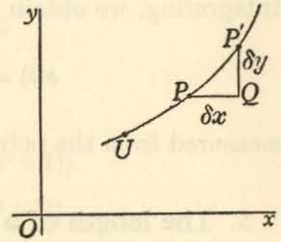


Fig. 77.

If P, P' are the points (x, y), $(x+\delta x, y+\delta y)$ respectively, then the length of the chord PP' is

$$\sqrt{\{(\delta x)^2 + (\delta y)^2\}},$$

whether $\delta x, \delta y$ are positive or negative.

Now the length of the arc UP is a function of p, which we may call $s(p)$. Thus, since arc PP' = arc UP' − arc UP, we have

$$\text{arc } PP' = s(p+\delta p) - s(p).$$

But
$$\frac{s(p+\delta p)-s(p)}{\delta p} = \frac{s(p+\delta p)-s(p)}{\text{chord } PP'} \cdot \frac{\text{chord } PP'}{\delta p}$$

$$= \frac{\text{arc } PP'}{\text{chord } PP'} \sqrt{\left\{\left(\frac{\delta x}{\delta p}\right)^2 + \left(\frac{\delta y}{\delta p}\right)^2\right\}}.$$

If we proceed to the limit, as $\delta p \to 0$ so that $P' \to P$, then

$$\lim_{\delta p\to 0} \frac{s(p+\delta p)-s(p)}{\delta p} = s'(p),$$

$$\lim_{P'\to P} \frac{\text{arc } PP'}{\text{chord } PP'} = 1,$$

$$\lim_{\delta p\to 0} \frac{\delta x}{\delta p} = \frac{dx}{dp}, \quad \lim_{\delta p\to 0} \frac{\delta y}{\delta p} = \frac{dy}{dp}.$$

Hence $$s'(p) = \sqrt{\left\{\left(\frac{dx}{dp}\right)^2 + \left(\frac{dy}{dp}\right)^2\right\}},$$

where the POSITIVE square root must be taken since s increases with p.

Replacing p by the current letter t, we have the relation

$$s'(t) = \sqrt{(\dot{x}^2 + \dot{y}^2)}.$$

Integrating, we obtain the formula

$$s(t) = \int_u^t \sqrt{\left\{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2\right\}}\, dt,$$

measured from the point U with parameter u.

5. The length of a curve in Cartesian coordinates. If the coordinate x is taken as the parameter t, then the formula of § 4 becomes

$$s'(x) = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}},$$

so that $$s(x) = \int_a^x \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}\, dx,$$

measured from the point where $x = a$, where the positive sense of the curve is determined by x increasing.

In terms of the coordinate y, we have similarly

$$s(y) = \int_b^y \sqrt{\left\{\left(\frac{dx}{dy}\right)^2 + 1\right\}}\, dy,$$

measured from the point where $y = b$, where the positive sense of the curve is determined by y increasing.

ILLUSTRATION 1. *To find the length of the arc of the parabola $y^2 = 4ax$ from the origin to the point (x, y), where y is taken to be positive.*

The parametric representation is

$$x = at^2, \quad y = 2at,$$

so that $$\dot{x} = 2at, \quad \dot{y} = 2a.$$

Hence $$s = \int_0^t 2a\sqrt{(t^2+1)}\, dt.$$

To evaluate the integral, write

$$I = \int \sqrt{(t^2+1)}\, dt$$

$$= t\sqrt{(t^2+1)} - \int t \cdot \frac{t}{\sqrt{(t^2+1)}}\, dt,$$

on integration by parts. Hence

$$I = t\sqrt{(t^2+1)} - \int \frac{(t^2+1)-1}{\sqrt{(t^2+1)}}\, dt$$

$$= t\sqrt{(t^2+1)} - I + \int \frac{dt}{\sqrt{(t^2+1)}}$$

$$= t\sqrt{(t^2+1)} - I + \log\{t + \sqrt{(t^2+1)}\}.$$

Hence $$I = \tfrac{1}{2}t\sqrt{(t^2+1)} + \tfrac{1}{2}\log\{t + \sqrt{(t^2+1)}\},$$

so that $$s = at\sqrt{(t^2+1)} + a\log\{t + \sqrt{(t^2+1)}\}.$$

6. The length of a curve in polar coordinates. Let the equation of the curve in polar coordinates be

$$r = f(\theta).$$

If θ is taken as the parameter, then the formula of § 4 becomes

$$s'(\theta) = \sqrt{\left\{\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2\right\}}.$$

Now $$x = r\cos\theta, \quad y = r\sin\theta,$$

where r is a function of θ. Hence

$$\frac{dx}{d\theta} = \frac{dr}{d\theta}\cos\theta - r\sin\theta, \quad \frac{dy}{d\theta} = \frac{dr}{d\theta}\sin\theta + r\cos\theta,$$

so that $$\left(\frac{dx}{d\theta}\right)^2 + \left(\frac{dy}{d\theta}\right)^2 = \left(\frac{dr}{d\theta}\right)^2 + r^2.$$

It follows that $$s'(\theta) = \sqrt{\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}},$$

so that $$s(\theta) = \int_\alpha^\theta \sqrt{\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}}\, d\theta,$$

measured from the point where $\theta = \alpha$, where the positive sense of the curve is determined by θ increasing.

ILLUSTRATION 2. *To find the length of the curve* (*a* CARDIOID) *given by the equation*

$$r = a(1+\cos\theta).$$

The shape of the curve is shown in the diagram (Fig. 78).

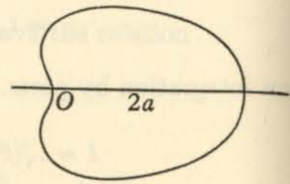


Fig. 78.

We have $\dfrac{dr}{d\theta} = -a\sin\theta$, so that

$$\left(\frac{dr}{d\theta}\right)^2 + r^2 = a^2\{\sin^2\theta + (1+2\cos\theta+\cos^2\theta)\}$$
$$= 2a^2(1+\cos\theta) = 4a^2\cos^2\tfrac{1}{2}\theta.$$

Hence the length of the curve is

$$\int_{-\pi}^{\pi} 2a\cos\tfrac{1}{2}\theta\, d\theta = 4a\Big[\sin\tfrac{1}{2}\theta\Big]_{-\pi}^{\pi}$$
$$= 4a[\sin(\tfrac{1}{2}\pi) - \sin(-\tfrac{1}{2}\pi)]$$
$$= 4a[1-(-1)]$$
$$= 8a.$$

Note. If we had taken the limits of integration as $0, 2\pi$, we should apparently have had the result that the length is

$$\int_0^{2\pi} 2a\cos\tfrac{1}{2}\theta\, d\theta = 4a\Big[\sin\tfrac{1}{2}\theta\Big]_0^{2\pi}$$
$$= 4a[\sin\pi - \sin 0]$$
$$= 0.$$

It is instructive to trace the source of error. This lies in our assumption that

$$\sqrt{(4a^2\cos^2\tfrac{1}{2}\theta)} = 2a\cos\tfrac{1}{2}\theta.$$

When θ lies between $-\pi$, π, this is true, since $\cos\frac{1}{2}\theta$ is then positive. But in the interval $\pi, 2\pi$, the value of $\cos\frac{1}{2}\theta$ is negative, so that

$$\sqrt{(4a^2\cos^2\tfrac{1}{2}\theta)} = -2a\cos\tfrac{1}{2}\theta.$$

Hence we must use the argument:

$$\int_0^{2\pi}\sqrt{(4a^2\cos^2\tfrac{1}{2}\theta)}\,d\theta$$
$$= \int_0^{\pi}\sqrt{(4a^2\cos^2\tfrac{1}{2}\theta)}\,d\theta + \int_{\pi}^{2\pi}\sqrt{(4a^2\cos^2\tfrac{1}{2}\theta)}\,d\theta$$
$$= \int_0^{\pi} 2a\cos\tfrac{1}{2}\theta\,d\theta - \int_{\pi}^{2\pi} 2a\cos\tfrac{1}{2}\theta\,d\theta$$
$$= 4a\Big[\sin\tfrac{1}{2}\theta\Big]_0^{\pi} - 4a\Big[\sin\tfrac{1}{2}\theta\Big]_{\pi}^{2\pi}$$
$$= 4a[1-0] - 4a[0-1]$$
$$= 8a.$$

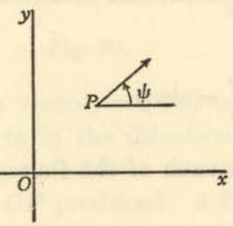
7. The 'gradient angle' ψ. If the tangent at a point P of the curve

$$y = f(x)$$

makes an angle ψ with the x-axis, then

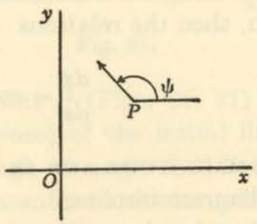
$$\frac{dy}{dx} = \tan\psi.$$

The diagrams (Fig. 79) represent the four ways (indicated by the arrows) in which a curve may 'leave' a point P on it, the parameter being such that the positive sense along the curve is that of the arrow.



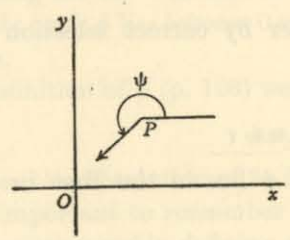
(i) FIRST QUADRANT

$dx+$; $dy+$; $\cos\psi+$; $\sin\psi+$.



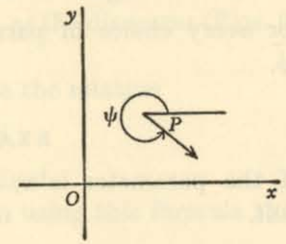
(ii) SECOND QUADRANT

$dx-$; $dy+$: $\cos\psi-$; $\sin\psi+$.



(iii) THIRD QUADRANT

$dx-$; $dy-$; $\cos\psi-$; $\sin\psi-$.



(iv) FOURTH QUADRANT

$dx+$; $dy-$; $\cos\psi+$; $\sin\psi-$.

Fig. 79.

Thus $\dfrac{dx}{ds}$ is positive for (i), (iv) and negative for (ii), (iii); while $\dfrac{dy}{ds}$ is positive for (i), (ii) and negative for (iii), (iv).

But $$\left(\frac{ds}{dx}\right)^2 = 1+\left(\frac{dy}{dx}\right)^2 = 1+\tan^2\psi = \sec^2\psi,$$

so that $$\left(\frac{dx}{ds}\right)^2 = \cos^2\psi;$$

similarly $$\left(\frac{dy}{ds}\right)^2 = \sin^2\psi.$$

Hence $\frac{dx}{ds}, \frac{dy}{ds}$ have the numerical values of $\cos\psi$, $\sin\psi$ respectively; and, if we define the angle ψ to be the angle $\left(\text{whose tangent is } \frac{dy}{dx}\right)$ from the positive direction of the x-axis round to the positive direction along the curve, in the usual counter-clockwise sense of rotation, then the relations

$$\frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = \sin\psi$$

hold IN MAGNITUDE AND IN SIGN for each of the four quadrants, as the diagram implies.

We therefore have the relations

$$\frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = \sin\psi,$$

true for every choice of parameter by correct selection of the angle ψ.

EXAMPLES I

1. If the parameter is x, then ψ lies in the first or fourth quadrant.

2. If the parameter is y, then ψ lies in the first or second quadrant.

3. If the parameter is θ, then ψ lies between θ and $\theta+\pi$ (reduced by 2π if necessary).

4. What modifications are required in the treatment given in § 7 if the curve is parallel to the y-axis? Prove that the formulæ $\frac{dx}{ds} = \cos\psi$, $\frac{dy}{ds} = \sin\psi$ are still true.

8. The angle from the radius vector to the tangent.

The direction of the tangent to the curve

$$r = f(\theta)$$

may be described in terms of the angle ϕ 'behind' the radius vector. For precision, we define ϕ as follows:

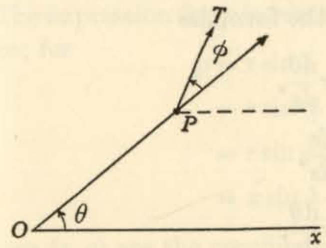


Fig. 80.

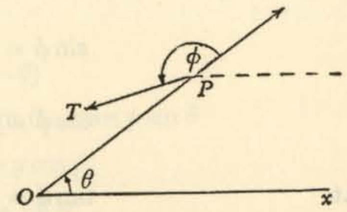


Fig. 81.

A radius vector, centred on the point P (Figs. 80, 81) of the curve, starts in the direction (and sense) of the initial line Ox; after counter-clockwise rotation through an angle θ, it lies along the radius OP produced. A further counter-clockwise rotation ϕ brings it to the tangent to the curve, in the POSITIVE sense; this defines ϕ.

If we assume, as usual, that θ is the parameter, then the positive sense along the curve is that in which the length increases with θ. Hence *the angle* ϕ *lies between* 0 *and* π, as the diagrams (Figs. 80, 81) indicate.

By definition of ψ (p. 108) we have the relation

$$\psi = \theta+\phi,$$

with possible subtraction of 2π if desired.

It is important to remember when using this formula that θ is the parameter used in defining ψ.

To find expressions for $\sin\phi$, $\cos\phi$, $\tan\phi$:

From the relations

$$x = r\cos\theta, \quad y = r\sin\theta,$$

we have $$\frac{dx}{ds} = \frac{dr}{ds}\cos\theta - r\frac{d\theta}{ds}\sin\theta,$$

$$\frac{dy}{ds} = \frac{dr}{ds}\sin\theta + r\frac{d\theta}{ds}\cos\theta,$$

so that (p. 108), when θ is the parameter,

$$\frac{dr}{ds}\cos\theta - r\frac{d\theta}{ds}\sin\theta = \cos\psi = \cos(\theta+\phi) = \cos\phi\cos\theta - \sin\phi\sin\theta,$$

$$\frac{dr}{ds}\sin\theta + r\frac{d\theta}{ds}\cos\theta = \sin\psi = \sin(\theta+\phi) = \cos\phi\sin\theta + \sin\phi\cos\theta.$$

Solving for $\sin\phi$, $\cos\phi$, we have the formulæ

$$\sin\phi = r\frac{d\theta}{ds},$$

$$\cos\phi = \frac{dr}{ds},$$

so that
$$\tan\phi = r\frac{d\theta}{dr}.$$

9. The perpendicular on the tangent. Let the line ON (Fig. 82) be drawn from the pole O on to the tangent at the point P of the curve

$$r = f(\theta).$$

Then, by elementary trigonometry,

$$p = r\sin\phi$$
$$= r^2\frac{d\theta}{ds}.$$

This formula may be cast into an alternative useful form:

$$\frac{1}{p^2} = \frac{1}{r^4}\left(\frac{ds}{d\theta}\right)^2$$
$$= \frac{1}{r^4}\left\{\left(\frac{dr}{d\theta}\right)^2 + r^2\right\}$$
$$= \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2}.$$

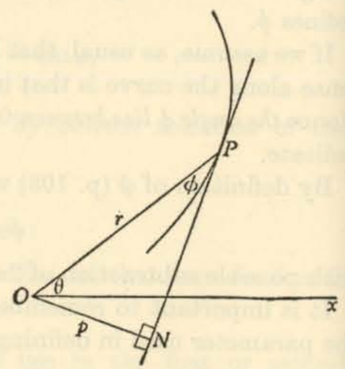


Fig. 82.

If we write
$$u = \frac{1}{r},$$

so that
$$\frac{du}{d\theta} = -\frac{1}{r^2}\frac{dr}{d\theta},$$

then
$$\frac{1}{p^2} = \left(\frac{du}{d\theta}\right)^2 + u^2.$$

Note. Since ϕ lies between $0, \pi$, the value of $\sin\phi$ is necessarily positive. We may retain the formula

$$p = r\sin\phi$$

generally if we allow p to take the sign of r. In any case, the numerical value of p is that of $r\sin\phi$.

The expression for p in terms of Cartesian coordinates follows at once; for

$$\begin{aligned} p &= r\sin\phi \\ &= r\sin(\psi-\theta) \\ &= r\sin\psi\cos\theta - r\cos\psi\sin\theta \\ &= x\sin\psi - y\cos\psi, \end{aligned}$$

where (x, y) are the coordinates of P.

It should be noticed that the step $\phi = \psi - \theta$ depends on the conventions adopted when θ is the parameter. If another parameter (for example, x) is used, the sign attached to p may require separate checking.

10. Other coordinate systems. The Cartesian coordinates (x, y) and the polar coordinates (r, θ) are by no means the only coordinates available for defining the position of a point of a curve. Others in use are the *intrinsic* coordinates (s, ψ) and the *pedal* coordinates (p, r). The passage between Cartesian and polar coordinates is familiar. To pass from Cartesian to intrinsic coordinates, we have the relations

$$\frac{dx}{ds} = \cos\psi, \quad \frac{dy}{ds} = \sin\psi,$$

or the equivalent

$$\frac{ds}{dx} = \sqrt{\left\{1 + \left(\frac{dy}{dx}\right)^2\right\}}, \quad \tan\psi = \frac{dy}{dx}.$$

To pass from polar to pedal coordinates, we have

$$\frac{1}{p^2} = \frac{1}{r^4}\left(\frac{dr}{d\theta}\right)^2 + \frac{1}{r^2}.$$

We do not propose to develop the theory of these new coordinate systems any further. The following illustration demonstrates the use of some of the formulæ.

Illustration 3. *The* **pedal curve** *of a given curve with respect to a given pole.*

The locus of the foot of the perpendicular from a given point O on to the tangent at a variable point P of a given curve is called the *pedal curve* of O with respect to the given curve.

Referring to the diagram (Fig. 82) on p. 110, we see that N is the point of the pedal curve which corresponds to P. The polar coordinates of N are (r_1, θ_1), where

$$r_1 = p,$$

$$\theta_1 = \psi - \tfrac{1}{2}\pi.$$

Let us find the pedal coordinates (p_1, r_1) of N in terms of the pedal coordinates (p, r) of P. We have at once

$$r_1 = p.$$

Also, if ϕ_1 is the angle 'behind' the radius vector for the locus of N,

$$\tan\phi_1 = r_1 \frac{d\theta_1}{dr_1}$$

$$= p\frac{d\psi}{dp} = p\frac{d\theta}{dp} + p\frac{d\phi}{dp}.$$

Now $\quad p = r\sin\phi, \quad r\dfrac{d\theta}{dr} = \tan\phi,$

so that
$$\tan\phi_1 = r\sin\phi\frac{d\theta}{dp} + r\sin\phi\frac{d\phi}{dp}$$

$$= \sin\phi \,.\, r\frac{d\theta}{dr}\frac{dr}{dp} + r\sin\phi\frac{d\phi}{dp}$$

$$= \sin\phi\tan\phi\frac{dr}{dp} + r\sin\phi\frac{d\phi}{dp}$$

$$= \tan\phi\left(\frac{dr}{dp}\sin\phi + r\cos\phi\frac{d\phi}{dp}\right)$$

$$= \tan\phi\frac{d}{dp}(r\sin\phi) = \tan\phi\frac{dp}{dp}$$

$$= \tan\phi.$$

Hence $\quad \phi_1 = \phi,$

since each angle lies between $0, \pi$ and θ is being used as parameter. It follows that

$$p_1 = r_1\sin\phi_1$$

$$= p\sin\phi$$

$$= p^2/r,$$

and so the pedal coordinates (p_1, r_1) of N are $(p^2/r, p)$.

Corollary. Since $\tan\phi_1 = p\dfrac{d\psi}{dp}$, and $\phi = \phi_1$, we have the relation

$$\tan\phi = p\frac{d\psi}{dp},$$

true for any curve.

11. Curvature. The instinctive idea of curvature, or bending, might be expressed in some such phrase as 'change of angle with distance', and it is just this conception which we use for our formal definition.

Definition. (See Fig. 83.) *The* **curvature** κ *at a point P of a curve is defined by the relation*

$$\kappa = \frac{d\psi}{ds}.$$

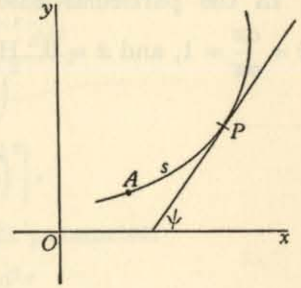


Fig. 83.

The value of κ may be positive or negative, according as ψ increases or decreases with s.

For many purposes, *the calculation of κ is best effected by the direct use of the definition.* It is, however, useful to be able to obtain the formulæ in the various systems of coordinates which we have described.

(i) Cartesian parametric form.

If x, y are functions $x(t), y(t)$ of a parameter t, the relations $\dfrac{dx}{ds} = \cos\psi$, $\dfrac{dy}{ds} = \sin\psi$ assume the form

$$\dot{x} = \dot{s}\cos\psi, \quad \dot{y} = \dot{s}\sin\psi.$$

Differentiating, $\ddot{x} = \ddot{s}\cos\psi - \dot{s}\dot{\psi}\sin\psi$

$$= \ddot{s}\cos\psi - \dot{s}^2\kappa\sin\psi,$$

since $\dot{\psi} = \dfrac{d\psi}{ds}\dot{s} = \kappa\dot{s};$

similarly $\ddot{y} = \ddot{s}\sin\psi + \dot{s}^2\kappa\cos\psi.$

Hence $\ddot{y}\cos\psi - \ddot{x}\sin\psi = \dot{s}^2\kappa,$

or
$$\frac{\ddot{y}\dot{x}}{\dot{s}} - \frac{\ddot{x}\dot{y}}{\dot{s}} = \dot{s}^2\kappa,$$

so that
$$\kappa = \frac{\ddot{y}\dot{x} - \ddot{x}\dot{y}}{\dot{s}^3},$$

where
$$\dot{s} = \sqrt{(\dot{x}^2 + \dot{y}^2)},$$

with (p. 104) POSITIVE square root.

Note. We must choose our parameter to avoid the possibility that $\dot{x} = 0, \dot{y} = 0$ simultaneously. This can be done, for example, by identifying t with x, when $\dot{x} = 1$.

(ii) CARTESIAN FORM.

In the particular case when x is the parameter, we have $\dot{x} = \dfrac{dx}{dx} = 1$, and $\ddot{x} = 0$. Hence, by (i),

$$\kappa = \frac{\dfrac{d^2y}{dx^2}}{\left(\dfrac{ds}{dx}\right)^3}$$

$$= \frac{\dfrac{d^2y}{dx^2}}{\left\{1 + \left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}};$$

the denominator is POSITIVE since $\dfrac{ds}{dx}$ is positive when x is the parameter.

It follows that, *with our conventions, the sign of κ is the same as the sign of* $\dfrac{d^2y}{dx^2}$. Thus (Vol. I, p. 54) the sign of κ is positive when the concavity of the curve is 'upwards', and negative when the concavity is 'downwards'.

(iii) POLAR FORM.

When θ is the parameter, we have the relation (p. 109)

$$\psi = \theta + \phi,$$

so that
$$\kappa = \frac{d\psi}{ds} = \frac{d\psi}{d\theta}\frac{d\theta}{ds}$$

$$= \left(1 + \frac{d\phi}{d\theta}\right)\frac{d\theta}{ds}.$$

Now (p. 110) $\cot\phi = \dfrac{1}{r}\dfrac{dr}{d\theta},$

so that
$$-\operatorname{cosec}^2\phi\frac{d\phi}{d\theta} = \frac{1}{r}\frac{d^2r}{d\theta^2} - \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2,$$

or
$$-\left\{1 + \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2\right\}\frac{d\phi}{d\theta} = \frac{1}{r}\frac{d^2r}{d\theta^2} - \frac{1}{r^2}\left(\frac{dr}{d\theta}\right)^2.$$

Hence
$$\frac{d\phi}{d\theta} = \frac{\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2},$$

and
$$1 + \frac{d\phi}{d\theta} = \frac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{r^2 + \left(\dfrac{dr}{d\theta}\right)^2}.$$

Also (p. 105)
$$\frac{ds}{d\theta} = +\sqrt{\left\{r^2 + \left(\frac{dr}{d\theta}\right)^2\right\}},$$

the positive sign being taken since θ is the parameter.

Hence
$$\kappa = \frac{r^2 + 2\left(\dfrac{dr}{d\theta}\right)^2 - r\dfrac{d^2r}{d\theta^2}}{\left\{r^2 + \left(\dfrac{dr}{d\theta}\right)^2\right\}^{\frac{3}{2}}}.$$

(iv) PEDAL FORM.

We use the Corollary (p. 113)

$$\tan\phi = p\frac{d\psi}{dp},$$

giving
$$\tan\phi = r\sin\phi\,.\frac{d\psi}{ds}\frac{ds}{dr}\frac{dr}{dp}.$$

But $\dfrac{dr}{ds} = \cos\phi$, so that $\dfrac{ds}{dr} = \sec\phi$. Hence

$$\tan\phi = \kappa r \tan\phi \frac{dr}{dp},$$

so that

$$\kappa = \frac{1}{r}\frac{dp}{dr}.$$

The sign conventions used in this proof imply that θ was originally taken as parameter on the curve; otherwise, the sign of κ may require independent examination.

12. A parametric form for a curve in terms of s. Suppose that O is a given point on a curve (Fig. 84). Choose the tangent at O as x-axis and the normal at O as y-axis. We seek to express x, y in terms of s as parameter, assuming that the conditions are such that a Maclaurin expansion is possible.

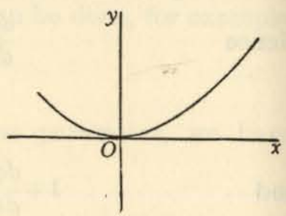


Fig. 84.

The Maclaurin formulæ (p. 49) are

$$x(s) = x(0) + sx'(0) + \frac{s^2}{2!}x''(0) + \ldots,$$

$$y(s) = y(0) + sy'(0) + \frac{s^2}{2!}y''(0) + \ldots.$$

Now

$$\frac{dx}{ds} = \cos\psi,$$

so that

$$\frac{d^2x}{ds^2} = -\sin\psi\frac{d\psi}{ds},$$

$$\frac{d^3x}{ds^3} = -\cos\psi\left(\frac{d\psi}{ds}\right)^2 - \sin\psi\frac{d^2\psi}{ds^2};$$

and

$$\frac{dy}{ds} = \sin\psi,$$

so that

$$\frac{d^2y}{ds^2} = \cos\psi\frac{d\psi}{ds},$$

$$\frac{d^3y}{ds^3} = -\sin\psi\left(\frac{d\psi}{ds}\right)^2 + \cos\psi\frac{d^2\psi}{ds^2}.$$

If we write κ_0, κ_0' to denote the values of κ, $\dfrac{d\kappa}{ds}$ at the origin, then, since $\psi = 0$ there,

$$x'(0) = 1; \qquad x''(0) = 0; \qquad x'''(0) = -\kappa_0^2,$$

$$y'(0) = 0; \qquad y''(0) = \kappa_0; \qquad y'''(0) = \kappa_0'.$$

We therefore have the expansions

$$x(s) = s - \tfrac{1}{6}\kappa_0^2 s^3 + \ldots,$$

$$y(s) = \tfrac{1}{2}\kappa_0 s^2 + \tfrac{1}{6}\kappa_0' s^3 + \ldots.$$

13. Newton's formula. *To prove that the curvature at the origin of a curve passing simply through it, and having $y = 0$ as tangent there, is*

$$\kappa_0 = \lim_{x\to 0}\frac{2y}{x^2}.$$

We use a method like that of the preceding paragraph to obtain the expansion, assumed possible, of y as a series of ascending powers of x. We have that, if

$$y = f(x),$$

then

$$f'(x) = \frac{dy}{dx} = \tan\psi,$$

$$f''(x) = \sec^2\psi\frac{d\psi}{dx}$$

$$= \sec^2\psi\frac{d\psi}{ds}\frac{ds}{dx}$$

$$= \sec^2\psi \, . \, \kappa\sec\psi$$

$$= \kappa\sec^3\psi.$$

Thus $\quad f(0) = 0, \quad f'(0) = 0, \quad f''(0) = \kappa_0,$

so that

$$y = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \ldots$$

$$= \tfrac{1}{2}\kappa_0 x^2 + \ldots.$$

Hence

$$\frac{2y}{x^2} = \kappa_0 + (\text{terms involving } x),$$

and so, assuming conditions to be such that the remainder tends to zero with x,

$$\lim_{x\to 0}\frac{2y}{x^2} = \kappa_0.$$

ILLUSTRATION 4. *To find the curvature of an ellipse, of semi-axes a, b, at an end of the minor axis.*

Take the required end of the minor axis as origin (Fig. 85) and the tangent there as the x-axis. The equation of the ellipse is

$$\frac{x^2}{a^2}+\frac{(y-b)^2}{b^2}=1.$$

Hence $$y-b=\pm b\sqrt{\left\{1-\frac{x^2}{a^2}\right\}}.$$

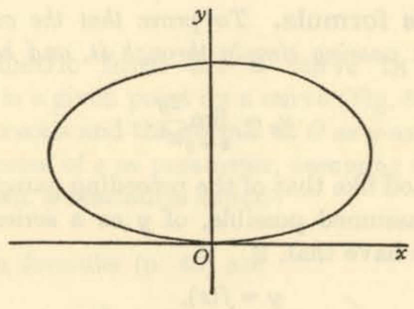


Fig. 85.

For values of y near the origin, the negative square root must be taken, giving

$$\frac{y}{b}=1-\sqrt{\left\{1-\frac{x^2}{a^2}\right\}}$$

$$=1-\left[1-\frac{1}{2}\frac{x^2}{a^2}-\frac{1}{8}\frac{x^4}{a^4}+\ldots\right]$$

$$=\frac{x^2}{2a^2}+\frac{x^4}{8a^4}+\ldots,$$

so that $$\frac{2y}{x^2}=\frac{b}{a^2}+\frac{bx^2}{4a^4}+\ldots.$$

Hence $$\kappa=\lim_{x\to 0}\frac{2y}{x^2}$$

$$=\frac{b}{a^2}.$$

14. The circle of curvature. The formulæ

$$\tan\psi=\frac{dy}{dx},$$

$$\kappa=\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}$$

show that, if the two curves

$$y=f(x),$$
$$y=g(x)$$

both pass through a point $P(\xi,\eta)$, so that

$$f(\xi)=g(\xi)=\eta,$$

and if, further, $$f'(\xi)=g'(\xi)$$

and $$f''(\xi)=g''(\xi),$$

then the two curves have the same gradient and curvature at P.

In particular, the circle which passes through P, touches the given curve $y=f(x)$ at P, and has the same curvature as the given curve at P, is called the *circle of curvature* of the given curve at P.

Denote by $$y_P,\quad y_P',\quad y_P''$$

the values of y, y', y'' for the given curve at the point $P(x_P, y_P)$. Suppose that the equation of the circle of curvature at P is

$$(x-\alpha)^2+(y-\beta)^2=\rho^2,$$

where $C\ (\alpha,\beta)$ is the centre of the circle of curvature and ρ its radius. By differentiation of this equation with respect to x, we obtain the relations

$$(x-\alpha)+(y-\beta)\,y'=0,$$
$$1+y'^2+(y-\beta)\,y''=0.$$

Since x, y, y', y'' are the same for the circle of curvature as for the given curve,

$$(x_P-\alpha)^2+(y_P-\beta)^2=\rho^2,$$
$$(x_P-\alpha)+(y_P-\beta)\,y_P'=0,$$
$$(1+y_P'^2)+(y_P-\beta)\,y_P''=0.$$

Hence
$$\beta = y_P + \frac{(1+y_P'^2)}{y_P''},$$

$$\alpha = x_P + \left\{-\frac{(1+y_P'^2)}{y_P''}\right\} y_P'$$

$$= x_P - \frac{y_P'(1+y_P'^2)}{y_P''},$$

$$\rho^2 = \frac{(1+y_P'^2)^2}{y_P''^2}(y_P'^2+1)$$

$$= \frac{(1+y_P'^2)^3}{y_P''^2}$$

$$= \frac{1}{\kappa_P^2},$$

where κ_P is the curvature of the given curve at P.

We therefore have the formulæ

$$\alpha = x_P - \frac{y_P'(1+y_P'^2)}{y_P''},$$

$$\beta = y_P + \frac{(1+y_P'^2)}{y_P''},$$

$$\rho = \pm\frac{(1+y_P'^2)^{\frac{3}{2}}}{y_P''}$$

$$= \pm\frac{1}{\kappa_P},$$

the sign being selected to make ρ positive.

If the coordinates x, y of a point on the curve are given as functions of a parameter t, then

$$\frac{dy}{dx} = \frac{dy}{dt}\bigg/\frac{dx}{dt},$$

$$\frac{d^2y}{dx^2} = \frac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{dy}{dt}\dfrac{d^2x}{dt^2}}{\left(\dfrac{dx}{dt}\right)^2}\frac{dt}{dx},$$

so that, if dots denote differentiations with respect to t,

$$\frac{dy}{dx} = \frac{\dot{y}}{\dot{x}},$$

$$\frac{d^2y}{dx^2} = \frac{\dot{x}\ddot{y} - \dot{y}\ddot{x}}{\dot{x}^3}.$$

Hence
$$\alpha = x_P - \frac{\dot{y}_P(\dot{x}_P^2 + \dot{y}_P^2)}{\dot{x}_P\ddot{y}_P - \dot{y}_P\ddot{x}_P},$$

$$\beta = y_P + \frac{\dot{x}_P(\dot{x}_P^2 + \dot{y}_P^2)}{\dot{x}_P\ddot{y}_P - \dot{y}_P\ddot{x}_P},$$

$$\rho = \pm\frac{(\dot{x}_P^2 + \dot{y}_P^2)^{\frac{3}{2}}}{\dot{x}_P\ddot{y}_P - \dot{y}_P\ddot{x}_P}$$

$$= \pm\frac{1}{\kappa_P}.$$

The point $C\ (\alpha, \beta)$ is called the *centre of curvature* of the given curve at P, and ρ its *radius of curvature*. We are adopting a convention of signs in which the radius of curvature is essentially positive.

ILLUSTRATION 5. *The curvature of a circle, and the sign of the curvature.*

Too much emphasis may easily be given to questions about the sign of curvature. Usually common sense and a diagram will settle all that is wanted. We give, however, an exposition for the case when the given curve is itself a circle, so that the reader may, if he wishes, be enabled to examine more elaborate examples.

The difficulties about sign arise, with our conventions, as a result of varying choices of the parameter used to determine the curve. We begin with the simplest case, in which the parameter is selected so that the circle is described completely in one definite sense as the parameter increases.

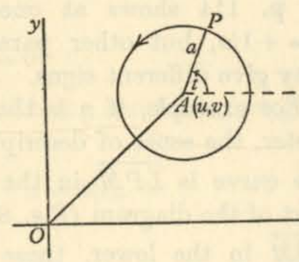


Fig. 86.

Let $A\ (u, v)$ be the centre of the given circle (Fig. 86), and a its radius. Take a variable point $P\ (x, y)$ of the circle, and denote by t

the angle which the radius vector AP makes with the positive direction of the x-axis. As t (the parameter) increases from 0 to 2π, the point P describes the circle completely in the counter-clockwise sense. Then

$$x = u + a\cos t, \quad y = v + a\sin t,$$

so that

$$\dot{x} = -a\sin t, \quad \dot{y} = a\cos t,$$

$$\ddot{x} = -a\cos t, \quad \ddot{y} = -a\sin t.$$

Hence

$$\dot{x}^2 + \dot{y}^2 = a^2,$$

$$\dot{x}\ddot{y} - \dot{y}\ddot{x} = a^2(\sin^2 t + \cos^2 t) = a^2.$$

We therefore have the relations

$$\alpha = (u + a\cos t) - \frac{(a\cos t)(a^2)}{a^2} = u,$$

$$\beta = (v + a\sin t) + \frac{(-a\sin t)(a^2)}{a^2} = v,$$

$$\rho = \pm\frac{a^3}{a^2} = a.$$

Hence for all points on a given circle, *the centre of curvature is at the centre of the circle, and the radius of curvature is equal to the radius of the circle.*

The curvature at any point is therefore $\pm 1/a$. With the present choice of parameter, the formula of p. 114 shows at once that $\kappa = +1/a$, but other parameters may give different signs.

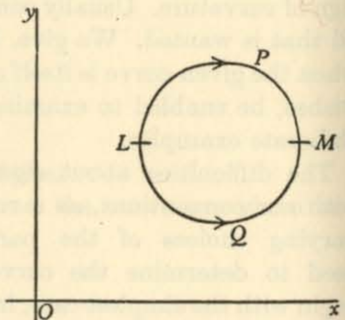


Fig. 87.

For example, if x is the parameter, the sense of description of the curve is $\overrightarrow{LPM}$ in the upper part of the diagram (Fig. 87) and $\overrightarrow{LQM}$ in the lower, these being the directions taken by P, Q respectively as x increases. We know (p. 114) that, when x is the parameter, κ is positive when the concavity is 'upwards' and negative where it is 'downwards'. Thus $\kappa = -1/a$ in the arc LPM, and $+1/a$ in the arc LQM. In fact, if $P(x,y)$ is in the arc LPM, then

$$y = v + \sqrt{\{a^2 - (x-u)^2\}},$$

where the positive sign is attached to the square root. Hence

$$y' = \frac{-(x-u)}{\{a^2 - (x-u)^2\}^{\frac{1}{2}}},$$

so that

$$1 + y'^2 = \frac{a^2}{a^2 - (x-u)^2},$$

and

$$y'' = \frac{-1}{\{a^2 - (x-u)^2\}^{\frac{1}{2}}} - \frac{(x-u)^2}{\{a^2 - (x-u)^2\}^{\frac{3}{2}}}$$

$$= -\frac{a^2}{\{a^2 - (x-u^2\}^{\frac{3}{2}}}.$$

Applying the formula (p. 114) for κ, we have

$$\kappa = -\frac{a^2}{\{a^2 - (x-u)^2\}^{\frac{3}{2}}} \bigg/ \frac{a^3}{\{a^2 - (x-u)^2\}^{\frac{3}{2}}}$$

$$= -\frac{1}{a}.$$

Similarly we may prove that $\kappa = +1/a$ in the arc LQM, where

$$y = v - \sqrt{\{a^2 - (x-u)^2\}}.$$

15. Envelopes. We have been considering a curve as the path traced out by a point whose coordinates are expressed in terms of a parameter. An analogous (*dual*) problem is the study of a system of straight lines

$$lx + my + n = 0$$

when the coefficients l, m, n, instead of being constants, are given to be functions of a parameter t, say

$$l = f(t), \quad m = g(t), \quad n = h(t).$$

A familiar example is the system

$$x - yt + at^2 = 0$$

consisting of the tangents to the parabola

$$y^2 = 4ax.$$

If we take a number of values of t, we obtain correspondingly a number of lines, which lie in some such way as that indicated in the diagram (Fig. 88). They look, in fact, as if a curve could be determined to which they are all tangents. More precisely, the diagram assumes that the individual lines are numbered in an

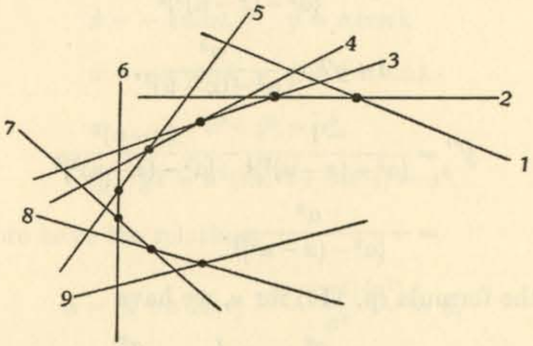


Fig. 88.

order corresponding to increasing values of the parameter, and the points of intersection of consecutive pairs $(1, 2)$, $(2, 3)$, $(3, 4)$,... have been emphasized by dots. These dots appear to lie on a curve, and it is easy to conceive of the lines as becoming tangents to that curve as their number increases indefinitely. In that case, the lines are said to *envelop* the curve, and we make the following formal definition:

DEFINITION. *Given a system of lines*

$$lx + my + n = 0,$$

whose coefficients l, m, n are functions of a parameter t, the locus of that point on a typical line of the system, which is the limiting position of the intersection of a neighbouring line tending to coincidence with it, is called the envelope of the system.

Consider, for example, the system of which a typical line is

$$x - yt + at^2 = 0.$$

Another line of the system is

$$x - yu + au^2 = 0.$$

They meet where $\quad x = aut, \quad y = a(u+t).$

That point on the typical line to which the intersection tends is found by putting $u = t$ in the expressions for the coordinates, giving

$$x = at^2, \quad y = 2at.$$

This is the parametric representation of the envelope, whose equation is therefore

$$y^2 = 4ax.$$

We may now find the rule for determining parametrically the envelope of the system of which a typical line is

$$xf(t) + yg(t) + h(t) = 0.$$

Another line of the system is

$$xf(u) + yg(u) + h(u) = 0.$$

Where these lines meet, it is also true that

$$x\{f(u) - f(t)\} + y\{g(u) - g(t)\} + \{h(u) - h(t)\} = 0,$$

or, on division by $u - t$, that

$$x\frac{f(u) - f(t)}{u - t} + y\frac{g(u) - g(t)}{u - t} + \frac{h(u) - h(t)}{u - t} = 0.$$

To emphasize the limiting approach of u to t, write $u = t + \delta t$. Then the point of intersection of the two lines 'u', 't' also lies on the line

$$x\frac{f(t+\delta t) - f(t)}{\delta t} + y\frac{g(t+\delta t) - g(t)}{\delta t} + \frac{h(t+\delta t) - h(t)}{\delta t} = 0.$$

In the limit, as $\delta t \to 0$, this is the line

$$xf'(t) + yg'(t) + h'(t) = 0.$$

Hence *the envelope is the locus, as t varies, of the point of intersection of the lines*

$$xf(t) + yg(t) + h(t) = 0,$$

$$xf'(t) + yg'(t) + h'(t) = 0.$$

ILLUSTRATION 6. *To find the envelope of the system*

$$x \cos t + y \sin t + a = 0.$$

The envelope is the locus of the point of intersection of this line with the line

$$-x \sin t + y \cos t = 0.$$

That point is $x = -a\cos t,\quad y = -a\sin t,$

and so the envelope is the circle

$$x^2+y^2 = a^2.$$

Note. Envelopes exist for many families of curves as well as for families of straight lines; but we are not in a position to give a treatment of the more general case.

EXAMPLES II

Find the envelopes of the following families of straight lines:

1. $(1+t^2)x+2ty+(1-t^2)a = 0.$
2. $x\sec t-y\tan t-a = 0.$
3. $x\cosh t-y\sinh t-a = 0.$
4. $2tx+(1-t^2)y+(1+t^2)a = 0.$
5. $tx+y-a(t^3+2t) = 0.$
6. $t^3x-ty-c(t^4-1) = 0.$

16. Evolutes. Particular interest attaches to the envelope of those lines which are the normals of a given curve. Using the notation of § 14, denote by

$$y_P,\quad y_P',\quad y_P''$$

the values of y, y', y'' (where dashes denote differentiations with respect to x) for the given curve at the point $P(x_P, y_P)$. The equation of the normal at P is

$$y-y_P = (-1/y_P')(x-x_P),$$

or

$$x+y_P'y = x_P+y_P'y_P.$$

Taking x_P as the parameter, the envelope of this line is the locus of its intersection with the line

$$y_P''y = 1+y_P''y_P+y_P'^2,$$

so that

$$y = y_P+\frac{(1+y_P'^2)}{y_P''},$$

$$x = x_P-\frac{y_P'(1+y_P'^2)}{y_P''}.$$

Comparison with the results in § 14 establishes the theorem: *The centre of curvature at a point P of a given curve is that point on the normal at P which corresponds to P on the envelope of the normals.*

Definition. The envelope of the normals, which is also the locus of the centres of curvature, is called the *evolute* of the given curve.

Illustration 7. *To find the evolute of the rectangular hyperbola*

$$xy = c^2.$$

Parametrically, the hyperbola is

$$x = ct,\quad y = c/t,$$

and the gradient at this point is $-1/t^2$. Hence the normal is

$$y-c/t-t^2(x-ct) = 0,$$

or

$$t^3x-ty = ct^4-c.$$

For the envelope, we have

$$3t^2x-y = 4ct^3.$$

The centre of curvature, being the point of intersection of these two lines, is given by

$$2t^3x = 3ct^4+c,$$

so that

$$x = \left(\frac{3t}{2}+\frac{1}{2t^3}\right)c,$$

$$y = \left(\frac{3}{2t}+\frac{t^3}{2}\right)c.$$

The evolute is the curve given parametrically by these two equations.

EXAMPLES III

Find the evolutes of the following curves:

1. The parabola $(at^2, 2at)$.
2. The ellipse $(a\cos t, b\sin t)$.
3. The rectangular hyperbola $(a\sec t, a\tan t)$.

17. The area of a closed curve. Consider the closed curve $PUQV$ shown in the diagram (Fig. 89). For simplicity, we suppose it to be oval in shape, and also, to begin with, to lie entirely in the first quadrant.

The coordinates of the points of the curve being expressed in the parametric form

$$x = x(t), \quad y = y(t),$$

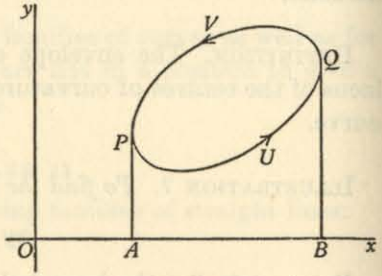


Fig. 89.

we suppose that the positive sense, namely that of t increasing, is COUNTER-CLOCKWISE round the curve, as implied by the arrows in the diagram. [If this is not so, replacement of t by $-t$ will reverse the sense.] Thus the curve is described once by the point (x, y) as t increases from a value t_0 to a value t_1. Moreover, since the curve is closed, the two values t_0, t_1 give rise to the SAME point, so that

$$x(t_0) = x(t_1); \quad y(t_0) = y(t_1).$$

A simple example is the circle

$$x = 5 + 3\cos t, \quad y = 4 + 3\sin t,$$

described once in the counter-clockwise sense as t increases from 0 to 2π. The values $t = 0, t = 2\pi$ both give the point $(8, 4)$.

In order to calculate the area, draw the ordinates AP, BQ which just contain the curve, touching it at two points P, Q whose parameters we write as t_P, t_Q, respectively. For reference, let U be a point in the lower arc PQ and V in the upper. Then the area enclosed by the curve can be expressed in the form

$$\text{area } APVQ - \text{area } APUQ.$$

The area $APVQ$ is given by the formula

$$\text{area } APVQ = \int_{x_P}^{x_Q} y\,dx$$

integrated over points (x, y) on the arc PVQ. Let us suppose first that the 'junction' point given by t_0 or t_1, does not lie on this arc.

Then the parameter t varies steadily along the arc, and the area is

$$\int_{t_P}^{t_Q} y\frac{dx}{dt}\,dt.$$

On the other hand, the 'junction' point J must then lie on the lower arc PUQ, so we write the formula for the area $APUQ$ in the form

$$\text{area } APUQ = \int_{x_P}^{x_Q} y\,dx$$

integrated over points (x, y) on the arc PUQ, giving

$$\text{area } APUQ = \int_{x_P}^{J} y\,dx + \int_{J}^{x_Q} y\,dx$$
$$= \int_{t_P}^{t_1} y\frac{dx}{dt}\,dt + \int_{t_0}^{t_Q} y\frac{dx}{dt}\,dt,$$

where the value t_1 or t_0 is taken for J according to the segment of the arc PUQ over which the respective integral is calculated, t_1 for PJ and t_0 for JQ. In all, the area enclosed by the curve is thus

$$\int_{t_P}^{t_Q} y\frac{dx}{dt}\,dt - \left\{\int_{t_P}^{t_1} y\frac{dx}{dt}\,dt + \int_{t_0}^{t_Q} y\frac{dx}{dt}\,dt\right\}$$
$$= -\left\{\int_{t_0}^{t_Q} y\frac{dx}{dt}\,dt + \int_{t_Q}^{t_P} y\frac{dx}{dt}\,dt + \int_{t_P}^{t_1} y\frac{dx}{dt}\,dt\right\}$$
$$= -\int_{t_0}^{t_1} y\frac{dx}{dt}\,dt.$$

Similarly, if J lies on the arc PVQ, we have the formulæ

$$\text{area } APVQ = \int_{x_P}^{J} y\,dx + \int_{J}^{x_Q} y\,dx$$
$$= \int_{t_P}^{t_0} y\frac{dx}{dt}\,dt + \int_{t_1}^{t_Q} y\frac{dx}{dt}\,dt,$$

$$\text{area } APUQ = \int_{x_P}^{x_Q} y\,dx$$
$$= \int_{t_P}^{t_Q} y\frac{dx}{dt}\,dt,$$

so that the area enclosed by the curve is

$$\left\{\int_{t_P}^{t_0} y\frac{dx}{dt}\,dt+\int_{t_1}^{t_Q} y\frac{dx}{dt}\,dt\right\}-\int_{t_P}^{t_Q} y\frac{dx}{dt}\,dt$$

$$=-\left\{\int_{t_0}^{t_P} y\frac{dx}{dt}\,dt+\int_{t_P}^{t_Q} y\frac{dx}{dt}\,dt+\int_{t_Q}^{t_1} y\frac{dx}{dt}\,dt\right\}$$

$$=-\int_{t_0}^{t_1} y\frac{dx}{dt}\,dt.$$

Hence, in both cases, *the area of the closed curve is*

$$-\int_{t_0}^{t_1} y\frac{dx}{dt}\,dt.$$

In the same way, if we draw the lines CR, DS parallel to Ox (Fig. 90), just containing the curve, to touch it at R, S, and if L, M are points on the left and right arcs RS respectively, then the area of the closed curve is

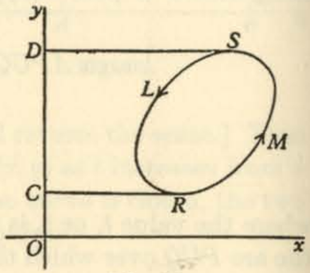


Fig. 90.

area $CRMS$ − area $CRLS$.

Now $$\text{area } CRMS=\int_{y_R}^{y_S} x\,dy$$

integrated over points (x,y) on the arc RMS. If the 'junction' point J is not on this arc, we have

$$\int_{t_R}^{t_S} x\frac{dy}{dt}\,dt.$$

For the area $CRLS$, the 'junction' point J must then lie on the arc RLS, so we have

$$\int_{t_R}^{t_0} x\frac{dy}{dt}+\int_{t_1}^{t_S} x\frac{dy}{dt}\,dt.$$

In all, the area of the closed curve is (in brief notation)

$$\int_{t_R}^{t_S}-\left\{\int_{t_R}^{t_0}+\int_{t_1}^{t_S}\right\}$$

$$=\int_{t_0}^{t_R}+\int_{t_R}^{t_S}+\int_{t_S}^{t_1}$$

$$=\int_{t_0}^{t_1} x\frac{dy}{dt}\,dt.$$

Similarly for the 'junction' point on the arc RMS, we have

$$\left\{\int_{t_R}^{t_1}+\int_{t_0}^{t_S}\right\}-\left\{\int_{t_R}^{t_S}\right\}$$

$$=\int_{t_0}^{t_S}+\int_{t_S}^{t_R}+\int_{t_R}^{t_1}$$

$$=\int_{t_0}^{t_1} x\frac{dy}{dt}\,dt.$$

Hence *the area of the closed curve may be expressed in either of the forms*

$$-\int_{t_0}^{t_1} y\frac{dx}{dt}\,dt,$$

$$+\int_{t_0}^{t_1} x\frac{dy}{dt}\,dt,$$

where the closed curve is described completely, in the counter-clockwise sense, as t increases from t_0 to t_1.

A useful alternative form is found by taking half from each of these:

The area of the closed curve is

$$\frac{1}{2}\int_{t_0}^{t_1}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt.$$

ILLUSTRATION 8. *To find the area of the ellipse*

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1.$$

The ellipse is traced out by the point

$$x=a\cos t,\quad y=b\sin t$$

as t moves from 0 to 2π. Then

$$dx=-a\sin t\,dt,\quad dy=b\cos t\,dt,$$

and $\frac{1}{2}\int(x\,dy-y\,dx)$

$$=\frac{1}{2}\int_0^{2\pi}(a\cos t\,.\,b\cos t+b\sin t\,.\,a\sin t)\,dt$$

$$=\frac{1}{2}ab\Big[t\Big]_0^{2\pi}$$

$$=\pi ab.$$

The restriction for the curve to lie in the first quadrant is not essential. For, if it does not, a transformation of the type

$$x' = x+a, \quad y' = y+b,$$

for suitable values of a, b, can always be employed to bring it into the first quadrant of a fresh set of axes. But this shows that the area enclosed is

$$\frac{1}{2}\int_{t_0}^{t_1}\left(x'\frac{dy'}{dt}-y'\frac{dx'}{dt}\right)dt = \frac{1}{2}\int_{t_0}^{t_1}\left\{(x+a)\frac{dy}{dt}-(y+b)\frac{dx}{dt}\right\}dt$$

$$= \frac{1}{2}\int_{t_0}^{t_1}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt+\tfrac{1}{2}a\int_{t_0}^{t_1}\frac{dy}{dt}\,dt-\tfrac{1}{2}b\int_{t_0}^{t_1}\frac{dx}{dt}\,dt.$$

But
$$\int_{t_0}^{t_1}\frac{dy}{dt}\,dt = \Big[y\Big]_{t_0}^{t_1} = 0,$$

$$\int_{t_0}^{t_1}\frac{dx}{dt}\,dt = \Big[x\Big]_{t_0}^{t_1} = 0,$$

since t_0, t_1 give the SAME point of the curve. Hence the area is

$$\frac{1}{2}\int_{t_0}^{t_1}\left(x\frac{dy}{dt}-y\frac{dx}{dt}\right)dt.$$

18.* Second theorem of Pappus. Suppose that a surface of revolution is obtained by rotating an arc PQ (Fig. 91) about the x-axis (assumed not to meet it). We proved (Vol. I, p. 130) that, if PQ is the curve

$$y = f(x),$$

the area S so generated is given by the formula

$$S = \int_a^b 2\pi y\,ds.$$

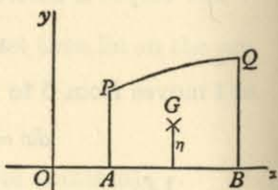


Fig. 91.

Now it is easy to prove that the y-coordinate of the centre of gravity of the arc is given by the formula

$$\eta = \frac{\int_a^b y\,ds}{l},$$

where l is the length of the arc PQ. (Compare the similar work in Chapter VI.) Hence

$$S = 2\pi\eta l.$$

* This paragraph may be postponed, if desired.

Thus *if a given curve, lying on one side of a given line, is rotated about that line as axis to form a surface of revolution, then the area of the surface so generated is equal to the product of the length of the curve and the distance rotated by its centre of gravity.*

ILLUSTRATION 9. *To find the centre of gravity of a semicircular arc.*

Suppose that the circle is of radius a (Fig. 92), and that the centre of gravity, lying on the axis of symmetry, is at a distance η from the centre.

On rotating the semicircle about its bounding diameter, we obtain the surface of a sphere, whose area is known to be

$$4\pi a^2.$$

Hence
$$4\pi a^2 = 2\pi\eta\,.\,\pi a,$$

so that
$$\eta = \frac{2a}{\pi}.$$

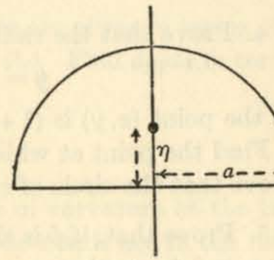


Fig. 92.

REVISION EXAMPLES V

'Advanced' Level

1. Find the equations of the tangent and normal at any point of the cycloid given by the equations

$$x = a(2\psi+\sin 2\psi), \quad y = a(1-\cos 2\psi).$$

Prove that ψ is the angle which the tangent makes with the axis of x; and verify that, if p and q are the lengths of the perpendiculars drawn from the origin to the tangent and normal respectively, then q and $dp/d\psi$ are numerically equal.

2. Define the length of an arc of a curve and obtain an expression for it.

A curve is given in the form

$$x = \cosh t - t, \quad y = \cosh t + t.$$

Express t in terms of the length s of the arc of the curve measured from the point $(1, 1)$.

The coordinates of any point of the curve are expressed in terms of a and are then expanded in series of ascending powers of s.

Prove that the first few terms of the expansions are

$$x = 1 - \frac{s}{\sqrt{2}} + \frac{s^2}{4} + \dots,$$

$$y = 1 + \frac{s}{\sqrt{2}} + \frac{s^2}{4} + \dots.$$

3. Find the radius of curvature of the parabola $y^2 = 4\lambda x$ at a point for which $x = c$, and deduce that, if λ varies (c remaining constant), this radius of curvature is a minimum when $\lambda = \frac{1}{2}c$.

4. Prove that the radius of curvature of the curve

$$y = \tfrac{1}{4}x^2 - \tfrac{1}{2}\log x \quad (x > 0)$$

at the point (x, y) is $(1+x^2)^2/4x$.

Find the point at which this curve is parallel to the x-axis, and prove that the circle of curvature at this point touches the y-axis.

5. Prove that, if ψ is the inclination to the x-axis of the tangent at a point on the curve $y = a \log \sec (x/a)$, then the radius of curvature at this point is $a \sec \psi$.

6. A particle moves in a plane so that, at time t, its coordinates referred to rectangular axes are given by

$$x = a \cos 2t + 2a \cos t, \quad y = a \sin 2t + 2a \sin t.$$

Find the components of the velocity parallel to the axes and the resultant speed of the particle.

Show that ρ, the radius of curvature at a point of the path, is proportional to the speed of the particle at that point.

7. The coordinates of the points of the curve $4y^3 = 27x^2$ are expressed parametrically in the form $(2t^3, 3t^2)$. By using this parametric representation, or otherwise, prove that the length of the curve between the origin and the point P with parameter t_1 is

$$2(1+t_1^2)^{\frac{3}{2}} - 2,$$

and that the radius of curvature at P is numerically equal to

$$6t_1(1+t_1^2)^{\frac{3}{2}}.$$

8. The tangent to a curve at the point (x, y) makes an angle ψ with the x-axis. Prove that the centre of curvature at (x, y) is

$$\left(x - \frac{dy}{d\psi}, \quad y + \frac{dx}{d\psi}\right).$$

A curve is given parametrically by the equations

$$x = 2a \cos t + a \cos 2t, \quad y = 2a \sin t - a \sin 2t.$$

Prove that $\psi = -\frac{1}{2}t$.

P is a variable point on the curve, and Q is the centre of curvature at P; the point R divides PQ internally so that $PR = \frac{1}{4}PQ$. Prove that the locus of R is the circle $x^2 + y^2 = 9a^2$.

Prove also that no point of the curve lies outside this circle.

Draw a rough sketch of the curve.

9. The coordinates of a point of a curve are given in terms of a parameter t by the equations $x = te^t$, $y = t^2e^t$. Find dy/dx in terms of t, and prove that

$$\frac{d^2y}{dx^2} = \frac{t^2+2t+2}{(t+1)^3}e^{-t}.$$

Prove also that the radii of the circles of curvature at the two points at which the curve is parallel to the x-axis are in the ratio $e^2 : 1$.

10. Express $\cosh^4 \theta$ in the form

$$a_0 + a_1 \cosh \theta + a_2 \cosh 2\theta + a_3 \cosh 3\theta + a_4 \cosh 4\theta.$$

11. The area lying between the curve

$$y = \cosh (x/2\lambda),$$

the ordinates $x = \pm \lambda^2$, and the x-axis is rotated about this axis. Prove that the volume of the solid of revolution so formed is

$$\pi\lambda(\lambda + \sinh \lambda).$$

Show that the volumes given by $\lambda = 1, \lambda = 1 + \delta$ differ by approximately $\pi(2+e)\,\delta$ when δ is small.

12. Prove that the two curves

$$y_1 = \tfrac{1}{2}\cosh 2x, \quad y_2 = \tanh 2x$$

touch at one point and have no other point in common.

Sketch the two curves in the same diagram and, with them, the curves

$$y_3 = \cosh x, \quad y_4 = \sinh x.$$

Prove that the four curves intersect in pairs for two values of x and indicate clearly in the diagram the relative positions of the four curves for all values of x.

13. If s be the length of the arc of the catenary $y = a\cosh(x/a)$ from the point $(0, a)$ to the point (x, y), show that

$$s^2 = y^2 - a^2.$$

Find the area of the surface generated when this arc is rotated about the y-axis.

14. Sketch the curve whose coordinates are given parametrically by the relations $x = a(t + \sin t), y = a(1 + \cos t)$ for values of t between $-\pi, \pi$, and find the length of the curve between these two points.

15. Find the radius of curvature of the parabola $y^2 = x$ at the point $(2, \sqrt{2})$.

16. Find the radius of curvature and the coordinates of the centre of curvature at the point $(0, 1)$ on the curve $y = \cosh x$.

17. For what value of λ does the parabola

$$y^2 = 4\lambda x$$

have the same circle of curvature as the ellipse

$$\left(\frac{x-a}{a}\right)^2 + \frac{y^2}{b^2} = 1$$

at the origin?

18. Find the radius of curvature of the curve

$$y = \sin ax^2$$

at the origin.

19. Draw a rough graph of the function $\cosh x$.

The tangent at a point $P(x, y)$ of the curve $y = c\cosh(x/c)$ makes an angle ψ with the x-axis. Show that $y = c\sec\psi$.

If the tangent at P cuts the y-axis at Q, show that

$$PQ = xy/c.$$

20. Find the radius of curvature of the curve

$$ay^2 = x^3$$

at the point (a, a).

21. Find the radius of curvature at the point $t = \frac{1}{3}\pi$ on the curve

$$x = a\sin t, \quad y = a\cos 2t.$$

22. Find the radius of curvature of the curve

$$y = x^2 + x - 1$$

at the point (x, y) and find a point on the curve for which the centre of curvature is on the y-axis.

23. Show that the radius of curvature of the epicycloid

$$x = 3\cos t - \cos 3t, \quad y = 3\sin t - \sin 3t$$

is given by $3\sin t$.

24. Sketch the curve whose equation is

$$r = a\sin 3\theta \quad (a > 0).$$

Find the area of the loop in the first quadrant and show that the radius of curvature at the point (r, θ) is

$$\frac{1}{2}a\frac{(5 + 4\cos 6\theta)^{\frac{3}{2}}}{(7 + 2\cos 6\theta)}.$$

25. Show that the two functions

$$\sinh^{-1}(\tan x), \quad \tanh^{-1}(\sin x)$$

have equal derivatives, and hence prove that the functions themselves are equal when $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$.

26. Find the point of maximum curvature on the curve

$$y = \log x,$$

and the curvature at this point.

27. If y is the function of x given by the relation

$$\sinh y = \tan x,$$

where $-\frac{1}{2}\pi < x < \frac{1}{2}\pi$, prove that

$$\cosh y = \sec x, \quad y = \log\tan(\tfrac{1}{2}x + \tfrac{1}{4}\pi).$$

Show that y has an inflexion at $x = 0$, and draw a rough sketch of the function in the given range.

REVISION EXAMPLES VI

'Scholarship' Level

1. Transform the equation

$$\frac{d^2u}{dz^2}-\frac{2z}{1-z^2}\frac{du}{dz}+\frac{u}{(1-z^2)^2}=0$$

to one in which y is the dependent and x the independent variable, where

$$u(1-z^2)^{\frac{1}{2}}=y \quad\text{and}\quad (1+z^2)/(1-z^2)=x,$$

obtaining the result in the form

$$(x^2-1)\frac{d^2y}{dx^2}+(2x-1)\frac{dy}{dx}+\tfrac{1}{2}y=0.$$

2. If $f(x)$ is a polynomial which increases as x increases, show that, when $x>0$,

$$g(x)\equiv\frac{1}{x}\int_0^x f(y)\,dy$$

is also a polynomial which increases as x increases.

Show that, when $x>0$, the expression

$$\frac{(x^2-2x+2)\,e^x}{x}-\frac{2}{x}$$

is an increasing function of x.

3. Prove that

$$(x+1)\left(\frac{d}{dx}\right)^{n+1}\{(x+1)^n(x-1)^{n+1}\}=(n+1)\left(\frac{d}{dx}\right)^n\{(x+1)^{n+1}(x-1)^n\}.$$

Prove also that the function

$$\left(\frac{d}{dx}\right)^n\{(x+1)^{n+1}(x-1)^n\}$$

satisfies the equation

$$(1-x^2)\frac{d^2y}{dx^2}-(1+x)\frac{dy}{dx}+(n+1)^2y=0.$$

4. If $y=\sin(m\sin^{-1}x)$, show that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}+m^2y=0.$$

Hence or otherwise show that the expansion of $\sin m\theta$ as a power series in $\sin\theta$ is

$$m\sin\theta\left\{1-\frac{(m^2-1)}{3!}\sin^2\theta+\frac{(m^2-1)(m^2-3^2)}{5!}\sin^4\theta-\dots\right.$$
$$\left.+(-1)^n\frac{(m^2-1)\dots[m^2-(2n-1)^2]}{(2n+1)!}\sin^{2n}\theta+\dots\right\}.$$

5. By induction, or otherwise, prove that, if $y=\cot^{-1}x$, then

$$\frac{d^ny}{dx^n}=(-1)^n(n-1)!\sin ny\sin^n y.$$

Hence, or otherwise, prove that, if

$$z=\tan^{-1}\left(\frac{x\sin\alpha}{1+x\cos\alpha}\right),$$

then $$\frac{d^nz}{dx^n}=(-1)^{n-1}\frac{(n-1)!}{\sin^n\alpha}\sin n(\alpha-z)\sin^n(\alpha-z).$$

6. The function $f_n(x)$ is defined by the relation

$$f_n(x)=\frac{1}{y}\frac{d^ny}{dx^n},$$

where $y=e^{x^2}$. Prove that

$$\frac{dy}{dx}=2xy$$

and deduce that

$$f_{n+2}(x)-2xf_{n+1}(x)-2(n+1)f_n(x)=0,$$

and that $f_n(x)$ is a polynomial in x of degree n.

Prove that $\quad f_{n+1}(x)=f_n{}'(x)+2xf_n(x),$

and hence express $f_{n+2}(x)$ in terms of $f_n(x)$, $f_n{}'(x)$ and $f_n{}''(x)$. Deduce that

$$f_n{}''(x)+2xf_n{}'(x)-2nf_n(x)=0.$$

7. If $y=\sin^{-1}x$, prove that

$$(1-x^2)\frac{d^2y}{dx^2}-x\frac{dy}{dx}=$$

Determine the values of y and its successive derivatives when $x=0$, and hence expand y in a series of ascending powers of x.

8. Prove that, if $y = \tan^{-1} x$, then

$$u \equiv \frac{d^n y}{dx^n} = (n-1)!\cos^n y \cos\{ny + \tfrac{1}{2}(n-1)\pi\}$$

for every positive integer n.

Deduce, or prove otherwise, that u satisfies the differential equation

$$(1+x^2)\frac{d^2u}{dx^2} + 2(n+1)x\frac{du}{dx} + n(n+1)u = 0.$$

9. If
$$y_n(x) = \frac{d^n}{dx^n}\{(x^2-1)^n\},$$

prove the relations

$$\frac{dy_{n+1}}{dx} = (x^2-1)\frac{d^2y_n}{dx^2} + 2(n+2)x\frac{dy_n}{dx} + (n+1)(n+2)y_n,$$

$$\frac{dy_{n+1}}{dx} = 2(n+1)x\frac{dy_n}{dx} + 2(n+1)^2 y_n.$$

10. If
$$y_n(x) = e^x\frac{d^n}{dx^n}(x^n e^{-x}),$$

prove that
$$y_n = x\frac{dy_{n-1}}{dx} + (n-x)y_{n-1},$$

$$\frac{1}{n}\frac{dy_n}{dx} = \frac{dy_{n-1}}{dx} - y_{n-1}.$$

Hence show that the polynomial y_n satisfies a certain linear differential equation of the second order. $\left[\text{i.e. linear in } \frac{d^2y_n}{dx^2}, \frac{dy_n}{dx}, y_n.\right]$

11. Show that, if the substitution $u = t^{a+1}\dfrac{d(\log y)}{dt}$ is made in the equation
$$\frac{du}{dt} = t^a - t^{-a-1}u^2,$$

then
$$t\frac{d^2y}{dt^2} + (a+1)\frac{dy}{dt} - y = 0.$$

Deduce that, if a is not a negative integer, then the value of $\dfrac{d^n y}{dt^n}$ when $t = 0$ is
$$\frac{A}{(a+n)(a+n-1)\dots(a+1)},$$

where A is the value of y when $t = 0$.

12. Prove that the polynomial $y = (x^2-2x)^n$ satisfies the equation

$$(2x-x^2)\frac{d^{n+2}y}{dx^{n+2}} - 2(x-1)\frac{d^{n+1}y}{dx^{n+1}} + n(n+1)\frac{d^n y}{dx^n} = 0.$$

13. A function $f(x)$ is defined for $0 \leqslant x \leqslant \frac{3}{2}$ as follows:

$$0 \leqslant x < 1, \quad f(x) = -\tfrac{1}{6}x^3 + x;$$
$$1 \leqslant x \leqslant \tfrac{3}{2}, \quad f(x) = -\tfrac{1}{2}x^2 + \tfrac{3}{2}x - \tfrac{1}{6}.$$

Discuss the continuity of $f(x)$ and its successive derivatives. Sketch the graphs of $f(x)$ and its derivatives throughout the whole interval $(0, \frac{3}{2})$.

Prove that the function
$$f(x) - \frac{3}{\pi}\sin\frac{\pi x}{3}$$

has a stationary value at $x = 1$, and determine whether it is a maximum or a minimum.

14. Prove that, if $y = e^{-x^2/2}$ and n is a positive integer, then

$$\frac{d^{n+2}y}{dx^{n+2}} + x\frac{d^{n+1}y}{dx^{n+1}} + (n+1)\frac{d^n y}{dx^n} = 0.$$

The functions $f_n(x)$ are defined by the formula

$$f_n(x) = (-1)^n e^{x^2/2}\frac{d^n}{dx^n}(e^{-x^2/2}).$$

Prove that (i) $f'_{n+1} = (n+1)f_n$,

(ii) $f_{n+1} = xf_n - f'_n$,

(iii) $f_{n+2} - xf_{n+1} + (n+1)f_n = 0$,

(iv) f_n is a polynomial in x of degree n.

15. Obtain the equations

$$x = s - \tfrac{1}{6}\kappa^2 s^3 + \dots, \quad y = \tfrac{1}{2}\kappa s^2 + \tfrac{1}{6}\kappa' s^3 + \dots$$

for a curve C.

Prove that the equation of the general conic having 4-point contact (i.e. 4 'coincident' intersections) with C at the origin O is

$$x^2 - \tfrac{2}{3}\rho' xy + \lambda y^2 - 2\rho y = 0,$$

where $\rho = 1/\kappa$ and λ is an arbitrary constant.

Deduce that the length of the latus rectum of a parabola having 4-point contact with a circle of radius a is $2a$.

16. Obtain the equations

$$x = s - \tfrac{1}{6}\kappa^2 s^3 + \ldots,\quad y = \tfrac{1}{2}\kappa s^2 + \tfrac{1}{6}\kappa' s^3 + \ldots.$$

The point Q on the curve is such that P is the middle point of the arc OQ; the tangent at the origin O cuts the chord PQ at T. Find the limiting value of the ratio $TP : TQ$ as Q tends to O.

Prove that, correct to the second order in s, the angle between the tangents to the curve at P and Q is $\kappa s\left(1 + \dfrac{3\kappa'}{2\kappa}s\right)$.

17. Show that, if $f(x)$ has a derivative throughout the interval $a \leqslant x \leqslant b$, then

$$f(b) - f(a) = (b-a)f'\{a + \theta(b-a)\},$$

where θ lies between 0 and 1.

Find the value of θ if $f(x) \equiv \tan x$, $a = \tfrac{1}{4}\pi$, $b = \tfrac{1}{3}\pi$.

Draw an accurate graph of the function $y = \tan x$ between the limits $\tfrac{1}{4}\pi, \tfrac{1}{3}\pi$, taking 1 in. to represent $\tfrac{1}{60}\pi$ as the x-axis and 1 in. to represent 0·2 on the y-axis.

Illustrate the theorem by means of your graph.

18. Given that $$e^{ay} = \cos x$$

and that y_n denotes $d^n y/dx^n$, prove that

$$ay_2 + a^2 y_1^2 + 1 = 0$$

and express y_{n+2} in terms of $y_1, y_2, \ldots, y_{n+1}$.

The expansion of y as a power series in x is

$$y = b_1 x + b_2 x^2 + \ldots + b_n x^n + \ldots.$$

Prove that b_n is zero when n is odd and that, if a is positive, b_n is negative when n is even.

19. Prove that the nth differential coefficient of $e^{x\sqrt{3}}\cos x$ is

$$2^n e^{x\sqrt{3}} \cos\left(x + \frac{n\pi}{6}\right).$$

Deduce, or obtain otherwise, the expansion of the function as a series of ascending powers of x giving the terms up to x^6 and the general term.

Find the most general function whose nth differential coefficient is $e^{x\sqrt{3}}\cos x$.

20. (i) Prove by induction that, if $x = \cot\theta$, $y = \sin^2\theta$, then

$$\frac{d^n y}{dx^n} = (-1)^n n!\, \sin^{n+1}\theta \sin(n+1)\theta.$$

(ii) Prove that $$\frac{d^n}{dx^n}\left(\frac{1}{x^2-1}\right) = \frac{f_n(x)}{(x^2-1)^{n+1}},$$

where $f_n(x)$ is a polynomial of degree n in x.

Prove further that

$$f_{n+1}(x) + 2(n+1)\,x f_n(x) + n(n+1)(x^2-1) f_{n-1}(x) = 0.$$

21. State, without proof, Rolle's theorem, and deduce that there is a number ξ between a, b such that

$$f(b) - f(a) = (b-a)f'(\xi),$$

explaining what conditions must be satisfied by the function $f(x)$ in order that the theorem may be valid.

If $f(x) \equiv \sin x$, find all the values of ξ when $a = 0, b = \tfrac{3}{2}\pi$. Illustrate the result with reference to the graph of $\sin x$.

22. If α is small, the equation $\sin x = \alpha x$ has a root nearly equal to π. Show that $$\pi\{1 - \alpha + \alpha^2 - (\tfrac{1}{6}\pi^2 + 1)\alpha^3\}$$

is a better approximation, if α is sufficiently small.

23. A plane curve is such that the tangent at any point P is inclined at an angle $(k+1)\theta$ to a fixed line Ox, where k is a positive constant and θ is the angle xOP. The greatest length of OP is a. Find a polar equation for the curve.

Sketch the curve for the cases $k = 2, k = \tfrac{1}{2}$.

24. Prove that

$$\frac{3\dfrac{d^2y}{dx^2}\dfrac{d^4y}{dx^4} - 5\left(\dfrac{d^3y}{dx^3}\right)^2}{\left(\dfrac{dy}{dx}\right)^4} = \frac{3\dfrac{d^2x}{dy^2}\dfrac{d^4x}{dy^4} - 5\left(\dfrac{d^3x}{dy^3}\right)^2}{\left(\dfrac{dx}{dy}\right)^4}.$$

25. Obtain the expansion of $\sin x$ in ascending powers of x. For what values of x is this series convergent.

A small arc PQ of a circle of radius 1 is of length x. The arc PQ is bisected at Q_1 and the arc PQ_1 is bisected at Q_2. The chords PQ, PQ_1, PQ_2 are of lengths $c, \tfrac{1}{2}c_1, \tfrac{1}{4}c_2$ respectively. Prove that $\tfrac{1}{45}(c - 20c_1 + 64c_2)$ differs from x by a quantity of order x^7.

26. Sketch the graph of the function

$$y = e^{-(ax+bx^3)}$$

where a, b are both positive. Prove that there are always at least two points of inflexion.

Find the abscissæ of the points of inflexion when $a = \frac{10}{27}, b = 5$.

27. If $x = f(t)$, $y = g(t)$, express $\frac{dy}{dx}, \frac{dx}{dy}, \frac{d^2y}{dx^2}, \frac{d^2x}{dy^2}$ in terms of $\frac{dx}{dt}, \frac{dy}{dt}, \frac{d^2x}{dt^2}, \frac{d^2y}{dt^2}$.

Prove that

$$\left(\frac{dx}{dy}\right)^2 \frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 \frac{d^3x}{dy^3} + 3\frac{d^2y}{dx^2}\frac{d^2x}{dy^2} = 0.$$

28. The functions $\phi(x), \psi(x)$ are differentiable in the interval $a \leqslant x \leqslant b$, and $\psi'(x) > 0$ for $a \leqslant x \leqslant b$. Prove that there is at least one number ξ between a, b such that

$$\frac{\phi(\xi) - \phi(a)}{\psi(b) - \psi(\xi)} = \frac{\phi'(\xi)}{\psi'(\xi)}.$$

If $\phi(x) \equiv x^2$, $\psi(x) \equiv x$, find a value of ξ in terms of a and b.

29. Prove by differentiation, or otherwise, that

$$xy \leqslant e^{x-1} + y \log y$$

for all real x and positive y.

When does the sign of equality hold?

30. (i) Prove that, for positive values of x,

$$\log(1+x) < \frac{x(2+x)}{2(1+x)}.$$

(ii) Find whether $e^{-x^2}\sec^2 x$ has a maximum or a minimum value for $x = 0$.

31. Show that, if $f(0) = 0$ and if $f'(x)$ is an increasing function of x, then $y = \frac{f(x)}{x}$ is an increasing function of x for $x > 0$.

32. Prove that, if $x > 0$,

$$(1 - \tfrac{1}{2}x^2 + \tfrac{1}{24}x^4)\sin x > (x - \tfrac{1}{6}x^3 + \tfrac{1}{120}x^5)\cos x.$$

33. If $x > 0$, prove that $(x-1)^2$ is not less than $x(\log x)^2$.

Discuss the general behaviour of the function

$$(\log x)^{-1} - (x-1)^{-1}$$

for positive values of x and with special reference to $x = 1$.

Sketch the graph of the function.

34. Prove that, if x is positive,

$$\frac{2x}{2+x} < \log(1+x) < x.$$

Prove also that, if a, h are positive, then

$$\log(a+\theta h) - \log a - \theta\{\log(a+h) - \log a\},$$

considered as a function of θ, has a maximum for a value of θ between 0 and $\frac{1}{2}$.

35. If the sum of the lengths of the hypotenuse and one other side of a right-angled triangle is given, find the angle between the hypotenuse and that side when the area of the triangle has its maximum value.

36. A pyramid consists of a square base and four equal triangular faces meeting at its vertex. If the total surface area is kept fixed, show that the volume of the pyramid is greatest when each of the angles at its vertex is 36° 52′.

37. Find the least volume of a right circular cone in which a sphere of unit radius can be placed.

38. Find the numerical values of

$$y = \sin(x + \tfrac{1}{4}\pi) + \tfrac{1}{4}\sin 4x$$

at its stationary values in the range $-\pi \leqslant x \leqslant \pi$. Distinguish between maxima, minima, and points of inflexion, and give a rough sketch of the curve.

39. Prove that, if

$$\theta = \cot^{-1} x \quad (0 < \theta < \pi),$$

then
$$\frac{d^n\theta}{dx^n} = (-1)^n (n-1)! \sin^n\theta \sin n\theta,$$

when n is any positive integer.

Show that the absolute value of $d^n\theta/dx^n$ never exceeds $(n-1)!$ if n is odd, or $(n-1)!\cos^{n+1}\left(\frac{\pi}{2n+2}\right)$ if n is even.

40. Find the two nearest points on the curves

$$y^2-4x=0,\quad x^2+y^2-6y+8=0,$$

and evaluate their distance.

41. If
$$y=\frac{x(1-x)}{1+x^2},$$

(i) find the maximum and minimum values of y;

(ii) find the points of inflexion of the curve;

(iii) sketch a graph showing clearly the points determined in (i), (ii), and also the position of the curve relative to the line $y=x$.

42. Prove that the maxima of the curve $y=e^{-kx}\sin px$, where k,p are positive constants, all lie on a curve whose equation is $y=Ae^{-kx}$, and find A in terms of k,p.

Draw in the same diagram rough sketches of the curves $y=e^{-kx}$, $y=-e^{-kx}$ and $y=e^{-kx}\sin px$ for positive values of x.

43. The tangent and normal to a curve at a point O are taken as axes. P is a point on the curve at an arcual distance s from O. Prove that the coordinates of P are approximately

$$x=s-\tfrac{1}{6}\kappa^2s^3,\quad y=\tfrac{1}{2}\kappa s^2+\tfrac{1}{6}\kappa' s^3,$$

where κ is the curvature at O, and κ' the value of $d\kappa/ds$ at O.

The tangents at O and P meet at T. A circle is drawn through O to touch PT at T. Prove that the limiting value of its radius as P tends to O is $1/(4\kappa)$.

44. Obtain the coordinates of the centre of curvature at a point (x,y) of a curve in the form $(x-\rho\sin\psi, y+\rho\cos\psi)$, where ρ is the radius of curvature and $\tan\psi$ the slope of the tangent at the point. Prove also that, when s is the length of arc between (x,y) and a fixed point on the curve, $dx/ds=\cos\psi$, $dy/ds=\sin\psi$.

The centres of curvature of a certain curve lie on a fixed circle. Prove that $\rho\dfrac{d\rho}{ds}$ is constant.

45. Obtain the formula $\rho=r\,dr/dp$ for the radius of curvature of a curve in terms of the radius vector and the perpendicular from the origin on the tangent.

Prove that a curve for which $\rho=p$ satisfies the equation $r^2=p^2+a^2$, where a is constant, and deduce that the polar equation of the curve is

$$\sqrt{(r^2-a^2)}=a\theta+a\cos^{-1}(a/r),$$

where the coordinate system is chosen so that the point $(a,0)$ is on the curve.

46. (i) Prove that the (p,r) equation of the cardioid

$$r=a(1+\cos\theta)$$

is
$$2ap^2=r^3.$$

Hence, or otherwise, prove that the radius of curvature is

$$\tfrac{4}{3}a\cos\tfrac{1}{2}\theta.$$

(ii) Prove that the equation of the circle of curvature at the origin of the curve

$$x+y=x^2+2y^2+3x^3$$

is
$$3(x^2+y^2)=2(x+y).$$

47. If y is a function of x, and

$$x=\xi\cos\alpha-\eta\sin\alpha,\quad y=\xi\sin\alpha+\eta\cos\alpha,$$

where α is constant, express $\dfrac{dy}{dx}$ and $\dfrac{d^2y}{dx^2}$ in terms of $\dfrac{d\eta}{d\xi}$ and $\dfrac{d^2\eta}{d\xi^2}$.

Deduce that
$$\frac{\dfrac{d^2y}{dx^2}}{\left\{1+\left(\dfrac{dy}{dx}\right)^2\right\}^{\frac{3}{2}}}=\frac{\dfrac{d^2\eta}{d\xi^2}}{\left\{1+\left(\dfrac{d\eta}{d\xi}\right)^2\right\}^{\frac{3}{2}}},$$

and interpret this result.

48. A curve is such that its arc length s, measured from a certain point, and ordinate y are related by

$$y^2=s^2+c^2,$$

where c is a constant. Show that referred to suitably chosen rectangular axes the curve has the equation

$$y=c\cosh(x/c).$$

If C is the centre of curvature at a point P of the curve and G is the point in CP produced such that $CP = PG$, prove that the locus of G is a straight line.

49. Find the radius of curvature at the origin of the curve

$$y = 2x + 3x^2 - 2xy + y^2 + 2y^3,$$

and show that the circle of curvature at the origin has equation

$$3(x^2+y^2) = 5(y-2x).$$

50. Find the values of x for which $y = x^2(x-2)^3$ has maximum and minimum values, and evaluate for these values of x the curvature of the curve given by the equation above.

51. Sketch the curve defined by the parametric equations

$$x = a\cos^3 t, \quad y = a\sin^3 t.$$

Show that the intercept made on a variable tangent by the coordinate axes is of constant length.

Find the radius of curvature at the point 't'.

52. Find the points on the curve $y(x-1) = x$ at which the radius of curvature is least.

53. Sketch the curves

$$y = x^2 - x^3, \quad y^2 = x^2 - x^3,$$

and find their radii of curvature at the origin.

54. Prove that all the curves represented by the equation

$$\frac{x^{n+1}}{a} + \frac{y^{n+1}}{b} = \left(\frac{ab}{a+b}\right)^n,$$

for different positive values of n, touch each other at the point

$$\left(\frac{ab}{a+b}, \frac{ab}{a+b}\right).$$

Prove that the radius of curvature at the point of contact is equal to

$$\frac{(a^2+b^2)^{\frac{3}{2}}}{n(a+b)^2}.$$

55. Establish the (p, r) formula for the radius of curvature of a plane curve. For a certain curve it is known that the radius of curvature is a^n/r^{n-1}, where $n \neq -1$ and a is positive, and also that $p = a/(n+1)$ when $r = a$. Show that it is possible to express the curve by the polar equation $r^n = (n+1)a^n \cos n\theta$.

56. Show that the curvature of the curve $a/r = \cosh n\theta$ has a stationary value provided that $3n^2$ is not less than 1. Determine whether this value is a maximum or a minimum.

57. O is the middle point of a straight line AB of length $2a$, and a point P moves so that $AP.BP = c^2$. Show that the radius of curvature at P of the locus is $2c^2r^3/(3r^4 + a^4 - c^4)$, where $r = OP$.

58. Find the integrals:

$$\int \frac{dx}{x^4+4}, \quad \int e^{ax}\cos bx\,dx \quad (a \neq 0, b \neq 0), \quad \int \frac{dx}{x+(x^2-1)^{\frac{1}{2}}}.$$

59. Prove that the area enclosed by the curve traced by the foot of the perpendicular from the centre to a variable tangent of the ellipse $b^2x^2 + a^2y^2 = a^2b^2$ is $\frac{1}{2}\pi(a^2+b^2)$.

60. (a) Prove that, if

$$F(x) = e^{2x}\int_0^x e^{-2t}f(t)\,dt - e^x\int_0^x e^{-t}f(t)\,dt,$$

then (i) $F(0) = 0$,

(ii) $F'(0) = 0$,

(iii) $F''(x) - 3F'(x) + 2F(x) = f(x)$.

(b) Find the most general function $\phi(x)$ which is such that

$$\int_0^x t\phi(t)\,dt = x^2\phi(x).$$

61. Sketch the curve whose polar equation is $r^2 = a^2(1+3\cos\theta)$, and find the area it encloses.

62. (i) By differentiating the expression $\dfrac{nx\cos x + \sin x}{\sin^{n+1} x}$, or otherwise, find

$$\int \frac{x\,dx}{\sin^6 x}.$$

(ii) Find

$$\int \frac{dx}{x^6(a+bx)}.$$

63. Sketch the general shape of the curve

$$x = at\cos t, \quad y = at\sin t$$

for positive values of the parameter t.

Find an expression for the length of the arc of the curve measured from the origin to the point t.

64. (i) Prove that, if n is a positive integer,

$$\int_0^{\frac{1}{2}\pi} e^x \cos nx\,dx = \frac{1}{n^2+1}\{\lambda e^{\frac{1}{2}\pi} - 1\},$$

where λ has one of the values ± 1, $\pm n$; and classify the cases.

(ii) Find the area bounded by the parabola $y^2 = ax$ and the circle $x^2+y^2 = 2a^2$.

65. Find the indefinite integral

$$\int \log(1+x^2)\tan^{-1}x\,dx.$$

66. Show that the four figures bounded by the circle $r = 3a\cos\theta$ and the cardioid $r = a(1+\cos\theta)$ have areas

$$\tfrac{1}{8}\pi a^2, \quad \tfrac{1}{8}\pi a^2, \quad \pi a^2, \quad \tfrac{5}{4}\pi a^2.$$

67. The polar equation of a curve is

$$r^2 - 2r\cos\theta + \sin^2\theta = 0.$$

Sketch the curve and prove that the area enclosed by it is $\pi/\sqrt{2}$.

68. Determine the function $\phi(x)$ such that

$$1 + \int_0^x \phi(t)e^t\,dt \equiv (1+x)^2 e^x.$$

Prove also that it is possible to find a pair of quadratic functions $f(x)$, $g(x)$ such that

$$(1+x)f(x) \equiv 1 + \int_0^x g(t)\,dt,$$

$$(1+x)g(x) \equiv 3 + 9\int_0^x f(t)\,dt,$$

and determine these functions.

69. A plane curvilinear figure is bounded by the parabola $x^2 = \frac{169}{5}y$ from the origin to the point $(13, 5)$; by the hyperbola $x^2 - y^2 = 144$ from the point $(13, 5)$ to the point $(15, 9)$; and by the parabola $y^2 = \frac{27}{5}x$ from the point $(15, 9)$ to the origin. Prove that the area of the figure is $33\frac{1}{3} + 72\log\frac{4}{3}$.

70. Prove that

$$\frac{1}{2}\int_0^1 x^4(1-x)^4\,dx < \int_0^1 \frac{x^4(1-x)^4}{1+x^2}\,dx < \int_0^1 x^4(1-x)^4\,dx.$$

Verify the identity

$$x^4(1-x)^4 = (1+x^2)(4 - 4x^2 + 5x^4 - 4x^5 + x^6) - 4,$$

and, by using this identity in the second of these integrals, prove that

$$\frac{22}{7} - \frac{1}{630} < \pi < \frac{22}{7} - \frac{1}{1260}.$$

71. Determine constants A, B, C, D such that

$$\frac{x^4+1}{(x^2+1)^4} = \frac{d}{dx}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) + \frac{D}{x^2+1}.$$

Hence or otherwise prove that

$$0{\cdot}502 < \int_0^1 \frac{x^4+1}{(x^2+1)^4}\,dx < 0{\cdot}503.$$

72. Determine A, B, C, D such that

$$\frac{x^2}{(x^2+1)^4} = \frac{d}{dx}\left(\frac{Ax^5+Bx^3+Cx}{(x^2+1)^3}\right) + \frac{D}{x^2+1},$$

and show that $$\int_0^1 \frac{x^2}{(x^2+1)^4}\,dx = \frac{1}{48} + \frac{\pi}{64}.$$

73. If $y_r(x)$ satisfies the equation

$$\frac{d}{dx}\left\{(1-x^2)\frac{dy}{dx}\right\} + r(r+1)y = 0,$$

show that $$\int_{-1}^1 y_m(x)y_n(x)\,dx = 0 \quad (m \neq n).$$

74. Polynomials $f_0(x), f_1(x), f_2(x), \ldots$ are defined by the relation

$$f_n(x) = \frac{d^n}{dx^n}(x^2-1)^n.$$

Prove that $\displaystyle\int_{-1}^{1} f_m(x) f_n(x)\,dx = 0 \quad (m \neq n),$

$$\int_{-1}^{1} \{f_n(x)\}^2\,dx = \frac{(n!)^2}{2n+1}\,2^{2n+1}.$$

Show that, if $\phi(x)$ is any polynomial of degree m,

$$\phi(x) = \sum_{0}^{m} a_n f_n(x),$$

where $\displaystyle a_n = \frac{2n+1}{(n!)^2\,2^{2n+1}}\int_{-1}^{1}\phi(x)\,f_n(x)\,dx.$

75. If m, n are positive integers greater than unity, prove that

$$I_{m,n} \equiv \int_0^{\frac{1}{2}\pi} \cos^m x \cos nx\,dx$$

$$= \frac{m}{m-n} I_{m-1,\,n+1} = \frac{m}{m+n} I_{m-1,\,n-1}.$$

Hence show, if p, q are positive integers, that

$$\int_0^{\frac{1}{2}\pi} \cos^{p+q} x \cos(p-q)x\,dx = \frac{\pi(p+q)!}{2^{p+q+1}\,p!\,q!}.$$

76. Find a reduction formula for

$$\int (1-x^2)^n \cosh ax\,dx.$$

Evaluate $\displaystyle\int_0^1 (1-x^2)^3 \cosh x\,dx.$

77. If $y = \sin^p x \cos^q x \sqrt{(1-k^2\sin^2 x)}$ and p, q, k are constants, find $\sqrt{(1-k^2\sin^2 x)}\,\dfrac{dy}{dx}$.

Hence, by taking suitable values for p, q, express

$$I_m = \int_0^{\frac{1}{2}\pi} \frac{\sin^m x\,dx}{\sqrt{(1-k^2\sin^2 x)}}$$

in terms of I_{m-2}, I_{m-4}.

78. State and prove the formula for integration by parts, and show that

$$\int_0^1 x^n(1-x)^m\,dx = \frac{m!\,n!}{(m+n+1)!}.$$

79. Find the volume and area of the surface of the solid obtained by rotating the portion of the cycloid

$$x = a(\theta + \sin\theta), \quad y = a(1+\cos\theta)$$

between two consecutive cusps about the axis of x.
[Consider the range $-\pi \leqslant \theta \leqslant \pi$.]

80. Prove that the envelope of the system of lines

$$(t^3 - 3t)x + (t^3+2)y = t^3$$

is the curve $y^2(x+y-1) = x^3$.

81. Find the envelope of the line

$$x \sin 3t - y \cos 3t = 3a \sin t,$$

expressing the coordinates of a point of the envelope in terms of a and t.

82. Through a variable point $P\ (at^2, 2at)$ of the parabola $y^2 = 4ax$ a line is drawn perpendicular to SP, when S is the focus $(a, 0)$. Prove that the envelope of the line is the curve $27ay^2 = x(x-9a)^2$.

83. Sketch the locus (the cycloid) given by

$$x = a(t+\sin t), \quad y = a(1+\cos t)$$

for values of the parameter t between $0, 4\pi$.

Prove that the normals to this curve all touch an equal cycloid, and draw the second curve in your diagram.

84. Find the envelope, as t varies, of the straight line whose equation is

$$x\cos^3 t + y \sin^3 t = a.$$

Sketch this envelope, and find the radius of curvature at one of the points of the curve nearest to the origin.

85. As t varies, the line

$$x - t^2 y + 2at^3 = 0$$

envelops a curve Γ. Show that for each value of t, other than $t = 0$, the line cuts Γ at a point P distinct from the point at which the line touches Γ.

Find the equation of the normal to Γ at P, and deduce that the centre of curvature at P is given by

$$x = -20at^3-(3a/4t), \quad y = 48at^5-3at.$$

Discuss the case $t = 0$.

86. Find the equation of the normal and the centre and radius of curvature of the curve $ay^2 = x^3$ at the point (at^2, at^3).

Show that the length of the arc of the evolute between the points corresponding to $t = 0, t = 1$ is $(13^{\frac{3}{2}}/6)a$.

87. Show how to find the envelope of the line

$$y = tx+f(t).$$

Show that the length of an arc of the envelope is given by

$$\int_{t_1}^{t} f''(t)\sqrt{(1+t^2)}\,dt.$$

Hence obtain a formula for the radius of curvature in terms of t.

88. Prove that the equation of the normal to the curve

$$x^{\frac{2}{3}}+y^{\frac{2}{3}} = a^{\frac{2}{3}}$$

may be written in the form

$$x\sin t-y\cos t+a\cos 2t = 0,$$

and find parametrically the envelope of the normal.

89. Find the equations of the tangent and normal at any point of the curve

$$x = 3\sin t-2\sin^3 t, \quad y = 3\cos t-2\cos^3 t,$$

Prove that the evolute is

$$x^{\frac{2}{3}}+y^{\frac{2}{3}} = 2^{\frac{2}{3}}.$$

90. The coordinates of any point on the curve $x^3+xy^2 = y^2$ can be expressed in the form

$$x = t^2/(1+t^2), \quad y = t^3/(1+t^2).$$

Find the equations of the tangent and normal at any point, and deduce that the coordinates of the corresponding centre of curvature are

$$(-t^2-\tfrac{1}{6}t^4, \tfrac{4}{3}t).$$

91. Prove that the complete area of the curve traced out by the point

$$(2a\cos t+a\cos 2t, \quad 2a\sin t-a\sin 2t)$$

is $2\pi a^2$.

92. Find the area inside the curve given by

$$x = a\cos t+c\sin kt, \quad y = b\sin t+d\cos kt \quad (0 \leqslant t \leqslant 2\pi),$$

where k is an integer and a, b, c, d are positive.

[It may be assumed that c, d are so small in comparison with a, b that the curve has no double points.]

93. Sketch the curve

$$x = a\sin 2t, \quad y = b\cos^3 t,$$

where $a > 0, b > 0$, and find the area it encloses.

94. Show that the curve

$$x = (t-1)e^{-t}, \quad y = tx$$

has a loop, and find its area.

95. Trace the curve $x = \cos 2t, y = \sin 3t$ for real values of t, and find the area of the loop.

96. Show that, for the range $-a \leqslant x \leqslant a$, the area between the curve

$$\left(\frac{x}{a}\right)^{\frac{2}{3}}+\frac{y}{b} = 1$$

and the x-axis is $\frac{4}{5}ab$.

97. Sketch the curve given by

$$x = 2a(\sin^3 t+\cos^3 t), \quad y = 2b(\sin^3 t-\cos^3 t),$$

and prove that its area is $3\pi ab$.

98. Find the area of the loop $(-1 \leqslant t \leqslant 1)$ of the curve

$$x = \frac{1-t^2}{1+t^2}, \quad y = \frac{t(1-t^2)}{1+t^2}.$$

99. Prove that the area of the curve

$$x = a\cos t+b\sin t+c, \quad y = a'\cos t+b'\sin t+c'$$

is $\pi(ab'-a'b)$.

100. Explain the reasons for using the formula

$$\int 2\pi y\, ds$$

to calculate the area of a surface of revolution.

Apply the formula to show that the area of the surface obtained by revolving the curve

$$r = a(1+\cos\theta)$$

about the line $\theta = 0$ is equal to $\frac{32}{5}\pi a^2$.

101. The curve traced out by the point

$$x = a\log(\sec t + \tan t) - a\sin t,$$
$$y = a\cos t,$$

as t increases from $-\frac{1}{2}\pi$ to $+\frac{1}{2}\pi$, is rotated about the axis of x. Prove that the whole surface generated is equal to the surface of a sphere of radius a, and that the whole volume generated is half the volume of a sphere of radius a.

102. Find the length of the arc of the catenary

$$y = c\cosh(x/c)$$

between the points given by $x = \pm a$.

Find also the area of the curved surface generated by rotating this arc about the x-axis.

103. Evaluate the area of the surface generated by the revolution of the cycloid

$$x = a(t-\sin t), \quad y = a(1-\cos t)$$

about the line $y = 0$.

104. Two points $A(0,c), P(\xi,\eta)$ lie on the curve whose equation is

$$y = c\cosh(x/c),$$

and s is the length of the arc AP. If the curve makes a complete revolution about the x-axis, prove that the area S of the curved surface, bounded by planes through A and P perpendicular to the x-axis, and the corresponding volume V are given by

$$cS = 2V = \pi c(c\xi + s\eta).$$

105. A torus is the figure formed by rotating a circle of radius a about a line in its own plane at a distance $h(>a)$ from its centre. Find the volume and surface area of the torus.

106. AB, CD are two perpendicular diameters of a circle. Find the mean value of the distance of A from points on the semicircle BCD, and also the mean value of the reciprocal of that distance.

Prove that the product of these means is

$$8\pi^{-2}\sqrt{2}\log(1+\sqrt{2}).$$

107. Prove that the mean distance of points on a sphere of radius a from a point distant $f(\leqslant a)$ from the centre is

$$a + (f^2/3a).$$

What is the mean distance if $f > a$?

108. Calculate the average value, over the surface of a sphere of centre O and radius a, of the function $(1/r^3)$, where r is the distance of the point on the surface of the sphere from a fixed point C not on the surface and such that $OC = f$. Distinguish between the two cases $f > a$, $f < a$.

109. The centre of a disc of radius r is O, and P is a point on the line through O perpendicular to the plane of the disc. Prove that, if $OP = p$, the mean distance (with respect to area) of points of the disc from P is

$$\tfrac{2}{3}\{(p^2+r^2)^{\frac{3}{2}} - p^3\}/r^2.$$

Find the mean distance (with respect to volume) of the interior points of a sphere of radius a from a fixed point of its surface.

110. Prove that the mean value with respect to area over the surface of a sphere, of centre O and radius a, of the reciprocal of the distance from a fixed point C is equal to the reciprocal of OC if C is outside the sphere, but equal to the reciprocal of the radius a if C is inside the sphere.

111. A point P is taken on an ellipse whose foci are S, H. The distance SP is denoted by r, and the angle HSP by θ. Show that the mean value of r with respect to arc is the semi-major axis a, and that the mean value of r with respect to θ is the semi-minor axis b.

CHAPTER XI

COMPLEX NUMBERS

1. Introduction. It may be helpful if we begin this chapter by reminding the reader of the types of elements which he is already accustomed to use in arithmetic. These are

(i) the *integer* (positive, negative or zero);

(ii) the *rational number*, that is, the ratio of two integers;

(iii) the *irrational number* (such as $\sqrt{2}, \pi, e$) which cannot be expressed as the ratio of integers.

It is now necessary to extend our scope and to devise a system of numbers not included in any of these three classes.

In order to exhibit the need for the new numbers, we solve in succession a series of quadratic equations, similar to look at, but essentially distinct.

I. $$x^2-6x+5=0.$$

Following the usual 'completing the square' process, we have the solution

$$x^2-6x \quad = -5,$$

$$x^2-6x+9 = -5+9,$$

$$= 4,$$

$$(x-3)^2 \quad = 4.$$

The important point at this stage is the existence of two integers, namely $+2$ and -2, whose square is equal to 4. Hence we can find *integral* values of x to satisfy the equation:

$$x-3=+2 \quad \text{or} \quad x-3=-2,$$

so that $$x=5 \quad \text{or} \quad x=1.$$

The equation $x^2-6x+5=0$ can therefore be solved by taking x as one or other of the *rational* numbers (integers, in fact) 5 or 1.

II. $$x^2-6x+7=0.$$

Proceeding as before, we have the solution

$$x^2-6x \quad = -7,$$

$$x^2-6x+9 = -7+9,$$

$$= 2,$$

$$(x-3)^2 \quad = 2.$$

Now comes a break; for there is no rational number whose square is 2, and so we cannot find a rational number x to satisfy the given equation. We must therefore extend our idea of number beyond the elementary realm of integers and rational numbers. This is an advance which the reader absorbed, doubtless unconsciously, many years ago, but it represents a step of fundamental importance. The theory of the irrational numbers will be found in text-books of analysis; for our purpose, knowledge of its existence is sufficient. In particular, we regard as familiar the concept of the irrational number, written as $\sqrt{2}$, whose value is $1{\cdot}414\ldots$, with the property that $(\sqrt{2})^2=2$.

Once the irrational numbers are admitted, the solution of the equation follows:

$$x-3=+\sqrt{2} \quad \text{or} \quad x-3=-\sqrt{2},$$

so that $$x=3+\sqrt{2} \quad \text{or} \quad x=3-\sqrt{2}.$$

The equation $x^2-6x+7=0$ can therefore be solved by taking x as one or other of the *irrational* numbers $3+\sqrt{2}$ or $3-\sqrt{2}$.

III. $$x^2-6x+10=0.$$

As before, $$x^2-6x \quad = -10,$$

$$x^2-6x+9 = -10+9,$$

$$= -1,$$

$$(x-3)^2 \quad = -1.$$

Again comes the break; for there is no number, rational or irrational, whose square is the NEGATIVE number -1. We have therefore two choices, to accept defeat and say that the equation has no solution, or to invent a new set of numbers from which a value of x can be selected. The second alternative is the purpose of this chapter.

To be strictly logical, we should now proceed to define these new numbers and then show how to frame the solution from among them. But their definition is, naturally, somewhat complicated, and it seems better to begin by merely postulating their 'existence'; we can then see what properties they will have to possess, and the reasons for the definition will become more apparent.

Just as, in earlier days, we learned to write the symbol '$\sqrt{2}$' for a number whose square is 2, so now we use the symbol '$\sqrt{(-1)}$' for a 'number' (in some sense of the word) whose square is -1; but, in practice, it is more convenient to have a single mark instead of the five marks $\sqrt{}, (, -, 1,)$, and so we write

$$i \equiv \sqrt{(-1)}$$

as a convenient abbreviation.

We shall subject this symbol i to all the usual laws of algebra, but first endow it further with the property that

$$i^2 = -1.$$

This will be its sole distinguishing feature—though, of course, that feature is itself so overwhelming as to introduce us into an entirely new number-world.

It may be noticed at once that the 'number' $-i$ also has -1 for its square, since, in accordance with the normal rules,

$$\begin{aligned}(-i)^2 = (-1 \times i)^2 = (-1)^2 \times (i)^2 &= (+1) \times (-1) \\ &= -1.\end{aligned}$$

We round off this introduction by displaying the solution of the above equation:

$$x-3 = i \quad \textit{or} \quad x-3 = -i,$$

so that

$$x = 3+i \quad \textit{or} \quad x = 3-i.$$

By selecting x from this extended range of 'numbers', we are therefore able to find two solutions of the equation.

It may, perhaps, make this statement clearer if we verify how $3+i$, say, does exactly suffice. By saying that $3+i$ is a solution of the equation $x^2-6x+10 = 0$, we mean that

$$(3+i)^2 - 6(3+i) + 10 = 0.$$

The left-hand side is

$$\begin{aligned}9+6i+i^2-18-6i+10 & \\ &= 1+i^2 \\ &= 1+(-1) \\ &= 0 \\ &= \text{right-hand side,}\end{aligned}$$

so that the solution is verified.

We now investigate the elementary properties of our extended number-system, and then, when the ideas are a little more familiar, return to put the basis on a surer foundation.

EXAMPLES I

Solve after the manner of the text the following sets of equations:

1. $x^2-4x+3 = 0, \quad x^2-4x+1 = 0, \quad x^2-4x+5 = 0.$
2. $x^2+8x+15 = 0, \quad x^2+8x+11 = 0, \quad x^2+8x+20 = 0.$
3. $x^2-2x-3 = 0, \quad x^2-2x-4 = 0, \quad x^2-2x+10 = 0.$

2. Definitions. The numbers of ordinary arithmetic (positive or negative integers, rational numbers and irrational numbers) are called *real* numbers.

If a, b are two real numbers, then numbers of the form

$$a+ib \quad (i^2 = -1)$$

are called *complex* numbers. When $b = 0$, such a number is real. When $a = 0$, the number

$$ib \quad (b \text{ real})$$

is called a *pure imaginary* number.

Two numbers such as

$$a+ib, \quad a-ib \quad (a, b \text{ real})$$

are called *conjugate* complex numbers.

A single symbol, such as c, is often used for a complex number.

If
$$c \equiv a+ib,$$
then a is called the *real part* of c and b the *imaginary part*. The notations
$$a = \mathscr{R}c, \quad b = \mathscr{I}c$$
are often used.

The conjugate of a complex number c is often denoted by $\bar{c}$, so that, if $c = a+ib$, then $\bar{c} = a-ib$. Clearly the conjugate of $\bar{c}$ is c itself.

If c is real, then $c = \bar{c}$.

The magnitude
$$+\sqrt{(a^2+b^2)} \quad (a, b \text{ real})$$
is called the *modulus* of the complex number $c \equiv a+ib$; it is often denoted by the notations
$$|c|, \quad |a+ib|.$$
It follows that
$$|\bar{c}| = |c|.$$

Moreover, we have the relation
$$c\bar{c} = |c|^2,$$
for
$$\begin{aligned} c\bar{c} &= (a+ib)(a-ib) \\ &= a^2-i^2b^2 = a^2-(-1)b^2 \\ &= a^2+b^2. \end{aligned}$$

Finally, since $\quad c = a+ib, \quad \bar{c} = a-ib,$

we have the relations
$$\mathscr{R}c = a = \tfrac{1}{2}(c+\bar{c}),$$
$$\mathscr{I}c = b = \frac{1}{2i}(c-\bar{c}).$$

Note. All the numbers now to be considered are actually complex, of the form $a+ib$, even when $b = 0$. Correctly speaking we should use the phrase 'real complex number' for a number such as 2, and 'pure imaginary complex number' for $2i$. But this becomes tedious once it has been firmly grasped that *all* numbers are complex anyway.

EXAMPLES II

The following examples are designed to make the reader familiar with the use of the symbol i. Use normal algebra, except that $i^2 = -1$.

Express the following complex numbers in the form $a+ib$, where a, b are both real:

1. $(3+5i)+(7-2i)$.
2. $(-2+3i)-(6-5i)$.
3. $(4+5i)(6-2i)$.
4. $(2+7i)(-1+2i)$.
5. $3+2i+i(5+i)$.
6. $4-3i+i(3+4i)$.
7. $(1+i)^2$.
8. $(2+i)^3$.
9. $(2-i)^2-(3+2i)^2$.
10. $(1+i)(1+2i)(1+3i)$.

Prove the following identities:

11. $\dfrac{1}{i} = -i$.
12. $\dfrac{1}{1+i} = \tfrac{1}{2}(1-i)$.
13. $\dfrac{1}{3-4i} = \dfrac{1}{25}(3+4i)$.
14. $\dfrac{1}{a+ib} = \dfrac{a-ib}{a^2+b^2}$.
15. $\dfrac{p+iq}{a+ib} = \dfrac{(ap+bq)+i(aq-bp)}{a^2+b^2}$.

3. Addition, subtraction, multiplication.

Let
$$c \equiv a+ib, \quad w \equiv u+iv \quad (a, b, u, v \text{ real})$$
be two given complex numbers. In virtue of the meaning which we have given to i, we obtain expressions for their sum, difference and product as follows:

(i) Sum $\quad c+w = (a+u)+i(b+v).$

(ii) Difference $\quad c-w = (a-u)+i(b-v).$

(iii) Product
$$\begin{aligned} cw &= (a+ib)(u+iv) \\ &= au+ibu+iav+i^2bv \\ &= (au-bv)+i(bu+av). \end{aligned}$$

The awkward one of these is the product. At present it is best not to remember the formula, but to be able to derive it when required.

EXAMPLES III

Express the following products in the form $a+ib$ (a, b real):

1. $(2+3i)(4+5i)$.
2. $(1+4i)(2-6i)$.
3. $(-1-i)(-3+4i)$.
4. $(a+ib)(a-ib)$.
5. $i(4+3i)$.
6. $(\cos A+i\sin A)(\cos B+i\sin B)$.
7. $(3-i)(3+i)$.
8. $(2+3i)^3$.

4. Division. As before, let

$$c \equiv a+ib, \quad w \equiv u+iv \quad (a, b, u, v \text{ real}).$$

Then the QUOTIENT is, by definition, the fraction

$$\frac{c}{w} \equiv \frac{a+ib}{u+iv}.$$

It is customary to express such fractions in a form in which the denominator is real. For this, multiply numerator and denominator by $\bar{w} \equiv u-iv$. Then

$$\begin{aligned}\frac{c}{w} &= \frac{(a+ib)(u-iv)}{(u+iv)(u-iv)} \\ &= \frac{(au+bv)+i(bu-av)}{u^2+v^2} \\ &= \frac{au+bv}{u^2+v^2}+i\frac{bu-av}{u^2+v^2}.\end{aligned}$$

It is implicit that u, v are not both zero.

For example, $(2+i)\div(3-i)$

$$\begin{aligned}&= \frac{2+i}{3-i} \\ &= \frac{(2+i)(3+i)}{(3-i)(3+i)} \\ &= \frac{6+5i+i^2}{9-i^2} = \frac{6+5i-1}{9+1} \\ &= \frac{5+5i}{10} \\ &= \tfrac{1}{2}+\tfrac{1}{2}i.\end{aligned}$$

EXAMPLES IV

Express the following quotients in the form $a+ib$ (a, b real):

1. $\dfrac{1+i}{1-i}$.
2. $\dfrac{3+4i}{4+3i}$.
3. $\dfrac{5i+6}{7i}$.
4. $\dfrac{2-3i}{4+i}$.
5. $\dfrac{3+5i}{1-6i}$.
6. $\dfrac{\cos 2\theta+i\sin 2\theta}{\cos\theta+i\sin\theta}$.

5. Equal complex numbers. To verify that, *if two complex numbers are equal, then their real parts are equal and their imaginary parts are equal.*

Let $$c \equiv a+ib, \quad r \equiv p+iq$$

be two given complex numbers (a, b, p, q all real), such that $c = r$. Then

$$a+ib = p+iq,$$

or $$a-p = i(q-b).$$

Squaring each side, we have

$$\begin{aligned}(a-p)^2 &= i^2(q-b)^2 \\ &= -(q-b)^2.\end{aligned}$$

But $a-p, q-b$ are real, so that their squares are positive or zero; hence this relation cannot hold unless each side is zero. That is,

$$a = p, \quad b = q,$$

as required.

COROLLARY. *If a, b are real and*

$$a+ib = 0,$$

then $$a = 0, \quad b = 0.$$

EXAMPLES V

Find the sum, difference, product and quotient of each of the following pairs of complex numbers:

1. $2+3i, \quad 3-5i$.
2. $-4+2i, \quad -3+7i$.
3. $4, \quad 2i$.
4. $3+2i, \quad -i$.
5. $2+3i, \quad 2-3i$.
6. $-3+4i, \quad -3-4i$.

Find, in the form $a+ib$ (a, b real) the reciprocal of each of the following complex numbers:

7. $3+4i$. 8. $-5+12i$.

9. $6i$. 10. $-6-8i$.

Solve the following quadratic equations, expressing your answers in the form $a \pm ib$:

11. $x^2+4x+13=0$. 12. $x^2-2x+2=0$.

13. $x^2+6x+10=0$. 14. $4x^2+9=0$.

15. $x^2-8x+25=0$. 16. $x^2+4x+5=0$.

6. The complex number as a number-pair. A complex number

$$c \equiv a+ib$$

consists essentially of the pair of real numbers a, b linked by the symbol i to which we have given a specific property (i^2 negative) not enjoyed by any real number. Hence c is of the nature of an *ordered pair* of real numbers, the ordering being an important feature since, for example, the two complex numbers $4+5i$ and $5+4i$ are quite distinct.

The concept of number-pair forms the foundation for the more logical development promised at the end of § 1, for it is precisely this concept which enables us to extend the range of numbers required for the solution of the third quadratic equation ($x^2-6x+10=0$). We therefore make a fresh start, with a series of definitions designed to exhibit the properties hitherto obtained by the use of the fictitious 'number' i.

We define a *complex number* to be an ordered number-pair (of *real* numbers), denoted by the symbol $[a, b]$, subject to the following rules of operation (chosen, of course, to fit in to the treatment given at the start of this chapter):

(i) The symbols $[a, b]$, $[u, v]$ are equal if, and only if, the two relations $a=u, b=v$ are satisfied;

(ii) The sum

$$[a, b]+[u, v]$$

is the number-pair $[a+u, b+v]$;

(iii) The product

$$[a, b] \times [u, v]$$

is the number-pair $[au-bv, bu+av]$.

Compare the formulæ in § 3.

A number $[a, 0]$, whose second component is zero, is called a *real complex number*; a number $[0, b]$, whose first component is zero, is called a *pure imaginary complex number*. These terms are usually abbreviated to *real* and *pure imaginary* numbers. (Compare p. 162.)

In order to achieve correlation with the normal notation for real numbers, and with the customary notation based on the letter i for pure imaginaries, we make the following conventions:

When we are working in a system of algebra requiring the use of complex numbers,

(i) the symbol a will be used as an abbreviation for the number-pair $[a, 0]$,

(ii) the symbol ib will be used as an abbreviation for the number-pair $[0, b]$.

It follows that the symbol $a+ib$ may be used for the number-pair $[a, 0]+[0, b]$, or $[a, b]$.

In particular, we write 1 for the unit $[1, 0]$ of real numbers, and $i \times 1$ for the unit $[0, 1]$ of pure imaginary numbers; but no confusion arises if we abbreviate $i \times 1$ to the symbol i itself.

We do not propose to go into much greater detail, but the reader may easily check that the complex numbers defined in this way by number-pairs have all the properties tentatively proposed for them in the earlier paragraphs of this chapter. We ought, however, to verify explicitly that the square of the number-pair i is the real number -1. For this, we appeal directly to the definition of multiplication:

$$[a, b] \times [u, v] = [au-bv, bu+av].$$

Write $a=u=0$, $b=v=1$. Then

$$[0, 1] \times [0, 1] = [-1, 0]$$

or, in terms of the abbreviated symbols,

$$i \times i = -1.$$

Finally, we remark that, although the ordinary language of real numbers is retained without apparent change, the symbols in complex algebra carry an entirely different meaning. For example, the statement

$$2 \times 2 = 4$$

remains true; but what we really mean is that

$$[2,0] \times [2,0] = [2 \times 2 - 0 \times 0, 0 \times 2 + 2 \times 0]$$
$$= [4,0].$$

From now on, however, we shall revert to the normal symbolism without square brackets, writing a complex number in the form $a+ib$ as before. It must always be remembered that number-pairs are intended.

7. The Argand diagram. Abstractly viewed, a complex number $a+ib$ is a 'number-pair' $[a,b]$ in which the first component of the pair corresponds to the real part and the second component to the imaginary. A number-pair is, however, also capable of a familiar *geometrical* interpretation, when the two numbers a,b in assigned order are taken to be the Cartesian coordinates of a point in a plane. We therefore seek next to link these two conceptions of number-pair so that we can incorporate the ideas of analytical geometry into the development of complex algebra.

With a change of notation, denote a complex number by the letter z, where

$$z \equiv x+iy.$$

We do this to strengthen the implication of the real and imaginary parts x and y as the Cartesian coordinates of a point $P(x,y)$ (Fig. 93).

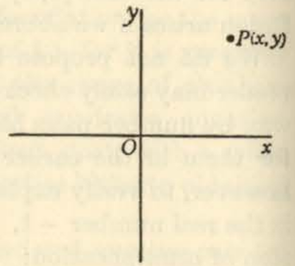


Fig. 93.

There is then an exact correspondence between the complex numbers $z \equiv x+iy$ and the points $P(x,y)$ of the plane:

(i) If z is given, then its real and imaginary parts are determined and so P is known;

(ii) If P is given, then its coordinates are determined, and so z is known.

We say that P *represents* the complex number z. The diagram in which the complex numbers are represented is called an *Argand diagram*, and the plane in which it is drawn is called the *complex plane*, or, when precision is required, the *z-plane*.

EXAMPLES VI

Mark on an Argand diagram the points which represent the following complex numbers:

1. $3+4i$. 2. $5-i$. 3. $6i$.

4. -3. 5. $1+i$. 6. $-2-3i$.

7. $\cos\theta + i\sin\theta$ for $\theta = 0°, 30°, 60°, \ldots, 330°, 360°$.

8. $2\cos^2\frac{1}{2}\theta + i\sin\theta$ for $\theta = 0, 45°, 90°, \ldots, 315°, 360°$.

8. Modulus and argument. Given the point $P(x,y)$ representing the complex number

$$z \equiv x+iy,$$

we may describe its position alternatively by means of polar coordinates r, θ (Fig. 94), where r is the distance OP, ESSENTIALLY POSITIVE and θ is the angle $\angle xOP$ which OP makes with the x-axis, measured in the counter-clockwise sense from Ox. Thus r is defined uniquely, but θ is ambiguous by multiples of 2π.

We know from elementary trigonometry that

$$x = r\cos\theta, \quad y = r\sin\theta,$$

whatever the quadrant in which P lies.

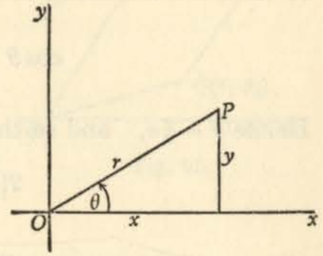


Fig. 94.

Squaring and adding these relations, we have

$$r^2 = x^2 + y^2,$$

so that, since r is positive,

$$r = +\sqrt{(x^2+y^2)}.$$

Dividing the relations, we have

$$\tan\theta = y/x,$$

or $$\theta = \tan^{-1}(y/x).$$

The choice of quadrant for θ depends on the signs of BOTH y AND x; if x, y are $+, +$, the quadrant is the first; if x, y are $-, +$, the second; if x, y are $-, -$, the third; if x, y are $+, -$, the fourth.

The two numbers
$$r = \sqrt{(x^2+y^2)},$$
$$\theta = \tan^{-1}(y/x),$$
with appropriate choice of the quadrant for θ, are called the *modulus* and *argument* respectively of the complex number z. If r, θ are given, then
$$z = r(\cos\theta + i\sin\theta).$$

We have already (p. 162) explained the notation $|z| = r$, and proved the formula $z\bar{z} = |z|^2$.

ILLUSTRATION 1. *To find the modulus and argument of the complex number* $-1+i\sqrt{3}$.

If the modulus is r, then
$$r^2 = (-1)^2 + (\sqrt{3})^2 = 1+3 = 4,$$
so that $r = 2$. The number is therefore
$$2\left(-\frac{1}{2} + i\frac{\sqrt{3}}{2}\right),$$
so that, if θ is the argument,
$$\cos\theta = -\tfrac{1}{2}, \quad \sin\theta = \frac{\sqrt{3}}{2}.$$

Hence $\theta = \frac{2}{3}\pi$, and so the number is
$$2\{\cos\tfrac{2}{3}\pi + i\sin\tfrac{2}{3}\pi\}.$$

EXAMPLES VII

1. Plot in an Argand diagram the points which represent the numbers $4+3i, -3+4i, -4-3i, 3-4i$, and verify that the points are at the vertices of a square.

2. Plot in an Argand diagram the points which represent the numbers $7+3i, 6i, -3-i, 4-4i$, and verify that the points are at the vertices of a square.

3. Find the modulus and argument in degrees and minutes of each of the complex numbers
$$\sqrt{3}-i, \quad 3+4i, \quad -5+12i, \quad 3, \quad 6-8i, \quad -2i.$$

4. Prove that, if the point z in the complex plane lies on the circle whose centre is the (complex) point a and whose radius is the real number k, then
$$|z-a| = k.$$

5. Find the locus in an Argand diagram of the point representing the complex number z subject to the relation
$$|z+2| = |z-5i|.$$

9. The representation in an Argand diagram of the sum of two numbers. Suppose that
$$z_1 \equiv x_1 + iy_1, \quad z_2 \equiv x_2 + iy_2$$
are two complex numbers represented in an Argand diagram by the points
$$P_1(x_1, y_1), \quad P_2(x_2, y_2)$$
respectively (Fig. 95).

Their sum is the number
$$z = z_1 + z_2$$
$$= (x_1+x_2) + i(y_1+y_2)$$
represented by the point $P(x, y)$, where
$$x = x_1 + x_2, \quad y = y_1 + y_2.$$

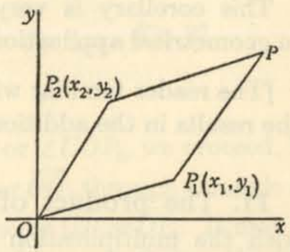


Fig. 95.

By elementary analytical geometry, the lines P_1P_2 and OP have the same middle point $\left(\frac{x_1+x_2}{2}, \frac{y_1+y_2}{2}\right)$, so that OP_1PP_2 is a parallelogram. Hence P (*representing* z_1+z_2) *is the fourth vertex of the parallelogram of which* OP_1, OP_2 *are adjacent sides.*

In particular, OP *is the modulus* $|z_1+z_2|$ *of the sum of the two numbers* z_1, z_2.

10. The representation in an Argand diagram of the difference of two numbers. To find the point representing the difference

$$z \equiv z_2 - z_1,$$

we proceed as follows:

The relation is equivalent to

$$z_2 = z + z_1,$$

so that OP_2 (Fig. 96) is the diagonal of the parallelogram of which OP_1, OP (where P represents z) are adjacent sides.

Hence $\overrightarrow{OP}$ *is the line through* O *parallel to* $\overrightarrow{P_1P_2}$, *the sense along the lines being indicated by the arrows, where* OP, P_1P_2 *are equal in magnitude.*

COROLLARY. *If* $P_1(x_1, y_1)$, $P_2(x_2, y_2)$ *represent the two complex numbers* $z_1 \equiv x_1 + iy_1, z_2 \equiv x_2 + iy_2$, *then the line* $\overrightarrow{P_1P_2}$ *represents the difference* $z_2 - z_1$, *in the sense that the length* P_1P_2 *is the modulus* $|z_2 - z_1|$ *and the angle which* $\overrightarrow{P_1P_2}$ *makes with the x-axis is the argument of* $z_2 - z_1$.

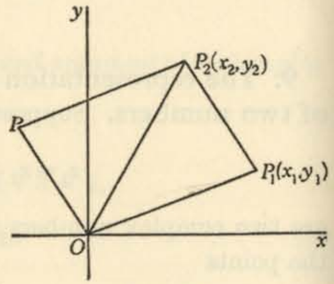


Fig. 96

This corollary is very important in geometrical applications.

[The reader familiar with the theory of vectors should compare the results in the addition and subtraction of two vectors.]

11. The product of two complex numbers. In dealing with the multiplication of complex numbers, it is often more convenient to express them in modulus-argument form. We therefore write

$$z_1 \equiv r_1(\cos\theta_1 + i\sin\theta_1), \quad z_2 \equiv r_2(\cos\theta_2 + i\sin\theta_2).$$

Then
$$\begin{aligned} z_1z_2 &= r_1r_2(\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2) \\ &= r_1r_2\{(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) \\ &\qquad + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)\} \\ &= r_1r_2\{\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\}, \end{aligned}$$

which is a complex number of modulus r_1r_2 and argument $\theta_1 + \theta_2$.

Hence

(i) *the modulus of a product is the product of the moduli;*

(ii) *the argument of a product is the sum of the arguments.*

We may obtain similarly the results for division:

(iii) *the modulus of a quotient is the quotient of the moduli;*

(iv) *the argument of a quotient is the difference of the arguments.*

12. The product in an Argand diagram. In the diagram (Fig. 97), let P_1, P_2, P represent the two complex numbers z_1, z_2 and their product $z \equiv z_1z_2$ respectively, and let U be the 'unit' point $z = 1$.

Then (§ 10)

$$\angle xOP = \theta_1 + \theta_2,$$

where θ_1, θ_2 are the arguments of z_1, z_2, and so

$$\begin{aligned} \angle P_2OP &= \angle xOP - \angle xOP_2 \\ &= (\theta_1 + \theta_2) - \theta_2 \\ &= \theta_1 \\ &= \angle UOP_1. \end{aligned}$$

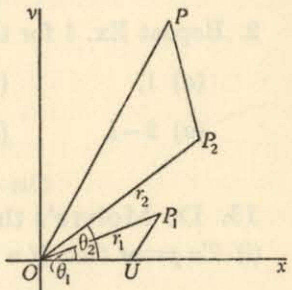


Fig. 97.

Moreover, if r_1, r_2 are the moduli of z_1, z_2,

$$\begin{aligned} OP/OP_2 &= (r_1r_2)/r_2 = r_1/1 \\ &= OP_1/OU. \end{aligned}$$

Now in measuring the angle $\angle P_2OP$ or $\angle UOP_1$, we proceed, by convention, from the radius vector $\overrightarrow{OP_2}$ or $\overrightarrow{OU}$, through an angle θ_1, in the counter-clockwise sense, to the vector $\overrightarrow{OP}$ or $\overrightarrow{OP_1}$. It may be that θ_1 itself does not lie between $0, \pi$, but nevertheless the fact that the angles $\angle P_2OP, \angle UOP_1$ so described are equal ensures that the Euclidean angles $\angle P_2OP$, $\angle UOP_1$ of the triangles ΔP_2OP, ΔUOP_1 are also equal. Moreover, the sides about these equal angles have been proved proportional, and so the triangles are similar.

Hence follows the construction for P:

Let P_1, P_2 *represent the two given numbers* z_1, z_2; *let* O *be the origin and* U *the unit point* $z = 1$. *Describe the triangle* POP_2 *similar to the triangle* P_1OU, *in such a sense that* $\angle P_2OP = \angle UOP_1$. *Then the point* P *represents the product* z_1z_2.

Corollary. *The effect of multiplying a complex number z by i is to rotate the vector OP representing z through an angle $\frac{1}{2}\pi$ in the counter-clockwise sense.* This follows at once if we put $z_1 = i$ in the preceding work, so that P_1 in the diagram becomes the point $(0, 1)$ or, in polar coordinates, $(1, \frac{1}{2}\pi)$.

Thus i acts in the Argand diagram like an *operator*, turning the radius vector through a right angle.

EXAMPLES VIII

1. Mark in an Argand diagram the points which represent (*a*) $3+2i$, (*b*) $2+i$, (*c*) their product. Verify from your diagram the theorem of the text.

2. Repeat Ex. 1 for the points

(*a*) 1, (*b*) $3-5i$, (*c*) their product;

(*a*) $2-i$, (*b*) $2+i$, (*c*) their product.

13. De Moivre's theorem.

(i) *To prove that, if n is a positive integer, then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

We use a proof by induction. Suppose that the theorem is true for a particular number N, a positive integer, so that

$$(\cos\theta + i\sin\theta)^N = \cos N\theta + i\sin N\theta.$$

Then

$$\begin{aligned}(\cos\theta + i\sin\theta)^{N+1} &= (\cos N\theta + i\sin N\theta)(\cos\theta + i\sin\theta)\\ &= (\cos N\theta\cos\theta - \sin N\theta\sin\theta)\\ &\quad + i(\cos N\theta\sin\theta + \sin N\theta\cos\theta)\\ &= \cos(N+1)\theta + i\sin(N+1)\theta.\end{aligned}$$

Hence if the theorem is true for any particular value N, it is true for $N+1, N+2, \ldots$ and so on. But it is clearly true for $N = 0$, and so the theorem is established.

(ii) *To prove that, if n is a negative integer, then*

$$(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta.$$

Write $$n = -m.$$

Then m is a positive integer, and so

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= (\cos\theta + i\sin\theta)^{-m}\\ &= \frac{1}{(\cos\theta + i\sin\theta)^m}\\ &= \frac{1}{\cos m\theta + i\sin m\theta} \quad \text{(as above)}\\ &= \frac{\cos m\theta - i\sin m\theta}{(\cos m\theta + i\sin m\theta)(\cos m\theta - i\sin m\theta)}\\ &= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta - i^2\sin^2 m\theta}\\ &= \frac{\cos m\theta - i\sin m\theta}{\cos^2 m\theta + \sin^2 m\theta}\\ &= \cos m\theta - i\sin m\theta\\ &= \cos(-n\theta) - i\sin(-n\theta)\\ &= \cos n\theta + i\sin n\theta,\end{aligned}$$

as required.

(iii) *To prove that, if n is a rational fraction, then*

$$(\cos\theta + i\sin\theta)^n = \cos n(\theta + 2k\pi) + i\sin n(\theta + 2k\pi),$$

where k is an integer, positive or negative.

Suppose that $$n = p/q,$$

where p, q are integers (q positive) without a common factor.

Consider the expression

$$(\cos\theta + i\sin\theta)^{1/q},$$

and suppose that it CAN be expressed in the form

$$\cos\phi + i\sin\phi.$$

If so, then $$\begin{aligned}\cos\theta + i\sin\theta &= (\cos\phi + i\sin\phi)^q\\ &= \cos q\phi + i\sin q\phi \quad \text{[by (i)]}.\end{aligned}$$

Equating real and imaginary parts, we have

$$\cos\theta = \cos q\phi, \quad \sin\theta = \sin q\phi.$$

These equalities hold simultaneously if, and only if,

$$q\phi = \theta + 2k\pi,$$

where k is any integer, positive or negative. Hence

$$\phi = \frac{1}{q}\theta + \frac{2k}{q}\pi,$$

so that

$$(\cos\theta + i\sin\theta)^{1/q} = \cos\left(\frac{1}{q}\theta + \frac{2k}{q}\pi\right) + i\sin\left(\frac{1}{q}\theta + \frac{2k}{q}\pi\right).$$

Raising each side to the power p, by (i) or (ii) above, and then writing $p/q = n$, we obtain the relation

$$(\cos\theta + i\sin\theta)^n = \cos n(\theta + 2k\pi) + i\sin n(\theta + 2k\pi).$$

The expression on the right-hand side takes various values according to the choice of k. That is to say, there are several values for $(\cos\theta + i\sin\theta)^n$ when n is a rational fraction; the case $n = \frac{1}{2}$ is familiar, leading to two distinct square roots. We see by elementary trigonometry that distinct values are obtained when k assumes in succession the values $0, 1, 2, \ldots, q-1$, where q is the denominator in the expression of n as a proper fraction in its lowest terms. Thereafter they repeat themselves, any block of q consecutive values of k giving the whole set.

SUMMARY. *The formula*

$$(\cos\theta + i\sin\theta)^n = \cos n(\theta + 2k\pi) + i\sin n(\theta + 2k\pi)$$

is true for all values of n, where n may be a positive or negative integer or rational fraction. When n is an integer, k may be taken as zero; when n is fractional, distinct values are obtained on the right-hand side for $k = 0, 1, 2, \ldots, q-1$, where q is the denominator in the expression of n as a proper fraction in its lowest terms.

EXAMPLES IX

1. Express $\frac{1}{2} + i\frac{\sqrt{3}}{2}$ in modulus-argument form $r(\cos\theta + i\sin\theta)$ and hence, with the help of your tables, find (i) its square, (ii) its square roots, (iii) its cube roots, (iv) its fourth roots.

2. Repeat Ex. 1 for the number $4 + 3i$.

3. Repeat Ex. 1 for the number $12 - 5i$.

14. The nth roots of unity. In virtue of the formula

$$1 = \cos 0 + i\sin 0,$$

it follows from De Moivre's theorem that, if n is any integer (assumed positive here) then an nth root of unity, that is

$$\sqrt[n]{1} \equiv 1^{1/n},$$

can be expressed in the form

$$\cos\left(\frac{1}{n}.2k\pi\right) + i\sin\left(\frac{1}{n}.2k\pi\right)$$

for $k = 0, 1, 2, \ldots, n-1$. Hence *for any given positive integer n, there are n distinct roots of unity, namely*

$$\cos\frac{2k\pi}{n} + i\sin\frac{2k\pi}{n},$$

where $k = 0, 1, 2, \ldots, n-1$.

For example:

When $n = 2$, there are two square roots, namely

$$\cos\frac{2k\pi}{2} + i\sin\frac{2k\pi}{2},$$

where $k = 0, 1$. When $k = 0$, this gives $\cos 0$, or 1; when $k = 1$, it gives $\cos\pi$, or -1.

When $n = 3$, there are three cube roots, namely

$$\cos\frac{2k\pi}{3} + i\sin\frac{2k\pi}{3},$$

where $k = 0, 1, 2$. When $k = 0$, this gives 1 itself. The values $k = 1, 2$ give

$$\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3} \equiv \tfrac{1}{2}(-1 + i\sqrt{3}),$$

$$\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3} \equiv \tfrac{1}{2}(-1 - i\sqrt{3}).$$

These complex cube roots of unity are usually denoted by the symbols ω, ω^2, either being the square of the other. They are also connected by the relation

$$1 + \omega + \omega^2 = 0.$$

EXAMPLES X

Use De Moivre's theorem to express the following powers in 'modulus-argument' form $r(\cos\theta + i\sin\theta)$:

1. $(1+i)^4$.
2. $\dfrac{1}{(1-i)^5}$.
3. $(1+i\sqrt{3})^6$.
4. $(\sqrt{3}-i)^{\frac{1}{2}}$.
5. $(1-i)^{\frac{1}{3}}$.
6. $(\sqrt{3}+i)^{\frac{3}{4}}$.
7. Find expressions for

$$\text{(i)}\quad 1^{\frac{1}{5}}, \quad \text{(ii)}\quad 1^{\frac{1}{8}}$$

analogous to the roots evaluated in the text.

15. Complex powers.

The reader will remember that, when seeking an interpretation for symbols such as

$$a^0, \quad a^{-n}, \quad a^{1/n}, \quad a^{-1/n}$$

for real positive a and positive integral n, he was guided by the principle that the formula of multiplication

$$a^p \times a^q = a^{p+q}$$

should hold for all values of p and q. This led to the interpretations

$$a^0 = 1, \quad a^{-n} = 1/a^n, \quad a^{1/n} = \sqrt[n]{a}, \quad a^{-1/n} = 1/\sqrt[n]{a}.$$

We now consider the meanings which should be given to a^n when a, n are complex numbers, still retaining the validity for the formula of multiplication.

(i) *When n is real and rational*, the treatment is comparatively simple, for we know that any complex number a can be expressed in the form

$$a \equiv r(\cos\theta + i\sin\theta),$$

and so, by De Moivre's theorem,

$$a^n = r^n\{\cos n(\theta + 2k\pi) + i\sin n(\theta + 2k\pi)\}$$

where the values $k = 0, 1, \ldots, q-1$ (q being the denominator in the expression of n as a proper fraction in its lowest terms) give distinct values for a^n. Thus a^n is, in general, a q-valued function. (For example, if $n = \frac{1}{2}$, there are two square roots.)

(ii) *When n is complex*, say

$$n = x + iy, \quad (x, y \text{ real})$$

then we must interpret a^n in such a way that

$$a^{x+iy} = a^x \times a^{iy}.$$

We have already dealt with the factor a^x for real and rational values of x, so that we are left to consider what meaning may be given to the expression a^{iy} when the power iy is pure imaginary.

16. Pure imaginary powers.

In the preceding paragraph we reduced the interpretation of a^n to the case, which we now consider, when n is pure imaginary. With a slight change of notation, we write

$$n = ix,$$

where x is real.

(i) *When a is real and positive.*

We note first that any real positive number a (other than zero) can be expressed in the form

$$a \equiv e^{\log_e a},$$

so that, if the laws of indices are to be preserved,

$$a^n \equiv (e^{\log_e a})^n \equiv e^{n\log_e a}.$$

Writing $\quad n\log_e a \equiv ix\log_e a \equiv ix' \quad (x' \text{ real}),$

we obtain the expression, as an imaginary power of e, in the form

$$a^n \equiv e^{ix'}$$

which is found to be more amenable to treatment. Dropping the dash, then, we consider what interpretation should be given to the number

$$e^{ix} \quad (x \text{ real}).$$

We take the hint from the expansion of p. 54, which, if valid for pure imaginary numbers, would yield the relation

$$e^{ix} = 1 + (ix) + \frac{(ix)^2}{2!} + \frac{(ix)^3}{3!} + \frac{(ix)^4}{4!} + \ldots$$

$$= \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots\right) + i\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots\right).$$

Now (p. 50) we have the relations, valid for all real x,

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \ldots,$$

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \ldots,$$

and so we are led to propose for consideration the relation

$$e^{ix} = \cos x + i \sin x.$$

Before we can accept this as a valid interpretation for e^{ix}, however, we must satisfy ourselves that it obeys the index law

$$e^{ix} \times e^{iy} = e^{i(x+y)} \quad (x, y \text{ real}).$$

Under the proposed interpretation, the left-hand side is

$$(\cos x + i \sin x) \times (\cos y + i \sin y)$$
$$= (\cos x \cos y - \sin x \sin y) + i(\sin x \cos y + \cos x \sin y)$$
$$= \cos (x+y) + i \sin (x+y),$$

and this is precisely the interpretation to be given to the right-hand side also.

The interpretation is therefore consistent with the series expansions and with the index law, and so we DEFINE the interpretation of e^{ix} (x real) to be given by the relation

$$e^{ix} = \cos x + i \sin x.$$

Note. The mere writing of ix in the expansion in series does not of itself allow us to use the notation e^{ix} under the index laws. The step that the product of the expressions proposed for e^{ix}, e^{iy} is indeed $e^{i(x+y)}$ is vital to the interpretation.

(ii) *When a is any complex number.*

We are now in a position to give an interpretation to the expression a^{ix} for any complex number a. We have seen that a may be written in the form

$$a \equiv r(\cos\theta + i \sin\theta) \quad (r \text{ positive})$$

and, by what we have just done, we have the two relations

$$r = e^{\log_e r} \quad (r \neq 0),$$

$$\cos\theta + i \sin\theta = e^{i\theta}.$$

Hence
$$a = e^{\log_e r} \times e^{i\theta}$$
$$= e^u \times e^{i\theta},$$

say, where u, or $\log_e r$, is a real number, positive or negative. For interpretations consistent with the laws of indices, we must therefore take

$$a^{ix} = (e^u)^{ix} \times (e^{i\theta})^{ix}$$
$$= e^{iux} \times e^{-\theta x}$$
$$= e^{-\theta x}(\cos ux + i \sin ux)$$
$$= e^{-\theta x}\{\cos (x \log_e r) + i \sin (x \log_e r)\}.$$

COROLLARY. *Any (non-zero) complex number can be expressed in the form e^{u+iv}, where u, v are real.*

17. Multiplicity of values. There is one difficulty which we have slurred over, as it should not be over-emphasized at the present stage. An example will illustrate the point.

Let us consider what interpretation we are to give to the symbol

$$i^i.$$

Our first task is to express the number to be raised to the power i (in this case, i itself) in the form $re^{i\theta}$, and this we do easily by noticing that

$$i = \cos (2k + \tfrac{1}{2})\pi + i \sin (2k + \tfrac{1}{2})\pi$$
$$= e^{(2k+\frac{1}{2})\pi i}$$

for any integer k, positive or negative. Hence the interpretation

$$i^i \equiv \{e^{(2k+\frac{1}{2})\pi i}\}^i$$
$$= e^{(2k+\frac{1}{2})\pi i \times i}$$
$$= e^{-(2k+\frac{1}{2})\pi}$$

gives an *infinite succession* of values, all of which, oddly enough, are real.

In a full discussion, we should therefore be on our guard to include many-valued powers. A safe way of doing this is to include the factor $e^{2k\pi i}$ (which is just $\cos 2k\pi + i \sin 2k\pi$, or 1, for any integer k) in the expression in exponential form of the number whose power we seek. That is to say, in order to evaluate a^n, we write a in the form

$$a \equiv e^{u+i(v+2k\pi)}$$

and allow k to take integral values.

It should be noticed that we established the interpretation

$$e^{ix} \equiv \cos x + i \sin x$$

under the conditions of De Moivre's theorem, namely that x is a real number, being a positive or negative integer or rational fraction. When x is not integral, there is really an ambiguity of interpretation, since

$$\begin{aligned} e^{ix} &= (e^i)^x \\ &= e^{i(1+2k\pi)x} \\ &= \cos(1+2k\pi)x + i\sin(1+2k\pi)x, \end{aligned}$$

where distinct values are obtained for $k = 0, 1, \ldots, q-1$ as usual, q being the denominator when the fraction x is expressed in its lowest terms. Our interpretation is therefore subject to the convention $k = 0$.

ILLUSTRATION 2. *To find an expression for*

$$(1+i\sqrt{3})^{4+\frac{1}{2}i}.$$

We first write $1+i\sqrt{3}$ in the form

$$\begin{aligned} 1+i\sqrt{3} &\equiv 2\left(\frac{1}{2}+i\frac{\sqrt{3}}{2}\right) \\ &= 2\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right) \\ &= 2e^{i(\frac{1}{3}\pi+2k\pi)} \quad (k \text{ integral}). \end{aligned}$$

We next use the fact that

$$(1+i\sqrt{3})^{4+\frac{1}{2}i} = (1+i\sqrt{3})^4 \times (1+i\sqrt{3})^{\frac{1}{2}i}.$$

Now
$$\begin{aligned} (1+i\sqrt{3})^4 &= 2^4 e^{4i(\frac{1}{3}\pi+2k\pi)} \\ &= 16e^{4\pi i/3}. \end{aligned}$$

Also
$$\begin{aligned} (1+i\sqrt{3})^{\frac{1}{2}i} &= 2^{\frac{1}{2}i} \times \{e^{i(\frac{1}{3}\pi+2k\pi)}\}^{\frac{1}{2}i} \\ &= 2^{\frac{1}{2}i} \times e^{-(\frac{1}{6}\pi+k\pi)}. \end{aligned}$$

Finally,
$$2 = e^{\log_e 2},$$

giving
$$2^{\frac{1}{2}i} = e^{\frac{1}{2}i\log_e 2},$$

so that, in all,

$$\begin{aligned} (1+i\sqrt{3})^{4+\frac{1}{2}i} &= 16e^{4\pi i/3} \times e^{\frac{1}{2}i\log_e 2} \times e^{-(\frac{1}{6}\pi+kn)} \\ &= 16e^{-(\frac{1}{6}\pi+k\pi)} \times e^{(\frac{4}{3}\pi+\frac{1}{2}\log_e 2)i} \\ &= 16e^{-(\frac{1}{6}\pi+k\pi)}\{\cos(\tfrac{4}{3}\pi+\tfrac{1}{2}\log_e 2)+i\sin(\tfrac{4}{3}\pi+\tfrac{1}{2}\log_e 2)\}, \end{aligned}$$

where k is any integer, positive, negative, or zero.

EXAMPLES XI

Find expressions for

1. $(1+i)^i$.
2. $(1-i)^{2-i}$.
3. $(\sqrt{3}+i)^{\frac{1}{2}i}$.
4. $(1-i\sqrt{3})^{3+2i}$.
5. $(\sqrt{3}-i)^{1+\frac{1}{2}i}$.
6. $(1+i)^{4+\frac{1}{2}i}$.

18. The logarithm of a complex number, to the base *e*.

We have seen (p. 181) that any (non-zero) complex number can be expressed in the form

$$a \equiv e^{u+iv}.$$

We DEFINE the logarithm of a to be the complex number

$$u+iv \quad (u, v \text{ real}).$$

This is consistent with the treatment already given for real numbers ($v = 0$) and, in virtue of the interpretation already given to indices, it retains the property

$$\log_e a + \log_e b = \log_e(ab).$$

We have now, of course, released the restriction (p. 5) that a must be positive in order to have a logarithm. It need not even be real.

The ambiguous determination of v, as seen by the equivalent formula

$$a \equiv e^{u+i(v+2k\pi)} \quad (k \text{ integral})$$

means that *the logarithm is ambiguous to the extent of additive multiples of* $2\pi i$.

Note. The relation (pp. 170, 180)

$$z = re^{i\theta},$$

where r is the modulus and θ the argument of z, gives the formula

$$\log z = \log|z| + i\arg z.$$

NOTATION. The notation

$$\mathrm{Log}_e a$$

is often used to denote a value of the logarithm when ambiguity may be present. Then the notation

$$\log_e a$$

denotes that determination of the logarithm whose imaginary part lies between $-\pi$ and π, and $\log_e a$ is called the *principal value* of $\mathrm{Log}_e\, a$.

This distinction, which is often of importance, will not, in fact, be used much in this book, and we shall usually take the principal value without further comment.

19. The sine and cosine. We have made no statements so far about the formula

$$e^{ix} = \cos x + i \sin x$$

except when x is real; naturally, because $\cos x, \sin x$ are otherwise undefined. Our next task is to define the trigonometric functions of a complex variable $z \equiv x+iy$ (x, y real), and this we do by what is in some ways a reversal of the process hitherto adopted. We first observe that, for real x,

$$e^{ix} = \cos x + i \sin x,$$

$$e^{-ix} = \cos x - i \sin x,$$

so that
$$\cos x = \tfrac{1}{2}(e^{ix}+e^{-ix}),$$

$$\sin x = \frac{1}{2i}(e^{ix}-e^{-ix}).$$

We now adopt these formulæ, and make them the basis of the following more general definitions:

If z is a complex number, then

$$\cos z = \tfrac{1}{2}(e^{iz}+e^{-iz}),$$

$$\sin z = \frac{1}{2i}(e^{iz}-e^{-iz}).$$

It follows at once that these definitions give the normal functions when z is real, and also that

$$e^{iz} = \cos z + i \sin z.$$

We must, however, now make sure that they retain the NORMAL properties of the trigonometric functions.

Firstly, we have

$$\begin{aligned} 4(\cos^2 z + \sin^2 z) &= (e^{iz}+e^{-iz})^2 - (e^{iz}-e^{-iz})^2 \\ &= (e^{2iz}+2+e^{-2iz}) - (e^{2iz}-2+e^{-2iz}) \\ &= 4, \end{aligned}$$

so that $\quad \cos^2 z + \sin^2 z = 1.$

Again,

$$\begin{aligned} 4(\cos z_1 \cos z_2 - \sin z_1 \sin z_2) &= (e^{iz_1}+e^{-iz_1})(e^{iz_2}+e^{-iz_2}) + (e^{iz_1}-e^{-iz_1})(e^{iz_2}-e^{-iz_2}) \\ &= 2(e^{i(z_1+z_2)}+e^{-i(z_1+z_2)}) \end{aligned}$$

on reduction, so that

$$\cos z_1 \cos z_2 - \sin z_1 \sin z_2 = \cos(z_1+z_2).$$

Also

$$\begin{aligned} 4i(\sin z_1 \cos z_2 + \cos z_1 \sin z_2) &= (e^{iz_1}-e^{-iz_1})(e^{iz_2}+e^{-iz_2}) + (e^{iz_1}+e^{-iz_1})(e^{iz_2}-e^{-iz_2}) \\ &= 2(e^{i(z_1+z_2)}-e^{-i(z_1+z_2)}), \end{aligned}$$

so that $\quad \sin z_1 \cos z_2 + \cos z_1 \sin z_2 = \sin(z_1+z_2).$

These are the formulæ on which the theory of real angles is based, and so the structure will stand for complex 'angles' also.

Further details are left to the reader.

Note. The functions $|\cos z|, |\sin z|$ are not now subject to the restriction of being less than unity. For example, the equation

$$\cos z = 2$$

is satisfied if $\quad e^{iz}+e^{-iz} = 4,$

or $\quad e^{2iz}-4e^{iz}+1 = 0,$

so that $\quad e^{iz} = 2 \pm \sqrt{3}.$

Thus $\quad iz = \log(2 \pm \sqrt{3})$

or $\quad z = -i \log(2 \pm \sqrt{3}).$

For reference we record the following formulæ:

(i) $\cosh iz = \frac{1}{2}(e^{iz}+e^{-iz})$
$= \cos z;$

(ii) $\sinh iz = \frac{1}{2}(e^{iz}-e^{-iz})$
$= i \sin z;$

(iii) $e^{2k\pi i} = \cos 2k\pi + i \sin 2k\pi = 1$ (k integral);

(iv) $e^{(2k+1)\pi i} = \cos(2k+1)\pi + i \sin(2k+1)\pi = -1;$

(v) $e^{(2k+\frac{1}{2})\pi i} = \cos(2k+\frac{1}{2})\pi + i \sin(2k+\frac{1}{2})\pi = i.$

20. The modulus of e^z. If z is complex, so that

$$z \equiv x+iy \quad (x, y \text{ real}),$$

then

$$e^z = e^{x+iy} = e^x e^{iy}$$
$$= e^x(\cos y + i \sin y).$$

Hence *the modulus of e^z is e^x and its argument is y*, the argument being undetermined to the extent of multiples of 2π.

COROLLARY. An important corollary, found by putting $x = 0$, is that

$$|e^{iy}| = 1,$$

when iy is a pure imaginary.

21. The differentiation and integration of complex numbers. We confine our attention to the only case which we shall use, the complex functions of a REAL variable x. The general theory for a complex variable is much beyond the scope of this book.

By a complex function of a real variable x, we shall mean a function $w(x)$ which is *either* given in the form

$$w(x) \equiv u(x) + iv(x),$$

where $u(x), v(x)$ are real functions of x, *or* can be reduced to that form. For example, the function e^{ix} can be reduced to

$$\cos x + i \sin x.$$

We then use the following DEFINITIONS:

(i) $$\frac{dw}{dx} = \frac{du}{dx} + i\frac{dv}{dx},$$

(ii) $$\int w\,dx = \int u\,dx + i\int v\,dx.$$

Particular interest is attached to the function e^{cx}, where c may be complex of the form $a+ib$. We have

$$\frac{d}{dx}(e^{cx}) = \frac{d}{dx}(e^{(a+ib)x})$$
$$= \frac{d}{dx}\{e^{ax}(\cos bx + i \sin bx)\}$$
$$= \frac{d}{dx}[\{e^{ax}\cos bx\} + i\{e^{ax}\sin bx\}].$$

Hence, by definition,

$$\frac{d}{dx}(e^{cx}) = (a\,e^{ax}\cos bx - b\,e^{ax}\sin bx)$$
$$+ i(a\,e^{ax}\sin bx + b\,e^{ax}\cos bx)$$
$$= a\,e^{ax}(\cos bx + i \sin bx) + ib\,e^{ax}(\cos bx + i \sin bx)$$
$$= (a+ib)\,e^{ax}.e^{ibx}$$
$$= (a+ib)e^{(a+ib)x}$$
$$= ce^{cx}.$$

Hence *the rule* $$\frac{d}{dx}(e^{cx}) = ce^{cx}$$

holds whether c is real or complex.

Similarly we may prove that the formula

$$\int e^{cx}\,dx = \frac{1}{c}\,e^{cx}$$

holds whether c is real or complex.

Because of its importance, we ought perhaps to mention also the differentiation and integration of x^c, where x is assumed to be real but where c may be complex. Since

$$x^c = e^{c\log x},$$

we have

$$\frac{d}{dx}(x^c) = e^{c\log x}.c.\frac{d}{dx}(\log x) \quad \text{(as above)}$$
$$= e^{c\log x}.\frac{c}{x}$$
$$= x^c.\frac{c}{x}$$
$$= cx^{c-1},$$

in accordance with the similar rule for real functions.

It follows that

$$\int x^c dx = \frac{1}{c+1}x^{c+1} \quad (c \neq -1)$$

whether c is real or complex.

Illustration 3. *To prove the rule*

$$\frac{d}{dx}\{f(x)\,g(x)\} = g(x)\frac{d}{dx}\{f(x)\} + f(x)\frac{d}{dx}\{g(x)\}$$

when $f(x), g(x)$ are complex functions of the real variable x.

Write
$$f(x) \equiv u(x) + iv(x) \equiv u + iv,$$
$$g(x) \equiv p(x) + iq(x) \equiv p + iq.$$

Then
$$f(x)g(x) = (up - vq) + i(uq + vp),$$

so that

$$\frac{d}{dx}\{f(x)g(x)\}$$
$$= [(u'p - v'q) + (up' - vq')] + i[(u'q + v'p) + (uq' + vp')]$$
$$= [u'g + iv'g] + [p'f + iq'f]$$
$$= f'g + g'f.$$

Reversal of this formula leads to the rule for *integration by parts*. Hence this method is also at our disposal for these functions.

The two illustrations which follow show how complex numbers may be used to sum series and to evaluate integrals.

Illustration 4. *To find the sum of the first n terms of the series*

$$1 + \cos\theta.\cos\theta + \cos^2\theta.\cos 2\theta + \ldots + \cos^{n-1}\theta.\cos(n-1)\theta,$$

where θ is a real angle.

Write

$$C \equiv 1 + \cos\theta.\cos\theta + \cos^2\theta.\cos 2\theta + \ldots + \cos^{n-1}\theta.\cos(n-1)\theta,$$
$$S \equiv \cos\theta.\sin\theta + \cos^2\theta.\sin 2\theta + \ldots + \cos^{n-1}\theta.\sin(n-1)\theta.$$

Then

$$C + iS = 1 + \cos\theta\, e^{i\theta} + \cos^2\theta\, e^{2i\theta} + \ldots + \cos^{n-1}\theta\, e^{(n-1)i\theta}.$$

This is a geometric progression, of n terms, with first term 1 and ratio $\cos\theta\, e^{i\theta}$. It may be proved, exactly as for real numbers, that the sum for first term a and ratio r is

$$\frac{a(1-r^n)}{1-r},$$

so that

$$C + iS = \frac{1 - \cos^n\theta\, e^{ni\theta}}{1 - \cos\theta\, e^{i\theta}}.$$

In order to find 'real and imaginary parts', multiply numerator and denominator by $1 - \cos\theta\, e^{-i\theta}$. The new denominator is

$$(1 - \cos\theta\, e^{i\theta})(1 - \cos\theta\, e^{-i\theta}) = 1 - \cos\theta(e^{i\theta} + e^{-i\theta}) + \cos^2\theta$$
$$= 1 - \cos\theta(2\cos\theta) + \cos^2\theta = 1 - \cos^2\theta$$
$$= \sin^2\theta.$$

Thus

$$C + iS = \frac{1}{\sin^2\theta}\{(1 - \cos^n\theta\, e^{ni\theta})(1 - \cos\theta\, e^{-i\theta})\}$$
$$= \frac{1}{\sin^2\theta}\{1 - \cos^n\theta\, e^{ni\theta} - \cos\theta\, e^{-i\theta} + \cos^{n+1}\theta\, e^{(n-1)i\theta}\}.$$

Equating real parts, we have

$$C = \frac{1}{\sin^2\theta}\{1 - \cos^n\theta\cos n\theta - \cos^2\theta + \cos^{n+1}\theta\cos(n-1)\theta\}$$
$$= \frac{1}{\sin^2\theta}\{\sin^2\theta - \cos^n\theta(\cos n\theta - \cos\theta\cos(n-1)\theta)\}$$
$$= \frac{1}{\sin^2\theta}\{\sin^2\theta - \cos^n\theta(-\sin\theta\sin(n-1)\theta)\}$$
$$= 1 + \frac{\cos^n\theta\sin(n-1)\theta}{\sin\theta}.$$

It is implicit in the working that $\sin\theta$ is not zero. That case can easily be given separate treatment.

ILLUSTRATION 5. *To find the real integral*

$$\int xe^x \cos x\,dx.$$

Write
$$C \equiv \int xe^x \cos x\,dx,$$
$$S \equiv \int xe^x \sin x\,dx,$$

so that
$$C+iS = \int xe^x e^{ix}\,dx$$
$$= \int xe^{(1+i)x}\,dx.$$

Integrating by parts (p. 188), we have

$$C+iS = \frac{1}{1+i}\,e^{(1+i)x}.x - \frac{1}{1+i}\int e^{(1+i)x}.1.dx$$
$$= \frac{1}{1+i}\,xe^{(1+i)x} - \frac{1}{(1+i)^2}\,e^{(1+i)x}.$$

Now
$$\frac{1}{1+i} = \frac{1-i}{1-i^2} = \tfrac{1}{2}(1-i),$$
$$e^{(1+i)x} = e^x(\cos x + i\sin x).$$

Hence
$$C+iS = [\tfrac{1}{2}(1-i)x - \tfrac{1}{4}(1-2i+i^2)]e^x(\cos x + i\sin x)$$
$$= [\tfrac{1}{2}x - \tfrac{1}{2}ix + \tfrac{1}{2}i]e^x(\cos x + i\sin x).$$

Equating real parts, we have
$$C = \tfrac{1}{2}e^x\{x\cos x + (x-1)\sin x\}.$$

EXAMPLES XII

Use the method of the two illustrations to find:

1. $1+\cos\theta+\cos 2\theta+\ldots+\cos n\theta.$
2. $\sin\theta - x\sin 2\theta + x^2\sin 3\theta - x^3\sin 4\theta + \ldots$ (to n terms).

Find the integrals:

3. $\int x\cos x\,dx.$ 4. $\int e^x\sin x\,dx.$ 5. $\int e^{4x}\sin 3x\,dx.$

6. $\int xe^{2x}\sin x\,dx.$ 7. $\int x\sin 3x\,dx.$ 8. $\int xe^{-4x}\cos 3x\,dx.$

Finally, we give an illustration to show how the use of complex numbers can help in dealing with the geometry of a plane curve. We use for the curve the notation of the preceding chapter.

ILLUSTRATION 6. Let $P(x,y)$ be a point of a plane curve, and let s,ψ denote as usual the length of the arc measured from a fixed point and the angle between the tangent at P and the x-axis. (It is assumed that x,y are real.)

Write
$$z = x+iy.$$

Then
$$\frac{dz}{ds} = \frac{dx}{ds} + i\frac{dy}{ds}$$
$$= \cos\psi + i\sin\psi \quad \text{(p. 108)}$$
$$= e^{i\psi} \quad \text{(p. 180).}$$

In particular,
$$\left|\frac{dz}{ds}\right| = 1.$$

We also have the relation
$$\frac{d^2z}{ds^2} = ie^{i\psi}\frac{d\psi}{ds}$$
$$= i\kappa e^{i\psi} \quad \text{(p. 113).}$$

This is equivalent to
$$\frac{d^2x}{ds^2} + i\frac{d^2y}{ds^2} = i\kappa e^{i\psi},$$

so that, taking moduli on each side,
$$|\kappa| = \sqrt{\left\{\left(\frac{d^2x}{ds^2}\right)^2 + \left(\frac{d^2y}{ds^2}\right)^2\right\}}.$$

Again,
$$\frac{dx}{ds} - i\frac{dy}{ds} = \cos\psi - i\sin\psi$$
$$= e^{-i\psi},$$

so that
$$\left(\frac{dx}{ds} - i\frac{dy}{ds}\right)\left(\frac{d^2x}{ds^2} + i\frac{d^2y}{ds^2}\right) = i\kappa.$$

Equating imaginary parts, we obtain the formula
$$\kappa = \frac{dx}{ds}\frac{d^2y}{ds^2} - \frac{dy}{ds}\frac{d^2x}{ds^2}.$$

REVISION EXAMPLES VII

'Scholarship' Level

1. Two complex numbers z, z' are connected by the relation $z' = (2+z)/(2-z)$. Show that, as the point which represents z on the Argand diagram describes the axis of y, from the negative end to the positive end, the point which represents z' describes completely the circle $x^2+y^2 = 1$ in the counter-clockwise sense.

2. Explain what is meant by the principal value of the logarithm of a complex number $x+iy$ as distinct from the general value.

Show that, considering only principal values, the real part of

$$(1+i)^{\log_e(1+i)}$$

is

$$2^{\frac{1}{4}\log_e 2}e^{-\frac{1}{16}\pi^2}\cos\left(\tfrac{1}{4}\pi\log_e 2\right).$$

3. Express $\tan(a+ib)$ in the form $A+iB$, where A, B are real when a, b are real.

Show that, if $x+iy = \tan\frac{1}{2}(\xi+i\eta)$, then

$$\tfrac{1}{2}x = e^{-\eta}\sin\xi - e^{-2\eta}\sin 2\xi + e^{-3\eta}\sin 3\xi - \ldots,$$

when η is positive; and that there is a like expansion with the sign of η changed, valid when η is negative.

4. Prove that, if

$$\sin(x+iy) = \operatorname{cosec}(u+iv),$$

where x, y, u, v are real, then

$$\cosh^2 v\tanh^2 y = \cos^2 u, \quad \cosh^2 y\tanh^2 v = \cos^2 x.$$

5. Solve the equation

$$(1-xi)^n + i(1+xi)^n = 0,$$

giving the roots as trigonometrical functions of angles between 0 and π.

6. Prove that, if $x+iy = c\cot(u+iv)$, then

$$\frac{x}{\sin 2u} = \frac{-y}{\sinh 2v} = \frac{c}{\cosh 2v - \cos 2u},$$

and show that, if x, y are coordinates of a point in a plane, then for a given value of v the point lies on the circle

$$x^2+y^2+2cy\coth 2v + c^2 = 0.$$

Also verify that, if a_1, a_2 denote the radii of the circles for the values v_1, v_2 of v, and d denotes the distance between their centres, then

$$\frac{a_1^2+a_2^2-d^2}{2a_1a_2} = \cosh 2(v_1-v_2).$$

7. Obtain an expression for $|\cos(x+iy)|^2$ in terms of trigonometric and hyperbolic functions of the real variables x, y.

Show that $|\cos(x+ix)|$ increases with x for all positive values of x.

8. Express

$$\frac{z_1+z_2}{z_1z_2-1}$$

in the form $X+iY$, where $z_1 = x_1+iy_1$, $z_2 = x_2+iy_2$.

Deduce that the points which represent three complex numbers z_1, z_2, z_3 in the Argand diagram cannot all lie on the same side of the real axis if

$$z_1+z_2+z_3 = z_1z_2z_3.$$

9. Prove that the roots of the equation

$$(x-1)^n = -(x+1)^n,$$

where n is a positive integer, are

$$i\cot\frac{(2k+1)\pi}{2n} \quad (k = 0, 1, 2, \ldots, n-1).$$

Deduce, or prove otherwise, that

$$\sum_{k=0}^{n-1}\cot\frac{(2k+1)\pi}{2n} = 0, \quad \sum_{k=0}^{n-1}\operatorname{cosec}^2\frac{(2k+1)\pi}{2n} = n^2.$$

10. Write down the complex numbers conjugate to $x+iy$ and to $r(\cos\theta+i\sin\theta)$.

Prove that, if a quadratic equation with real coefficients has one complex root, the other root is the conjugate complex number. Deduce a similar result for a cubic equation with real coefficients.

Given that $x = 1+3i$ is one solution of the equation

$$x^4+16x^2+100 = 0,$$

find all the solutions.

11. Express each of the complex numbers

$$z_1 = (1+i)\sqrt{2}, \quad z_2 = 4(-1+i)\sqrt{2}$$

in the form $r(\cos\theta + i\sin\theta)$, where r is positive.

Prove that $z_1^3 = z_2$, and find the other cube roots of z_2 in the form $r(\cos\theta + i\sin\theta)$.

12. The complex number $z = x+iy = r(\cos\theta+i\sin\theta)$ is represented in the Argand diagram by the point (x,y). Prove that, if three variable points z_1, z_2, z_3 are such that $z_3 = \lambda z_2 + (1-\lambda)z_1$, where λ is a complex constant, then the triangle whose vertices are z_1, z_2, z_3 is similar to the triangle with vertices at the points $0, 1, \lambda$.

ABC is a triangle. On the sides BC, CA, AB triangles BCA', CAB', ABC' are described similar to a given triangle DEF. Prove that the median points of the triangles ABC, $A'B'C'$ are coincident.

13. The complex numbers $a, b, (1-k)a+kb$ are represented in the Argand diagram by the points A, B, C. Prove that the vector $\overrightarrow{AB}$ represents $b-a$, and that

(i) when k is real, C lies on AB and divides AB in the ratio $k:1-k$;

(ii) when k is not real, ABC is a triangle in which the sides AB, AC are in the ratio $1:|k|$ and the angle of turn from $\overrightarrow{AB}$ to $\overrightarrow{AC}$ is θ, where $k = |k|(\cos\theta+i\sin\theta)$.

The vertices of two triangles ABC and XYZ represent the complex numbers a,b,c and x,y,z. Prove that a necessary and sufficient condition for the triangles to be similar and similarly situated is

$$\begin{vmatrix} 1 & 1 & 1 \\ a & b & c \\ x & y & z \end{vmatrix} = 0.$$

14. [In this problem small letters stand for complex numbers and capital letters for the corresponding points in the Argand diagram.]

The circumcentre, centroid, and orthocentre of a triangle ABC are S, G, H respectively. Prove that

$$g = \tfrac{1}{3}(a+b+c), \quad 2s+h = a+b+c.$$

ABC is a triangle with its circumcentre at the origin. The internal bisectors of the angles A, B, C of the triangle meet the circumcircle again at L, M, N respectively. Prove (by pure geometry if you wish) that the incentre P of ABC is the orthocentre of LMN, and deduce that

$$p = l+m+n.$$

Prove also that the excentres P_1, P_2, P_3 of the triangle ABC are given by

$$p_1 = l-m-n, \quad p_2 = -l+m-n, \quad p_3 = -l-m+n,$$

and that the circumcentre K of $P_1P_2P_3$ is given by

$$k = -(l+m+n).$$

Prove, finally, that KP_1 is perpendicular to BC.

15. Points A, B in the Argand diagram represent complex numbers a, b respectively; O is the origin, and P represents one of the values of $\sqrt{(ab)}$. Prove that, if $OA = OB = r$, then also $OP = r$, and OP is perpendicular to AB.

A, B, C lie on a circle with centre O, and represent complex numbers a, b, c respectively. Prove that the point D which represents $-bc/a$ also lies on the circle, and that AD is perpendicular to BC.

The perpendiculars from B, C to CA, AB meet the circle again at E, F respectively. Prove that OA is perpendicular to EF.

16. Prove by means of an Argand diagram, that

$$|z_1+z_2| \leqslant |z_1|+|z_2|.$$

Find all the points in the complex plane which represent numbers satisfying the equations

(i) $z^2-2z=3$, (ii) $|z-2|=3$, (iii) $|z-1|+|z-2|=3$.

17. Find the fifth roots of unity, and hence solve the equation

$$(2x-1)^5 = 32x^5.$$

18. Prove that, if α, β are the roots of the equation

$$t^2-2t+2=0,$$

then

$$\frac{(x+\alpha)^n-(x+\beta)^n}{\alpha-\beta} = \frac{\sin n\theta}{\sin^n\theta},$$

where

$$\cot\theta = x+1.$$

19. A point $z = x+iy$ in the Argand diagram is such that $|z| = 2, x = 1$, and $y > 0$. Determine the point and find the distance between the point and $\frac{1}{2}z^2$.

Show also that the points $2, z, \frac{1}{2}z^2, \frac{1}{4}z^3, \frac{1}{8}z^4, \frac{1}{16}z^5$ are the vertices of a regular hexagon.

20. The points z_1, z_2, z_3 form an equilateral triangle in the Argand diagram, and $z_1 = 4+6i, z_2 = (1-i)z_1$. Show that z_3 must have one of two values, and determine these values.

What are the vertices of the regular hexagon of which z_1 is the centre, and z_2 is one vertex?

21. A regular pentagon $ABCDE$ is inscribed in the circle $x^2+y^2 = 1$, the vertex A being the point $(1, 0)$. Obtain the complex numbers $x+iy$ of which the points A, B, C, D, E form a representation in an Argand diagram.

22. Use De Moivre's theorem to find all the roots of the equation

$$(2x-1)^5 = (x-2)^5$$

in the form $a+ib$, where a, b are real numbers.

23. Mark on an Argand diagram the points $\surd 3+i, 2+2i\surd 3$, and their product. What is the general relation between the positions of the points $a+ib$ and $(\surd 3+i)(a+ib)$?

A triangle ABC has its vertices at the points $0, 2+2i\surd 3$, $-1+i\surd 3$ respectively. A similar triangle $A'B'C'$ has its vertices A', B' at the points $0, 8i$ respectively. Find the position of C'.

24. Find all values of $(5-12i)^{\frac{1}{2}}, (2i-2)^{\frac{1}{3}}$, expressing the answers in the form $a+ib$, where a, b are fractions or surds.

25. Prove that the triangle formed by the three points representing the complex numbers f, g, h is similar to the triangle formed by the points representing the numbers u, v, w if

$$fv+gw+hu = fw+gu+hv.$$

26. By expressing $3+4i$, $1+2i$ in the form $r(\cos\theta+i\sin\theta)$, or otherwise, evaluate

$$(3+4i)^{30}/(1+2i)^{50}$$

in the form $a+ib$, obtaining each of the real numbers a, b correct to two significant figures.

27. Identify all the points in the z plane which satisfy the following relations:

(i) $z^2+2 = 2z$, (ii) $\left|\dfrac{z-2}{z+2}\right| \leqslant 1$,

(iii) $\left|\dfrac{z-1}{z+1}\right| \geqslant 2$, (iv) $|z-1|+|z+1| \leqslant 4$.

28. Prove that three points z_1, z_2, z_3 in the complex plane are collinear if, and only if, the ratio

$$(z_3-z_1)/(z_2-z_1)$$

is real.

29. Find the real and imaginary parts and the modulus for each of the expressions

(i) $\dfrac{1}{1+2i}$, (ii) $\dfrac{a+ib}{b+ia}$, (iii) $\dfrac{1+\cos\alpha+i\sin\alpha}{1+\cos\beta+i\sin\beta}$.

30. Find the logarithms of $-1, 2-i, 10^{x+iy}$.

31. The three points in an Argand diagram which correspond to the roots of the equation

$$z^3-3pz^2+3qz-r = 0$$

are the vertices of a triangle ABC. Prove that the centroid of the triangle is the point corresponding to p.

If the triangle ABC is equilateral, prove that $p^2 = q$.

32. Find the cube roots of $4-3i$.

33. If the vertices A, B, C of an equilateral triangle represent the numbers z_1, z_2, z_3 respectively in an Argand diagram, state the number represented by the mid-point M of AB, and show that

$$z_3 = \tfrac{1}{2}(z_1+z_2) \pm \frac{\sqrt{3}}{2} i(z_2-z_1).$$

If $z_1 = 2+2i, \quad z_2 = 4+(2+2\surd 3)i$,

and if C is on the same side of AB as the origin, find the number z_3.

34. If $z = \cos\theta+i\sin\theta$, express $\cos\theta, \sin\theta, \cos n\theta, \sin n\theta$ in terms of z, where n is an integer.

Hence express $\sin^7\theta$ in the form

$$A\sin\theta+B\sin 3\theta+C\sin 5\theta+D\sin 7\theta.$$

35. Prove that

$$\left(\frac{1+\sin\theta-i\cos\theta}{1+\sin\theta+i\cos\theta}\right)^6=-\cos 6\theta-i\sin 6\theta.$$

36. Prove that

$$\left(\frac{1+\sin\theta+i\cos\theta}{1+\sin\theta-i\cos\theta}\right)^n=\cos(\tfrac{1}{2}n\pi-n\theta)+i\sin(\tfrac{1}{2}n\pi-n\theta).$$

37. Show that the equation

$$|z-1-2i|=3$$

represents a circle in an Argand diagram.

If z lies on $|z-1-2i|=3$, what is the locus of the point $u=z+4-3i$?

Find the greatest and least values of $|z-4-6i|$ if z is subject to the inequality $|z-1-2i|\leqslant 3$.

38. Find the modulus of

$$\frac{3z+i}{(3z-i)^2}$$

when $|z|=1$.

39. Two complex numbers z, z_1 are connected by the relation

$$z_1=\frac{2+z}{2-z}.$$

Prove that, if $z=iq$, where q is real, then the locus of z_1 in an Argand diagram is a circle. Describe the variation in the amplitude of z_1 as q increases from $-\infty$ to $+\infty$.

40. If a, b are real and n is an integer, prove that

$$\sqrt[n]{(a+ib)}+\sqrt[n]{(a-ib)}$$

has n real values, and find those of

$$\sqrt[3]{(1+i\sqrt{3})}+\sqrt[3]{(1-i\sqrt{3})}.$$

41. If x, y are real, separate

$$\sec(x+iy)$$

into its real and imaginary parts.

42. If a, b, c, d are real, express each of a^2+b^2 and c^2+d^2 as a product of linear factors.

Deduce that the product of two factors, each of which is a sum of two squares, is itself a sum of two squares.

43. The intrinsic coordinates of a point on a plane curve are s, ψ, and the cartesian coordinates are x, y. The complex coordinate $x+iy$ is denoted by z and the curvature by κ. Prove that

$$\frac{dz}{ds}=e^{i\psi},\quad \frac{d^2z}{ds^2}=i\kappa e^{i\psi}.$$

Hence, or otherwise, prove that

$$\kappa=(x''^2+y''^2)^{\frac{1}{2}}=(x'y''-x''y'),$$

where dashes denote differentiations with respect to s.

Prove further that

$$\begin{vmatrix} x' & y' & 1 \\ x'' & y'' & \kappa \\ x''' & y''' & \kappa^2 \end{vmatrix}=\kappa(2\kappa^2-\kappa').$$

44. Any point P is taken on a given curve. The tangent at P is drawn in the direction of increasing s, and a point Q is taken at a constant distance l along this tangent. In this way Q describes a curve specified by X, Y, S, Z analogous to the specification x, y, s, z for P. [Notation of previous problem.] Show that

(i) $\dfrac{dS}{ds}=(1+l^2\kappa^2)^{\frac{1}{2}}$;

(ii) the curvature K of the derived curve at Q is given by the formula

$$K(1+l^2\kappa^2)^{\frac{3}{2}}=\kappa(1+l^2\kappa^2)+l\kappa',$$

where $\kappa'=d\kappa/ds$.

CHAPTER XII

SYSTEMATIC INTEGRATION

At first sight the integration of functions seems to depend as much upon luck as upon skill. This is largely because the teacher or author must, in the early stages, select examples which are known to 'come out'. Nor is it easy to be sure, even with years of experience, that any particular integral is capable of evaluation; for example, $x \sin x$ can be integrated easily, whereas $\sin x/x$ cannot be integrated at all in finite terms by means of functions studied hitherto.

The purpose of this chapter is to explain how to set about the processes of integration in an orderly way. This naturally involves the recognition of a number of 'types', followed by a set of rules for each of them. But first we make two general remarks.

(i) The rules will ensure that an integral of given type MUST come out; but it is always wise to examine any particular example carefully to make sure that an easier method (such as substitution) cannot be used instead.

(ii) It is probably true to say that more integrals remain unsolved through faulty manipulation of algebra and trigonometry than through difficulties inherent in the integration itself. The reader is urged to acquire facility in the normal technique of these subjects. For details a text-book should be consulted.

1. Polynomials. The first type presents no difficulty. If $f(x)$ is the polynomial

$$f(x) \equiv a_0 x^n + a_1 x^{n-1} + \ldots + a_n,$$

then $$\int f(x)\,dx = \frac{a_0 x^{n+1}}{n+1} + \frac{a_1 x^n}{n} + \ldots + a_n x.$$

2. Rational functions. (Compare also p. 11.) A rational function $f(x)$ of the variable x is defined to be the ratio of two *polynomials*, so that

$$f(x) = \frac{P(x)}{Q(x)}$$

for polynomials $P(x), Q(x)$. If the degree of $P(x)$ is not less than that of $Q(x)$, we can divide out, getting a polynomial (easily integrated) together with a rational fraction in which the degree of the numerator is less than that of the denominator.

We therefore confine our attention to the case in which the degree of $P(x)$ is less than that of $Q(x)$. It is assumed, too, that all coefficients are real.

It is a theorem of algebra that any (real) polynomial, and, in particular, $Q(x)$, can be expressed as a product of factors, of which typical terms are

$$(\alpha x + \beta)^m,$$

$$(ax^2 + 2hx + b)^n,$$

where α, β, a, h, b are real constants, but where

$$h^2 < ab$$

so that the quadratic $ax^2 + 2hx + b$ cannot be further resolved into real factors.

[We ought to add that, for a given polynomial, the difficulty of factorization may be very great indeed.]

It is a further theorem of algebra that the rational function may then be expressed in the form

$$\Sigma\left\{\frac{A_1}{(\alpha x+\beta)^m} + \frac{A_2}{(\alpha x+\beta)^{m-1}} + \ldots + \frac{A_m}{\alpha x+\beta}\right\}$$
$$+ \Sigma\left\{\frac{B_1 x + C_1}{(ax^2+2hx+b)^n} + \frac{B_2 x + C_2}{(ax^2+2hx+b)^{n-1}} + \ldots + \frac{B_n x + C_n}{ax^2+2hx+b}\right\},$$

where the first summation extends over all linear factors $\alpha x + \beta$, and the second over all quadratic factors $ax^2 + 2hx + b$.

The integrals from the first summation are of the type

$$\int \frac{A\,dx}{(\alpha x+\beta)^k},$$

which give $$\begin{cases} \dfrac{A}{(1-k)\,\alpha(\alpha x+\beta)^{k-1}} & (k \neq 1) \\ \dfrac{A}{\alpha}\log|\alpha x+\beta| & (k=1). \end{cases}$$

The integrals from the second summation are of the type

$$\int \frac{Bx+C}{(ax^2+2hx+b)^p}dx,$$

and require more detailed consideration.

Note first that

$$\int \frac{ax+h}{(ax^2+2hx+b)^p}dx$$

$$= \begin{cases} \dfrac{1}{2(1-p)(ax^2+2hx+b)^{p-1}} & (p \neq 1) \\ \tfrac{1}{2}\log|ax^2+2hx+b| & (p=1), \end{cases}$$

and so, by writing

$$Bx+C \equiv (B/a)(ax+h)+(aC-hB)/a,$$

we can reduce our problem to the evaluation of integrals such as

$$\int \frac{dx}{(ax^2+2hx+b)^p}.$$

Write $\qquad a(ax^2+2hx+b) \equiv (ax+h)^2+ab-h^2,$

and make the substitution (remembering that $ab-h^2$ is positive by assumption)

$$ax+h = t\sqrt{(ab-h^2)}.$$

Then $$\int \frac{dx}{(ax^2+2hx+b)^p} = \int \frac{a^{p-1}dt\sqrt{(ab-h^2)}}{\{t^2(ab-h^2)+(ab-h^2)\}^p}$$

$$= \frac{a^{p-1}}{(ab-h^2)^{p-\frac{1}{2}}}\int \frac{dt}{(t^2+1)^p}.$$

Our final problem is therefore to evaluate

$$I_p \equiv \int \frac{dt}{(t^2+1)^p},$$

and for this we need a formula of reduction. On integration by parts, we have

$$I_p = \frac{t}{(t^2+1)^p} - \int \frac{t(-2pt)}{(t^2+1)^{p+1}}dt$$

$$= \frac{t}{(t^2+1)^p} + 2p\int \frac{(t^2+1)-1}{(t^2+1)^{p+1}}dt$$

$$= \frac{t}{(t^2+1)^p} + 2p(I_p - I_{p+1}).$$

Hence $$2pI_{p+1}-(2p-1)I_p = \frac{t}{(t^2+1)^p},$$

or, replacing p by $p-1$,

$$2(p-1)I_p-(2p-3)I_{p-1} = \frac{t}{(t^2+1)^{p-1}}.$$

By applying this formula successively, we make the evaluation of I_p depend on that of $I_{p-1}, I_{p-2}, \ldots$, and, ultimately, on I_1. But

$$I_1 = \int \frac{dt}{t^2+1}$$

$$= \tan^{-1} t,$$

and so the whole integration is effected.

3. Integrals involving $\sqrt{(ax+b)}$. Rational functions of x and $\sqrt{(ax+b)}$ may be integrated readily by means of the substitution

$$t = \sqrt{(ax+b)}$$

or $$x = \frac{1}{a}(t^2-b).$$

The result is the integration of a rational function of t.

Note. The reader is unlikely to remember all the details to be given in § 4 following. The methods should be thoroughly understood, but it may well be found necessary to refer to the book for details in actual examples. If desired, §§ 5, 6 and 7, which will be of more immediate practical value, may be read next.

4. Integrals involving $\sqrt{(ax^2+2hx+b)}$. We first take steps to simplify the quadratic expression under the square root sign.

(i) *Suppose that a is positive.* Then

$$a(ax^2+2hx+b) = (ax+h)^2+ab-h^2.$$

Write $$ax+h = x';$$

then $$a(ax^2+2hx+b) = \begin{cases} x'^2+p^2 & (ab>h^2) \\ x'^2-q^2 & (ab<h^2) \end{cases}$$

where $p^2 = ab-h^2$, $q^2 = h^2-ab$ respectively.

(ii) *Suppose that a is negative.* Then $-a$ is positive, so we write

$$-a(ax^2+2hx+b) = h^2-ab-(ax+h)^2.$$

If $h^2 - ab$ is negative, the right side is necessarily negative, so that $ax^2 + 2hx + b$ is also negative and (in real algebra) has no square root. We therefore take $h^2 - ab$ to be positive, and write

$$r^2 = h^2 - ab;$$

thus, if
$$ax + h = x',$$
as before, we have
$$-a(ax^2 + 2hx + b) = r^2 - x'^2.$$

If, then, we have to evaluate a rational function of x and $\sqrt{(ax^2 + 2hx + b)}$, we may first apply the transformation

$$ax + h = x'.$$

The integrand becomes a rational function of x' and of a surd which may assume one or other of the three forms

$$\text{(i)}\quad \sqrt{(x'^2 + p^2)} \quad a > 0,\ ab - h^2 > 0,$$

$$\text{(ii)}\quad \sqrt{(x'^2 - q^2)} \quad a > 0,\ ab - h^2 < 0,$$

$$\text{(iii)}\quad \sqrt{(r^2 - x'^2)} \quad a < 0,\ ab - h^2 < 0.$$

We may now drop the dashes and treat x as the variable.

(i) *The surd* $\sqrt{(x^2 + p^2)}$.

Consider the transformation

$$x = \frac{2pt}{1 - t^2}.$$

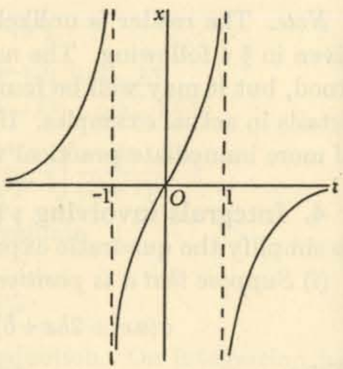


Fig. 98.

The graph (Fig. 98) indicates (what can also be proved algebraically) that all values of x are obtained by allowing t to vary continuously from -1 to $+1$. We therefore impose the restriction

$$-1 < t < 1$$

on the values of t which we select.

We have
$$dx = \frac{2p(1 + t^2)}{(1 - t^2)^2}.dt,$$

$$x^2 + p^2 = \left(\frac{1 + t^2}{1 - t^2}\right)^2 p^2.$$

Since $t^2 < 1$, we have, without ambiguity (assuming, as we may, that p is positive),
$$\sqrt{(x^2 + p^2)} = +\frac{1 + t^2}{1 - t^2}p.$$

Hence a rational function of $x, \sqrt{(x^2 + p^2)}$ is transformed into a rational function of t, and may be integrated accordingly.

(ii) *The surd* $\sqrt{(x^2 - q^2)}$.

Consider the transformation

$$x = \frac{t^2 + 1}{t^2 - 1}q.$$

The graph (Fig. 99) indicates (what can also be proved algebraically) that all values of x for which $x > q$ are obtained by allowing t to vary continuously from $+1$ to $+\infty$.

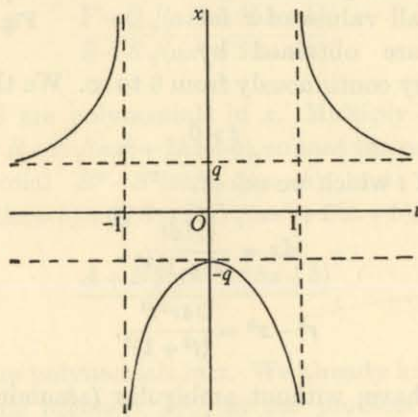


Fig. 99.

[We can restrict ourselves to positive values of x, since a range of integration which involved the two signs for x would have to pass through the region $-q < x < q$ where $\sqrt{(x^2 - q^2)}$ is undefined. We could, of course, equally restrict ourselves to negative values of x if necessary.]

We may therefore impose the restriction

$$t > 1.$$

Now
$$dx = \frac{-4qt\,dt}{(t^2 - 1)^2},$$

$$x^2 - q^2 = \frac{4t^2q^2}{(t^2 - 1)^2}.$$

Since $t > 1$, we have, without ambiguity (assuming, as we may, that q is positive),

$$\sqrt{(x^2-q^2)} = +\frac{2tq}{t^2-1}.$$

Hence a rational function of x, $\sqrt{(x^2-q^2)}$ is transformed into a rational function of t, and may be integrated accordingly.

(iii) *The surd* $\sqrt{(r^2-x^2)}$.

Consider the transformation

$$x = \frac{t^2-1}{t^2+1}r.$$

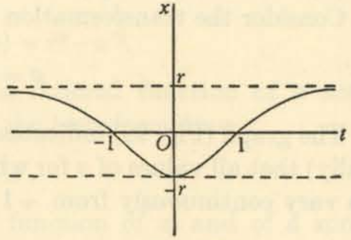


Fig. 100.

The graph (Fig. 100) indicates (what can also be proved algebraically) that all values of x for which $x^2 < r^2$ are obtained by allowing t to vary continuously from 0 to ∞. We therefore impose the restriction

$$t > 0$$

on the values of t which we select.

We have
$$dx = \frac{4rt\,dt}{(t^2+1)^2},$$
$$r^2-x^2 = \frac{4r^2t^2}{(t^2+1)^2}.$$

Since $t > 0$, we have, without ambiguity (assuming, as we may, that r is positive),

$$\sqrt{(r^2-x^2)} = +\frac{2rt}{t^2+1}.$$

Hence a rational function of x, $\sqrt{(r^2-x^2)}$ is transformed into a rational function of t, and may be integrated accordingly.

Note. The work just completed proves that the integrations are POSSIBLE; it does not necessarily give the easiest method. For example, the quadratic surds may also be subjected to the following substitutions:

(i) For $\sqrt{(x^2+p^2)}$, let $x = p\tan\theta$, or $x = p\sinh\theta$;

(ii) For $\sqrt{(x^2-q^2)}$, let $x = q\sec\theta$, or $x = q\cosh\theta$;

(iii) For $\sqrt{(r^2-x^2)}$, let $x = r\sin\theta$.

5. Integrals involving $\sqrt{(ax^2+2hx+b)}$: alternative treatment.

Integrals of rational functions of x, $\sqrt{(ax^2+2hx+b)}$ may also be treated by methods which, by leaving transformation until the last stages, retain the identity of the surd. We begin by reducing such a rational function to a more amenable form.

Since even powers of $\sqrt{(ax^2+2hx+b)}$ are polynomials and odd powers are polynomials multiplying the square root, we may express any polynomial in x, $\sqrt{(ax^2+2hx+b)}$ in the form

$$P+Q\sqrt{(ax^2+2hx+b)},$$

where P, Q are polynomials in x. Hence any rational function, being by definition the quotient of two polynomials, is

$$\frac{P+Q\sqrt{(ax^2+2hx+b)}}{R+S\sqrt{(ax^2+2hx+b)}},$$

where P, Q, R, S are polynomials in x. Multiply numerator and denominator by $R-S\sqrt{(ax^2+2hx+b)}$, so that the new denominator is the polynomial $R^2-S^2(ax^2+2hx+b)$ and the numerator $PR-QS(ax^2+2hx+b)+(QR-PS)\sqrt{(ax^2+2hx+b)}$, and we obtain the form

$$\frac{A+B\sqrt{(ax^2+2hx+b)}}{C},$$

where A, B, C are polynomials in x. We already know how to deal with the rational function A/C, so our problem reduces to the integration of

$$\frac{B\sqrt{(ax^2+2hx+b)}}{C}.$$

It is found (surprisingly, perhaps) more convenient to have the surd on the denominator, so we multiply numerator and denominator by $\sqrt{(ax^2+2hx+b)}$, obtaining

$$\frac{U}{V\sqrt{(ax^2+2hx+b)}},$$

or

$$\frac{F}{\sqrt{(ax^2+2hx+b)}},$$

where F is a rational function of x.

By § 2 above, the problems to which F gives rise involve as typical terms

$$x^m \quad (m \geqslant 0),$$

$$\frac{1}{(x-k)^n} \quad (n \geqslant 1),$$

$$\frac{Ax+B}{(px^2+2qx+r)^n} \quad (n \geqslant 1,\ q^2 < pr).$$

We have therefore to consider the three types of integral:

$$\text{(i)} \int \frac{x^m\,dx}{\sqrt{(ax^2+2hx+b)}} \quad (m \geqslant 0),$$

$$\text{(ii)} \int \frac{dx}{(x-k)^n\sqrt{(ax^2+2hx+b)}} \quad (n \geqslant 1),$$

$$\text{(iii)} \int \frac{(Ax+B)\,dx}{(px^2+2qx+r)^n\sqrt{(ax^2+2hx+b)}} \quad (n \geqslant 1,\ q^2 < pr).$$

(i) *The evaluation of the integral*

$$I_m \equiv \int \frac{x^m\,dx}{\sqrt{(ax^2+2hx+b)}} \quad (m \geqslant 0).$$

Preparing the ground for an integration by parts, we observe the identity

$$aI_{m+2} + 2hI_{m+1} + bI_m = \int \frac{(ax^2+2hx+b)\,x^m\,dx}{\sqrt{(ax^2+2hx+b)}}$$

$$= \int x^m \sqrt{(ax^2+2hx+b)}\,dx.$$

Now performing the integration, we have, on the right-hand side,

$$\frac{x^{m+1}}{m+1}\sqrt{(ax^2+2hx+b)} - \frac{1}{m+1}\int \frac{x^{m+1}(ax+h)}{\sqrt{(ax^2+2hx+b)}}\,dx$$

$$= \frac{x^{m+1}}{m+1}\sqrt{(ax^2+2hx+b)} - \frac{1}{m+1}(aI_{m+2}+hI_{m+1}).$$

Equating the two sides, we have the *recurrence relation*

$$(m+2)\,aI_{m+2} + (2m+3)\,hI_{m+1} + (m+1)\,bI_m = x^{m+1}\sqrt{(ax^2+2hx+b)}.$$

This enables us, once I_0, I_1 have been determined, to calculate $I_2, I_3, I_4, \ldots$ successively, and so to evaluate I_m for any positive integral value of m.

For I_0, we must reduce the surd to one or other of the forms enumerated earlier in this section, giving

$$\int \frac{dx}{\sqrt{(x^2+p^2)}} = \log\{x+\sqrt{(x^2+p^2)}\}$$

or

$$\int \frac{dx}{\sqrt{(x^2-q^2)}} = \log\{x+\sqrt{(x^2-q^2)}\}$$

or

$$\int \frac{dx}{\sqrt{(r^2-x^2)}} = \sin^{-1}(x/r).$$

For I_1, we have to consider the integral

$$I_1 \equiv \int \frac{x\,dx}{\sqrt{(ax^2+2hx+b)}}.$$

Now

$$aI_1 + hI_0 = \int \frac{(ax+h)\,dx}{\sqrt{(ax^2+2hx+b)}} = \sqrt{(ax^2+2hx+b)},$$

so that

$$I_1 = \frac{1}{a}\{\sqrt{(ax^2+2hx+b)} - hI_0\}.$$

(ii) *The evaluation of the integral*

$$\int \frac{dx}{(x-k)^n\sqrt{(ax^2+2hx+b)}}.$$

We can reduce this type to the form of type (i) and evaluate it at once by the substitution.

$$x - k = \frac{1}{t},$$

$$dx = -\frac{1}{t^2}\,dt.$$

Then

$$ax^2+2hx+b = a\left(k+\frac{1}{t}\right)^2 + 2h\left(k+\frac{1}{t}\right) + b$$

$$= \frac{1}{t^2}(\alpha t^2 + 2\beta t + \gamma),$$

where

$$\alpha = ak^2+2hk+b, \quad \beta = ak+h, \quad \gamma = a.$$

The integral is therefore

$$\int \frac{-\dfrac{dt}{t^2}}{\dfrac{1}{t^n}\cdot\dfrac{1}{t}\sqrt{(\alpha t^2+2\beta t+\gamma)}}$$

$$= -\int \frac{t^{n-1}\,dt}{\sqrt{(\alpha t^2+2\beta t+\gamma)}} \quad (n-1 \geqslant 0),$$

which is of the first form

$$\int \frac{x^m\,dx}{\sqrt{(ax^2+2hx+b)}} \quad (m \geqslant 0).$$

(iii) *The evaluation of the integral*

$$I_n \equiv \int \frac{(Ax+B)\,dx}{(px^2+2qx+r)^n\sqrt{(ax^2+2hx+b)}},$$

where $n \geqslant 1$, $q^2 < pr$.

The general case is very difficult. It is, of course, possible to avoid it by expressing the rational function as a sum of *complex* partial fractions of the type $1/(x-k)^n$, where k is complex. This reduces the integration to type (ii)

$$\int \frac{dx}{(x-k)^n\sqrt{(ax^2+2hx+b)}} \quad (k \text{ complex})$$

already discussed. The final return to real form is an added point of difficulty.

We begin with an algebraic lemma, designed to reduce two quadratic expressions *simultaneously* to simpler form.

LEMMA. *To establish the existence of a (real) transformation of the type*

$$x = \frac{\alpha t+\beta}{t+1}$$

which reduces the quadratic expressions

$$ax^2+2hx+b,$$

$$px^2+2qx+r$$

to the forms

$$\frac{ut^2+v}{(t+1)^2}$$

$$\frac{u't^2+v'}{(t+1)^2}$$

simultaneously.

Since $(t+1)^2(ax^2+2hx+b)$

$$= a(\alpha t+\beta)^2+2h(\alpha t+\beta)(1+t)+b(1+t)^2$$

under the transformation, we see that the coefficient of t vanishes on the right-hand side if α, β are chosen so that

$$a\alpha\beta+h(\alpha+\beta)+b = 0.$$

Similarly the coefficient of t from $px^2+2qx+r$ vanishes if

$$p\alpha\beta+q(\alpha+\beta)+r = 0.$$

Also α, β necessarily satisfy the equation in θ

$$\alpha\beta-\theta(\alpha+\beta)+\theta^2 = 0.$$

On eliminating the ratios $\alpha\beta:\alpha+\beta:1$ between these three relations, we obtain the equation

$$\begin{vmatrix} a & h & b \\ p & q & r \\ 1 & -\theta & \theta^2 \end{vmatrix} = 0,$$

which is a quadratic in θ with (by definition) roots α, β. Hence α, β may be found and the transformation determined. (If, exceptionally, $a/h = p/q$, the two quadratics differ only by a constant. The substitution $ax+h = at$ reduces the integral I_n to one or other of the types discussed on p. 213.)

Moreover α, β are *real*. If not, they must be conjugate complex numbers, since all coefficients are real. Suppose that

$$\alpha = \lambda+i\mu, \quad \beta = \lambda-i\mu \quad (\lambda, \mu \text{ real}).$$

Since

$$p\alpha\beta+q(\alpha+\beta)+r = 0,$$

we have the relation

$$p(\lambda^2+\mu^2)+2q\lambda+r = 0.$$

Multiply by p (which is not zero, by the nature of the problem). Then

$$p^2\lambda^2+2pq\lambda+pr+p^2\mu^2 = 0,$$

or

$$(p\lambda+q)^2+(pr-q^2)+(p\mu)^2 = 0.$$

Since p, q, λ, μ are all real, the two squares are positive; and, by hypothesis, $pr-q^2 > 0$. Hence the left-hand side is positive. We are therefore led to a contradiction, and so the supposition that α, β are not real is untenable.

Moreover, the condition $pr-q^2 > 0$ shows that α, β are DIFFERENT numbers.

SUMMARY. We can find two distinct real numbers, α, β, the roots of the quadratic equation

$$\begin{vmatrix} a & h & b \\ p & q & r \\ 1 & -\theta & \theta^2 \end{vmatrix} = 0,$$

which enable us, by means of the transformation

$$x = \frac{\alpha t+\beta}{t+1},$$

to reduce the two quadratic forms

$$ax^2+2hx+b = 0,$$

$$px^2+2qx+r = 0 \quad (q^2 < pr)$$

to the forms
$$\frac{ut^2+v}{(t+1)^2},$$

$$\frac{u't^2+v'}{(t+1)^2}$$

simultaneously.

Let us now return to the integral. On substituting for x in the expression $Ax+B$, we obtain an expression of the form

$$\frac{Ct+D}{t+1},$$

where $C = A\alpha+B$, $D = A\beta+B$; also

$$dx = \frac{\alpha-\beta}{(t+1)^2}dt.$$

Hence
$$I_n = \int \frac{\dfrac{Ct+D}{t+1}\cdot\dfrac{\alpha-\beta}{(t+1)^2}dt}{\dfrac{(u't^2+v')^n}{(t+1)^{2n}}\cdot\dfrac{\sqrt{(ut^2+v)}}{t+1}}$$

$$= \int\frac{(\alpha-\beta)(Ct+D)(t+1)^{2n-2}dt}{(u't^2+v')^n\sqrt{(ut^2+v)}}.$$

The numerator, which is a polynomial in t, can be expressed in the form $P+Qt$, where P, Q are polynomials in t^2; and so, writing $u't^2+v' = s$, or

$$t^2 = \frac{s-v'}{u'},$$

we obtain P and Q as polynomials in s. Hence the numerator can be expressed as a polynomial in $s \equiv (u't^2+v')$, together with t times another polynomial in $(u't^2+v')$, the order of the whole numerator being $2n-1$ in t, which is less than that of the denominator. Hence I_n is found as a sum of integrals of the form

$$U_k \equiv \int\frac{dt}{(u't^2+v')^k\sqrt{(ut^2+v)}}, \quad V_k \equiv \int\frac{t\,dt}{(u't^2+v')^k\sqrt{(ut^2+v)}}.$$

For the latter, put
$$y^2 = ut^2+v,$$

so that
$$y\,dy = ut\,dt;$$

hence
$$V_k = \frac{1}{u}\int\frac{y\,dy}{\left\{\dfrac{u'}{u}(y^2-v)+v'\right\}^k y}$$

$$= u^{k-1}\int\frac{dy}{\{u'y^2+(uv'-u'v)\}^k},$$

which reduces the problem to the integration of a rational function.

Finally, consider

$$U_k \equiv \int\frac{dt}{(u't^2+v')^k\sqrt{(ut^2+v)}}.$$

Write
$$u't^2+v' = \frac{1}{z},$$

$$2u't\,dt = -\frac{1}{z^2}dz,$$

so that
$$t = \sqrt{\left(\frac{1-v'z}{u'z}\right)},$$

$$ut^2+v = \frac{u-(uv'-u'v)z}{u'z}.$$

Then
$$U_k \equiv \int\left\{-\frac{dz}{2u'z^2}\sqrt{\left(\frac{u'z}{1-v'z}\right)}\right\}.z^k.\sqrt{\left\{\frac{u'z}{u-(uv'-u'v)z}\right\}}$$

$$= -\frac{1}{2}\int\frac{z^{k-1}dz}{\sqrt{[(1-v'z)\{u-(uv'-u'v)z\}]}}.$$

But the integral is of the form

$$\int\frac{z^{k-1}dz}{\sqrt{(a+2hz+bz^2)}},$$

whose evaluation we considered in the preceding section.

The whole integration may therefore be effected.

6. Trigonometric functions. Since

$$\cos x = \frac{1-\tan^2 \frac{1}{2}x}{1+\tan^2 \frac{1}{2}x}, \quad \sin x = \frac{2\tan \frac{1}{2}x}{1+\tan^2 \frac{1}{2}x},$$

trigonometric integrals may be reduced by means of the substitution

$$t = \tan \tfrac{1}{2}x.$$

Then
$$\cos x = \frac{1-t^2}{1+t^2}, \quad \sin x = \frac{2t}{1+t^2},$$

$$dx = \frac{2dt}{1+t^2}.$$

Hence a rational function of $\sin x$ and $\cos x$ is transformed into a rational function of t, and may be integrated by the methods of § 2.

For reduction formulæ for

$$\int \sin^m x \cos^n x\, dx$$

and similar integrals, see Vol. I, pp. 106–10.

The substitution $t = \tan \frac{1}{2}x$ should not be applied blindly. For example, if the integrand has period π, the alternative

$$t = \tan x$$

may be better.

Thus, consider the integral

$$I \equiv \int \frac{dx}{(\sec x + \cos x)^2}.$$

Since
$$\frac{1}{\{\sec(x+\pi)+\cos(x+\pi)\}^2} = \frac{1}{(\sec x+\cos x)^2},$$

we may use the substitution

$$t = \tan x.$$

Now
$$I = \int \frac{\sec^2 x\, dx}{(\sec^2 x + 1)^2}$$
$$= \int \frac{dt}{(t^2+2)^2}.$$

This is the integral of a rational function, and may be evaluated according to the usual rules. Alternatively, the substitution $t = \sqrt{2}\tan\theta$ makes the integral trigonometric again, but leads to an easy solution.

7. Exponential times polynomial in x. A polynomial is a sum of multiples of powers of x, and so, in order to evaluate the integral

$$\int e^{px} f(x)\, dx,$$

where $f(x)$ is a polynomial, we need only consider integrals of the type

$$u_n \equiv \int e^{px} x^n\, dx \quad (n \geqslant 0).$$

Integrating by parts, we have the relation

$$u_n = \frac{1}{p} e^{px} x^n - \frac{n}{p}\int e^{px} x^{n-1}\, dx,$$

leading to the recurrence formula

$$u_n = \frac{1}{p} e^{px} x^n - \frac{n}{p} u_{n-1} \quad (n \geqslant 1)$$

which enables us to express u_n in terms of

$$u_0 \equiv \int e^{px}\, dx = \frac{1}{p} e^{px}.$$

8. Exponential times polynomial in sines and cosines of multiples of x. Consider the integration of a sum of terms of the type

$$e^{px} \sin a_1 x \sin a_2 x \ldots \sin a_m x \cos b_1 x \cos b_2 x \ldots \cos b_n x,$$

involving $m+n$ factors in sines and cosines. The use of the formulæ such as

$$2\sin ux \cos vx = \sin(u+v)x + \sin(u-v)x,$$
$$2\sin ux \sin vx = \cos(u-v)x - \cos(u+v)x,$$

and so on, enables us to reduce the number of terms in a typical product to $m+n-1, m+n-2, m+n-3, \ldots$ successively, while retaining the same type of expression. Ultimately we reach one or other of the forms

$$C \equiv \int e^{px} \cos qx\, dx,$$

$$S \equiv \int e^{px} \sin qx\, dx,$$

whose solution should now be familiar. We find

$$C = \frac{1}{p^2+q^2} e^{px}(p\cos qx + q\sin qx),$$

$$S = \frac{1}{p^2+q^2} e^{px}(p\sin qx - q\cos qx).$$

WARNING. The work of this chapter will enable the reader to evaluate (ultimately) any integral that comes within its scope. But there are many functions which cannot be integrated in terms yet known to him. Some of these are very innocent to look at; for example,

$$\int \frac{\sin x}{x}\,dx, \quad \int e^{x^2}\,dx,$$

$$\int \frac{dx}{\sqrt{\{(1-x^2)(1-k^2x^2)\}}}.$$

Thorough familiarity with the FORMS of the integrals that can be evaluated should help in the avoidance of these pitfalls.

[For examples on the work of this chapter, see Revision Examples VIII and IX, pp. 226, 227.]

CHAPTER XIII

INTEGRALS INVOLVING 'INFINITY'

1. 'Infinite' limits of integration. It is often necessary to evaluate an integral such as

$$\int_a^b f(x)\,dx$$

under conditions where b tends to infinity; we then speak about 'the integral from a to infinity'

$$\int_a^\infty f(x)\,dx.$$

Similarly we meet the integral

$$\int_{-\infty}^b f(x)\,dx,$$

or, combining both possibilities,

$$\int_{-\infty}^\infty f(x)\,dx.$$

Our problem is to discuss what is meant by such integrals, and to show how to evaluate them. The general theory is difficult, so we confine ourselves to the simplest cases. We also assume that the function $f(x)$ is continuous throughout the range of integration.

We define the *integral to infinity*

$$\int_a^\infty f(x)\,dx$$

by means of the relation

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty}\int_a^N f(x)\,dx.$$

In the cases with which we shall be concerned, the indefinite integral $F(x)$ of $f(x)$ is considered to be known, so that

$$\int_a^N f(x)\,dx = F(N) - F(a).$$

We can therefore evaluate the integral to infinity in the form

$$\int_a^\infty f(x)\,dx = \lim_{N\to\infty} F(N) - F(a).$$

It may be added that it is often possible to evaluate the definite integral $\int_a^\infty f(x)\,dx$ even when the indefinite integral $\int f(x)\,dx$ cannot be found. A well-known example is $\int_0^\infty \frac{\sin x}{x}\,dx$ whose value is $\frac{1}{2}\pi$.

ILLUSTRATION 1. *To evaluate*

$$\int_1^\infty x^n\,dx.$$

Consider the integral

$$\int_1^N x^n\,dx \quad (n \neq -1)$$

which, by elementary theory, is

$$\left[\frac{1}{n+1}x^{n+1}\right]_1^N$$

or

$$\frac{N^{n+1}}{n+1} - \frac{1}{n+1}.$$

If $n+1$ is POSITIVE, then N^{n+1} increases indefinitely with N, so that

$$\lim_{N\to\infty}\left\{\frac{N^{n+1}}{n+1} - \frac{1}{n+1}\right\}$$

does not exist.

If $n+1$ is NEGATIVE, then N^{n+1} tends to zero as N increases indefinitely, so that

$$\lim_{N\to\infty}\left\{\frac{N^{n+1}}{n+1} - \frac{1}{n+1}\right\} = -\frac{1}{n+1}.$$

Hence $$\int_1^\infty x^n\,dx = -\frac{1}{n+1} \quad (n < -1).$$

ILLUSTRATION 2. *To evaluate*

$$\int_0^\infty xe^{-x}\,dx.$$

We know that, on integrating by parts,

$$\int xe^{-x}\,dx = -xe^{-x} + \int 1\,.\,e^{-x}\,dx$$
$$= -xe^{-x} - e^{-x},$$

so that

$$\int_0^N xe^{-x}\,dx = \Big[-(x+1)e^{-x}\Big]_0^N$$
$$= -(N+1)e^{-N} + 1.$$

Now $$(N+1)e^{-N} = \frac{N+1}{e^N} = \frac{N+1}{1+N+\frac{N^2}{2!}+\ldots},$$

and, for positive values of N, the denominator is certainly greater than $\frac{1}{2}N^2$, so that

$$(N+1)\,e^{-N} < \frac{2(N+1)}{N^2}.$$

Thus $$(N+1)e^{-N} \to 0$$

as N increases indefinitely. Hence

$$\int_0^\infty xe^{-x}\,dx = \lim_{N\to\infty}\{-(N+1)e^{-N}+1\}$$
$$= 0+1$$
$$= 1.$$

ILLUSTRATION 3. *To evaluate*

$$\int_{-\infty}^\infty \frac{dx}{1+x^2}.$$

Consider the integral

$$\int_{-M}^N \frac{dx}{1+x^2},$$

where M, N are large positive numbers. The value of this integral is

$$\Big[\tan^{-1}x\Big]_{-M}^N$$

or $$\tan^{-1}N - \tan^{-1}(-M).$$

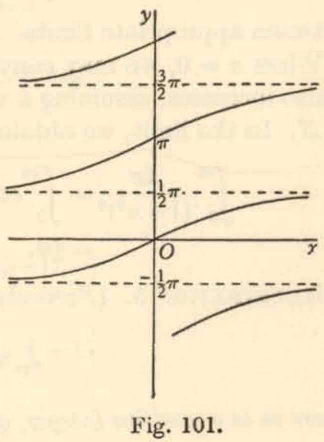


Fig. 101.

Care must be exercised in selecting the correct angles from the many-valued inverse tangents. The graph

$$y = \tan^{-1} x,$$

shown in the diagram (Fig. 101), illustrates the 'parallel' curves along which y must run, and the values selected must be confined to one of them. The most natural choice is to work with the curve through the origin O. Then for large negative values of x, the value $\tan^{-1}(-M)$ just exceeds $-\frac{1}{2}\pi$; as x increases, $\tan^{-1} x$ rises continuously through, say, $-\frac{1}{3}\pi, -\frac{1}{4}\pi, -\frac{1}{6}\pi, 0, \frac{1}{3}\pi, \frac{1}{6}\pi$ and so on, approaching the value $\frac{1}{2}\pi$ for the limit of $\tan^{-1} N$.

Hence
$$\int_{-\infty}^{\infty} \frac{dx}{1+x^2} = \lim_{N\to\infty} \{\tan^{-1} N\} - \lim_{M\to\infty} \{\tan^{-1}(-M)\}$$
$$= \tfrac{1}{2}\pi - (-\tfrac{1}{2}\pi)$$
$$= \pi.$$

ILLUSTRATION 4. (*Change of variable.*) *To evaluate*

$$\int_0^{\infty} \frac{dx}{(1+x^2)^2}.$$

Consider the integral $$u_N \equiv \int_0^N \frac{dx}{(1+x^2)^2}.$$

Make the substitution $$x = \tan\theta,$$
$$dx = \sec^2\theta\, d\theta.$$

Then $$u_N = \int \frac{\sec^2\theta\, d\theta}{\sec^4\theta} = \int \cos^2\theta\, d\theta$$

between appropriate limits.

When $x = 0$, we may conveniently take $\theta = 0$. As x increases, θ also increases, assuming a value very near to $\frac{1}{2}\pi$ for large values of N. In the limit, we obtain the value $\frac{1}{2}\pi$. Hence

$$\int_0^{\infty} \frac{dx}{(1+x^2)^2} = \int_0^{\frac{1}{2}\pi} \cos^2\theta\, d\theta = \frac{1}{2}\int_0^{\frac{1}{2}\pi} (1+\cos 2\theta)\, d\theta$$
$$= \tfrac{1}{4}\pi.$$

ILLUSTRATION 5. (*Formula of reduction.*) *To evaluate*

$$J_m \equiv \int_0^{\infty} x^m e^{-x}\, dx,$$

where m is a positive integer, greater than zero.

Let
$$I_m = \int_0^N x^m e^{-x}\, dx$$
$$= \Big[-x^m e^{-x}\Big]_0^N + m\int_0^N x^{m-1} e^{-x}\, dx.$$

Now
$$N^m e^{-N} = \frac{N^m}{e^N} = \frac{N^m}{1+N+\dfrac{N^2}{2!}+\dfrac{N^3}{3!}+\dots};$$

and the denominator, being certainly greater than

$$N^{m+1}/(m+1)!,$$

greatly exceeds the numerator for large values of N, whatever m may be, so that
$$\lim_{N\to\infty} N^m e^{-N} = 0.$$

Also $$x^m e^{-x} = 0$$

when $x = 0$, since $m > 0$.

Hence $$\lim_{N\to\infty} \Big[-x^m e^{-x}\Big]_0^N = 0,$$

and so, as $N\to\infty$, $$J_m = mJ_{m-1}.$$

The formula of reduction enables us to make the value of J_m depend on that of J_0; for

$$J_m = mJ_{m-1}$$
$$= m(m-1)J_{m-2}$$
$$\dots\dots\dots$$
$$= m(m-1)\dots 2.1 J_0$$
$$= m!\, J_0.$$

Moreover, $$J_0 = \int_0^{\infty} e^{-x}\, dx$$
$$= \lim_{N\to\infty} \Big[-e^{-x}\Big]_0^N$$
$$= \lim_{N\to\infty} \Big[-e^{-N}+1\Big]$$
$$= 1.$$

Hence $$\int_0^\infty x^m e^{-x}\,dx = m!$$

EXAMPLES I

Evaluate:

1. $\displaystyle\int_1^\infty \frac{dx}{x^2}$. 2. $\displaystyle\int_2^\infty \frac{dx}{x\sqrt{x}}$. 3. $\displaystyle\int_0^\infty x^2 e^{-x}\,dx$.

4. $\displaystyle\int_1^\infty \frac{dx}{1+x^2}$. 5. $\displaystyle\int_{-\infty}^1 \frac{dx}{1+x^2}$. 6. $\displaystyle\int_0^\infty e^{-x}\sin x\,dx$.

7. $\displaystyle\int_0^\infty e^{ax}$, stating for what values of a the integration is possible.

2. 'Infinite' integrand. It may happen that, in evaluating the function
$$\int_a^b f(x)\,dx,$$
we find that the integrand $f(x)$ tends to infinity—or, indeed, has some other discontinuity—in the range of integration. Suppose, for example, that $f(x)$ increases without bound near $x = c$. If c is inside the interval (that is, not equal to a or b) we 'cut it out' by considering the sum
$$\int_a^{c-\epsilon} f(x)\,dx + \int_{c+\eta}^b f(x)\,dx,$$
where ϵ, η are small positive constants, over ranges which just miss c on either side. *We then define the value of the integral to be*
$$\lim_{\epsilon\to 0}\int_a^{c-\epsilon} f(x)\,dx + \lim_{\eta\to 0}\int_{c+\eta}^b f(x)\,dx,$$
supposing that these limits exist.

When c is at a or b the modification is obvious; only one limiting value is then required.

ILLUSTRATION 6. *To determine the values of n for which the integral*
$$\int_0^1 \frac{dx}{x^n} \quad (n > 0)$$
exists.

Consider the integral
$$\int_\epsilon^1 \frac{dx}{x^n}$$
$$= \left[\frac{1}{1-n}x^{1-n}\right]_\epsilon^1$$
$$= \frac{1}{1-n}\{1-\epsilon^{1-n}\}.$$

As ϵ tends to zero, the term ϵ^{1-n} also tends to zero if the exponent $1-n$ is positive; otherwise ϵ^{1-n} increases indefinitely. Hence the integral exists provided that
$$n < 1,$$
and its value is then $\dfrac{1}{1-n}\{1-0\}$
$$= \frac{1}{1-n}.$$

The case $n = 1$ requires separate treatment. We then have
$$\int_\epsilon^1 \frac{dx}{x} = \Big[\log x\Big]_\epsilon^1 = \log 1 - \log \epsilon,$$
so that (compare p. 4) the integral does not exist.

The required condition is therefore
$$n < 1.$$

ILLUSTRATION 7. *To evaluate*
$$\int_1^\infty \frac{dx}{(x+3)\sqrt{(x-1)}}.$$

This integral involves two infinities; the upper limit of integration is infinite, and the integrand is infinite when $x = 1$. The two phenomena must be kept separate.

We consider the integral
$$I \equiv \int_{1+\epsilon}^N \frac{dx}{(x+3)\sqrt{(x-1)}}$$

Write
$$x-1 = t^2,$$
$$dx = 2t\,dt.$$

Then
$$I = \int \frac{2t\,dt}{(t^2+4)t}$$

between appropriate limits.

Now we have cut out the value $x = 1, t = 0$ by making the lower limit $1+\epsilon$. Thus t is never zero, so that the factor t may be cancelled from numerator and denominator of the integrand. Hence
$$I = \int \frac{2\,dt}{t^2+4}$$
$$= \left[\tan^{-1}\left(\frac{t}{2}\right)\right]$$

between appropriate limits.

Consider the range of variation of t as x increases continuously from $1+\epsilon$ to N. We have
$$t = +\sqrt{(x-1)},$$

having committed ourselves to the positive square root by putting $\sqrt{(x-1)} = t$ in the integrand during the substitution. Hence t increases continuously from the small value $\sqrt{\epsilon}$ to the large value $\sqrt{(N-1)}$; and as it does so, $\tan^{-1}(\frac{1}{2}t)$ increases continuously from just above zero to just short of $\frac{1}{2}\pi$. Thus, in the limit,
$$I = \tfrac{1}{2}\pi - 0$$
$$= \tfrac{1}{2}\pi.$$

ILLUSTRATION 8. *To evaluate*
$$\int_a^b \frac{dx}{\sqrt{\{(x-a)(b-x)\}}} \quad (0 < a < b).$$

The denominator in the integrand becomes zero at $x = a$ and at $x = b$, and so we must consider the integral
$$I \equiv \int_{a+\epsilon}^{b-\eta} \frac{dx}{\sqrt{\{(x-a)(b-x)\}}}.$$

Make the substitution
$$x = a\cos^2 t + b\sin^2 t,$$
$$dx = 2(b-a)\sin t\cos t\,dt.$$

Then
$$x-a = (b-a)\sin^2 t,$$
$$b-x = (b-a)\cos^2 t.$$

Consider the range of integration in the variable t. When $x = a$, we have $a\sin^2 t = b\sin^2 t$, so that $\sin t = 0$; and when $x = b$, we have $b\cos^2 t = a\cos^2 t$, so that $\cos t = 0$. We must select as starting-point for t a value for which $\sin t = 0$ (corresponding to $x = a$), and the obvious value is $t = 0$. The relation
$$\frac{dx}{dt} = 2(b-a)\sin t\cos t$$

shows that, at any rate for small values of t, the variables x, t increase together; and this process continues until $t = \frac{1}{2}\pi$, at which point x has the value b. The range is therefore $0, \frac{1}{2}\pi$. But we have had to exclude these points themselves because of trouble with the integrand, and so the range must run from just above zero to just short of $\frac{1}{2}\pi$; say from ϵ' to $\frac{1}{2}\pi - \eta'$. Thus
$$I = \int_{\epsilon'}^{\frac{1}{2}\pi-\eta'} \frac{2(b-a)\sin t\cos t\,dt}{\sqrt{\{(b-a)^2\sin^2 t\cos^2 t\}}}.$$

Moreover the POSITIVE value of $\sqrt{\{(b-a)^2\sin^2 t\cos^2 t\}}$ in the interval $0, \frac{1}{2}\pi$ is $(b-a)\sin t\cos t$, so that
$$I = \int_{\epsilon'}^{\frac{1}{2}\pi-\eta'} \frac{2(b-a)\sin t\cos t\,dt}{(b-a)\sin t\cos t}.$$

We have excluded the end-points, so that $\sin t$ and $\cos t$ are not zero in the range of integration; we may therefore cancel factors in the numerator and denominator of the integrand, giving
$$I = \int_{\epsilon'}^{\frac{1}{2}\pi-\eta'} 2\,dt$$
$$= 2\{(\tfrac{1}{2}\pi - \eta') - \epsilon'\}.$$

Proceeding to the limit as ϵ', η' tend to zero independently, we have
$$I = 2(\tfrac{1}{2}\pi)$$
$$= \pi.$$

REVISION EXAMPLES VIII

'Advanced' Level

1. Find the following integrals:

$$\int \frac{dx}{(x-1)(x-3)}, \quad \int \frac{x^2\,dx}{(x-1)(x-3)}, \quad \int \frac{dx}{\sqrt{\{(x-1)(3-x)\}}}.$$

2. Integrate $\dfrac{(x^2+1)}{x^2-4x+3}, \quad \dfrac{x}{\sqrt{(x^2+2x+2)}}.$

Show that $\displaystyle\int_0^{\frac{1}{2}\pi} \frac{dx}{a+b\sin x} = \frac{1}{\sqrt{(a^2-b^2)}}\cos^{-1}\left(\frac{b}{a}\right)(b<a).$

3. Evaluate the integral

$$\int_a^b \frac{x\,dx}{\sqrt{\{(x-a)(b-x)\}}}$$

by means of the substitution $x = a\cos^2\theta + b\sin^2\theta$.

Prove that the integrals

$$\int_0^1 x^7(1-x)^8\,dx, \quad \int_0^1 x^8(1-x)^7\,dx$$

are equal, and show that their common value is

$$\frac{7!\,8!}{16!}.$$

4. Integrate

$$\frac{x^3}{(x^2+1)(x-2)}, \quad (x+a)(x+b)^{\frac{1}{2}}, \quad \sin^5 x.$$

5. Integrate

$$\frac{x}{(1+x^2)(1-x)}, \quad \frac{x}{\sqrt{(x^2+4x+5)}}, \quad \frac{x^5}{(a^2+x^2)^2}.$$

Prove that, when a, b are positive,

$$\int_0^{\pi} \frac{\cos^2 x\,dx}{a^2\cos^2 x + b^2\sin^2 x} = \frac{\pi}{a(a+b)}.$$

6. Integrate

$$\frac{(2x+3)}{x^3+x^2-2x}, \quad \frac{1}{x\sqrt{(x^2+2x-1)}}, \quad \frac{xe^x}{(x+1)^2}.$$

7. Integrate

$$\frac{(x+1)}{x^3-x^2-6x}, \quad \sqrt{(x^2+x+1)}, \quad (\log x)^2.$$

8. Find a reduction formula for

$$\int_0^1 (1+x^2)^{n+\frac{1}{2}}\,dx,$$

and evaluate the integral when $n = 2$.

9. By integration by parts, show that, if $0<m<n$, and

$$I = \int_0^1 x^m \frac{d^n}{dx^n}\{x^n(1-x)^n\}\,dx,$$

then $$I = -m\int_0^1 x^{m-1}\frac{d^{n-1}}{dx^{n-1}}\{x^n(1-x)^n\}\,dx.$$

Deduce that $$I = 0.$$

10. Taking $$y = \frac{\sqrt{(2x^2+6x+5)}}{x+2},$$

verify that the result of changing the variable from y to x in the integral

$$\int \frac{dy}{\sqrt{(y^2-1)}}$$

is $$\int \frac{dx}{(x+2)\sqrt{(2x^2+6x+5)}}.$$

Deduce that $$\int_{-1}^{2} \frac{dx}{(x+2)\sqrt{(2x^2+6x+5)}} = \log_e 2.$$

REVISION EXAMPLES IX

'Scholarship' Level

1. Show that, if $$u_n = \int_0^{\pi} \frac{\cos nx\,dx}{5-4\cos x},$$

then, provided that $n \neq 1$,

$$2u_n - 5u_{n-1} + 2u_{n-2} = 0.$$

Hence, or otherwise, show that, if n is a positive integer,

$$u_n = \pi/(3.2^n).$$

2. Prove that

$$\int_0^{\frac{1}{2}\pi} \frac{d\theta}{a^2\cos^2\theta + b^2\sin^2\theta} = \frac{\pi}{2ab}, \quad \int_0^{\frac{1}{2}\pi} \log\tan\theta\,d\theta = 0,$$

$$\int_0^{\frac{1}{2}\pi} \sin^2\theta\log\tan\theta\,d\theta = \tfrac{1}{4}\pi.$$

3. (i) Prove that values of a, b can be chosen so that the substitution

$$x = \frac{at+b}{t+1}$$

yields the result

$$\int \frac{dx}{(2x^2-6x+5)\sqrt{(5x^2-12x+8)}} = -\int \frac{(t+1)\,dt}{(t^2+1)\sqrt{(t^2+4)}}.$$

(ii) Find the indefinite integral of

$$\frac{t+1}{(t^2+1)\sqrt{(t^2+4)}}.$$

4. Evaluate $\displaystyle\int_0^{\frac{1}{2}} (\sin^{-1}x)^2\,dx, \quad \int_1^2 \sqrt{\left(\frac{x-1}{x+1}\right)}\frac{dx}{x}.$

5. Prove that

$$\int_0^{\frac{1}{3}\pi} \sqrt{(\cos 2x - \cos 4x)}\,dx = \tfrac{1}{2}\sqrt{6} - \tfrac{1}{4}\sqrt{2}\log(2+\sqrt{3}).$$

6. Evaluate $\displaystyle\int_0^1 \frac{dx}{x+\sqrt{(1-x)}}, \quad \int_{-1}^1 \frac{x^4\,dx}{\sqrt{(1-x^2)}}.$

7. Evaluate

$$\int_0^\infty \frac{x\tan^{-1}x}{(1+x^2)^2}\,dx, \quad \int_0^{\frac{1}{2}\pi} \frac{\cos^2\theta\,d\theta}{a^2\cos^2\theta + b^2\sin^2\theta}.$$

Find $\displaystyle\int \frac{dx}{(1-x)\sqrt{(1+x)}}.$

8. Evaluate $\displaystyle\int_0^\infty \frac{x\,dx}{x^3+x^2+x+1}.$

Find $\displaystyle\int \frac{dx}{(x^2-1)^{\frac{3}{2}}}, \quad \int x^3\sin^2 x\,dx.$

9. If $\displaystyle I_n \equiv \int_0^{2\pi} \frac{\cos(n-1)x - \cos nx}{1-\cos x}\,dx,$

show that I_n is independent of n, where n is a positive integer.

Hence evaluate I_n and prove that

$$\int_0^{2\pi} \left(\frac{\sin\frac{1}{2}nx}{\sin\frac{1}{2}x}\right)^2 dx = 2n\pi.$$

10. Evaluate

$$\int_0^a \frac{x\,dx}{x+\sqrt{(a^2-x^2)}}, \quad \int_0^{\frac{1}{2}\pi} \frac{3+2\cos x}{(3+\cos x)^2}\,dx.$$

11. Evaluate the integrals:

$$\int_0^1 \frac{\sin^{-1}x}{(1+x)^2}\,dx, \quad \int_0^{\frac{3}{4}\pi} \frac{dx}{2\cos^2 x + 2\cos x\sin x + \sin^2 x},$$

$$\int_{-\infty}^\infty \frac{dx}{(e^{x-a}+1)(1+e^{-x})}.$$

12. Evaluate $\displaystyle\int_2^\infty \frac{x^2\,dx}{(1-x^2)(1+x^2)^2}.$

13. Evaluate $\displaystyle\int_0^\infty \frac{dx}{(x^2+a^2)(x^2+b^2)(x^2+c^2)},$

where a, b, c are positive.

14. Prove that

$$\int_0^\infty \frac{dx}{x^2+2x\cos\alpha+1} = \frac{\alpha}{\sin\alpha} \quad (0<\alpha<\pi).$$

Evaluate

$$\int_0^\infty \frac{dx}{x^4+2x^2\cos\alpha+1}, \quad \int_0^\infty \frac{(x^2+1)\,dx}{x^4+2x^2\cos\alpha+1}.$$

15. Prove that, when $b>a>0$,

$$\int_0^\pi \frac{\sin\theta\,d\theta}{\sqrt{(a^2+b^2-2ab\cos\theta)}} = \frac{2}{b},$$

$$\int_0^\pi \frac{\sin\theta\cos\theta\,d\theta}{\sqrt{(a^2+b^2-2ab\cos\theta)}} = \frac{2a}{3b^2}.$$

Make it clear at what points of your proofs you use the condition $b>a>0$.

16. Evaluate the definite integrals:

$$\int_{\sqrt{2}}^\infty \frac{dx}{(x^2-1)\sqrt{(1+x^2)}}, \quad \int_0^1 (1+x^2)\tan^{-1}x\,dx.$$

17. Evaluate

$$I=\int_{-\delta}^{\delta}\frac{1-r\cos\theta}{1-2r\cos\theta+r^2}\,d\theta\quad(0<\delta<\pi)$$

when (i) $0<r<1$, (ii) $r>1$.

Prove that, δ being fixed, I tends to one limit as $r\to1$ through values less than 1, and to a different limit as $r\to1$ through values greater than 1.

Show, also, that neither limit is equal to the value of I when

$$r=1.$$

18. Show that, by proper choice of a new variable, the integration of any rational function of $\sin x$ and $\cos x$ can be reduced to the integration of a rational algebraic function of that variable.

Integrate

$$\frac{\sin x}{\sin(x-\alpha)},\quad\frac{1}{\sin x\cos x+\sin x+\cos x-1}.$$

19. Integrate $$\frac{(x^2-1)}{x\sqrt{(x^4+x^2+1)}}.$$

Prove that

$$\int_1^\infty\frac{dx}{(x+\cos\alpha)\sqrt{(x^2-1)}}=\frac{\alpha}{\sin\alpha}\quad(0<\alpha<\pi),$$

$$\int_0^\pi\frac{x\,dx}{1+\cos\alpha\sin x}=\frac{\pi\alpha}{\sin\alpha}\quad(0<\alpha<\tfrac{1}{2}\pi).$$

20. Integrate $$\frac{1}{(x^2+1)\sqrt{(x^2+b^2)}}\,(b<a).$$

Evaluate

$$\int_a^b(x-a)^{\frac{3}{2}}(b-x)^{\frac{3}{2}}\,dx,\quad\int_0^\pi\frac{x\sin x\,dx}{\sqrt{(1-a^2\sin^2x)}}\quad|a|<1.$$

Prove that

$$\int_0^{\frac{1}{2}\pi}f(\sin2x)\sin x\,dx=\sqrt{2}\int_0^{\frac{1}{4}\pi}f(\cos2x)\cos x\,dx.$$

21. Evaluate the integrals:

$$\int_6^7\sqrt{\left(\frac{x-3}{x-6}\right)}\,dx,\quad\int_{\frac{1}{2}}^1\frac{\sqrt{(1-x^2)}}{x}\,dx,\quad\int_1^2\frac{\sqrt{(x^2-1)}}{x}\,dx.$$

22. Find the integrals:

$$\int\frac{x^2\,dx}{x^4-x^2-12},\quad\int e^x(1+x)\log x\,dx.$$

Evaluate $$\int_0^1\frac{dx}{(1+x)\sqrt{(1+2x-x^2)}}.$$

23. Prove that, if a is positive, the value of the integral

$$\int_0^{\frac{1}{2}\pi}\frac{1-a\cos\theta}{1-2a\cos\theta+a^2}\,d\theta$$

is $\pi-\cot^{-1}a$ or $-\cot^{-1}a$, according as $a<1$ or $a>1$, where the value of $\cot^{-1}$ is taken between $0,\frac{1}{2}\pi$.

What is the value of the integral when $a=1$?

24. Show that, if P,Q are polynomials in s,c (where $s=\sin\theta$, $c=\cos\theta$), of which Q contains only even powers of both s and c, then

$$\int\frac{P}{Q}\,d\theta$$

can be expressed as a sum of integrals of the form $\int R_1(s)\,ds$, $\int R_2(c)\,dc$, $\int R_3(t)\,dt$, where $t=\tan\theta$ and each of the functions R_1,R_2,R_3 is a rational function of its argument.

Apply this method to obtain

$$\text{(i)}\int\frac{(1+\sin\theta)(1+\cos\theta)}{1+\cos^2\theta}\,d\theta,\quad\text{(ii)}\int\frac{1}{1+\sin\theta}\,d\theta.$$

In case (ii) obtain the integral also by the substitution

$$x=\tan\tfrac{1}{2}\theta,$$

and reconcile the results obtained by the two methods.

25. Show that the substitution

$$4x=-\frac{b}{a}\left(t+\frac{1}{t}\right)^2+\frac{d}{c}\left(t-\frac{1}{t}\right)^2$$

reduces the integral

$$\int F\{x,\sqrt{(ax+b)},\sqrt{(cx+d)}\}\,dx,$$

where $F(x,y,z)$ is a rational function of x,y,z, to the integral of a rational function of t.

Hence, or otherwise, find

$$\int \frac{(x-1)^{\frac{1}{2}}}{(x+1)^{\frac{3}{2}}}\,dx.$$

26. Find a reduction formula for

$$I_n = \int \frac{dx}{(5+4\cos x)^n}$$

in terms of I_{n-1}, I_{n-2} $(n \geqslant 2)$, and use it to show that

$$\int_0^{\frac{2}{3}\pi} \frac{dx}{(5+4\cos x)^2} = \frac{1}{81}(5\pi - 6\sqrt{3}).$$

27. Find a reduction formula for

$$\int_1^\infty \frac{dx}{x^n\sqrt{(x^2-1)}}.$$

Evaluate the integral when $n = 1$ and when $n = 2$.

28. If $$I_{p,q} = \int_0^{\frac{1}{2}\pi} \sin^p x \cos^q x\,dx,$$

show that $$(p+q)I_{p,q} = \begin{cases}(q-1)\,I_{p,q-2} & (q \geqslant 2),\\ (p-1)\,I_{p-2,q} & (p \geqslant 2),\end{cases}$$

and evaluate $I_{\alpha-1,5}$ where α is any positive real number.

29. If $$I_n = \int_0^\infty x^n e^{-ax}\cos bx\,dx, \quad J_n = \int_0^\infty x^n e^{-ax}\sin bx,$$

where n is a positive integer and a, b are positive, prove that

$$I_n(a^2+b^2) = n(aI_{n-1} - bJ_{n-1}),$$
$$J_n(a^2+b^2) = n(bI_{n-1} + aJ_{n-1}).$$

Show that $$(a^2+b^2)^{\frac{1}{2}(n+1)}I_n = n!\cos(n+1)\alpha,$$
$$(a^2+b^2)^{\frac{1}{2}(n+1)}J_n = n!\sin(n+1)\alpha,$$

where $\tan\alpha = b/a$ and $0 < \alpha < \frac{1}{2}\pi$.

30. (i) Prove that, if

$$u_n = \int_0^{\frac{1}{4}\pi} \frac{dx}{(a+b\tan x)^n} \quad (n > 1),$$

then $$(a^2+b^2)\,u_n - 2au_{n-1} + u_{n-2} = \frac{b}{(n-1)\,a^{n-1}}.$$

(ii) Prove that, if n is an integer and $m > 1$, then

$$(m^2-4n^2)\int_0^{\frac{1}{2}\pi} \sin^m x\cos 2nx\,dx = m(m-1)\int_0^{\frac{1}{2}\pi} \sin^{m-2} x\cos 2nx\,dx.$$

31. Find the reduction formula for

$$\int \frac{dx}{(ax^2+2hx+b)^{n+1}}.$$

Prove that, if a and $ab-h^2$ are positive and n is a positive integer, then

$$\int_{-\infty}^{\infty} \frac{dx}{(ax^2+2hx+b)^{n+1}} = \frac{2n-1}{2n}\cdot\frac{a}{ab-h^2}\int_{-\infty}^{\infty} \frac{dx}{(ax^2+2hx+b)^n}$$
$$= \frac{1.3\ldots(2n-1)}{2.4\ldots 2n}\cdot\frac{\pi a^n}{(ab-h^2)^{n+\frac{1}{2}}}.$$

32. Show that

$$\int \frac{(x-p)^{2n+1}\,dx}{(ax^2+2hx+b)^{n+\frac{3}{2}}} = \frac{1}{(h^2-ab)^{n+1}}\int (y^2-q^2)^n\,dy,$$

where $$q^2 = ap^2 + 2hp + b,$$

$$y\sqrt{(ax^2+2hx+b)} = (ap+h)x + (hp+b).$$

Deduce that, when $a, b, h+\sqrt{(ab)}$ are positive,

$$\int_0^\infty \frac{dx}{(ax^2+2hx+b)^{\frac{3}{2}}} = \frac{1}{\{h+\sqrt{(ab)}\}\sqrt{b}},$$

$$\int_0^\infty \frac{x\,dx}{(ax^2+2hx+b)^{\frac{3}{2}}} = \frac{1}{\{h+\sqrt{(ab)}\}\sqrt{a}}.$$

33. Find a reduction formula for the integral

$$\int \frac{dx}{(1+x)^n\sqrt{(1+x^2)}}.$$

Prove that

$$\int_0^\infty \frac{dx}{(1+x)^2\sqrt{(1+x^2)}} = \tfrac{1}{2}\sqrt{2}\log(1+\sqrt{2}),$$

$$\int_0^\infty \frac{dx}{(1+x)^3\sqrt{(1+x^2)}} = \tfrac{1}{8}\{2+\sqrt{2}\log(1+\sqrt{2})\}.$$

34. (i) Integrate $\dfrac{1}{(x^2+4)\sqrt{(x^2+1)}}$.

(ii) Find a formula of reduction for

$$\int (1+x^2)^n e^{ax}\,dx.$$

35. (i) Evaluate

$$\int_{-1}^{1} \frac{dx}{(a-x)\sqrt{(1-x^2)}} \quad (a>1).$$

(ii) If $$u_n = \int_0^1 \frac{x^n\,dx}{\sqrt{(1-2x\cos\alpha+x^2)}},$$

prove that

$$nu_n-(2n-1)\,u_{n-1}\cos\alpha+(n-1)\,u_{n-2} = 2\sin\tfrac{1}{2}\alpha$$

when $n \geqslant 2$, and evaluate u_0, u_1.

36. Obtain reduction formulæ for the integrals

$$\int_1^\infty \frac{(\log x)^n}{x^2}\,dx, \quad \int_0^\pi x^n \sin^2 x\,dx,$$

and evaluate the first integral for any positive integer n.

37. If $$I_p = \int_0^1 \frac{x^p\,dx}{\sqrt{(x^q+1)}} \quad (p, q \text{ real}, p>q-1),$$

prove that

$$(2p-q+2)\,I_p+(2p-2q+2)\,I_{p-q} = 2\sqrt{2}.$$

Hence, or otherwise, prove that

$$\int_0^1 \frac{x^8\,dx}{\sqrt{(x^3+1)}} = \frac{1}{45}(14\sqrt{2}-16).$$

38. If $$I_n = \int_a^b \frac{dx}{(x^2+k)^n\sqrt{(x^2+l)}},$$

verify by differentiation that, when n is a positive integer or zero,

$$2(n+1)\,k(k-l)\,I_{n+2}-(2n+1)(2k-l)\,I_{n+1}+2nI_n = \left[\frac{-x\sqrt{(x^2+l)}}{(x^2+k)^{n+1}}\right]_a^b.$$

Show further that I_1 can be integrated by the substitution

$$\sqrt{(x^2+l)} = xt.$$

Hence, or otherwise, find

$$\int_0^1 \frac{dx}{(x^2+2)^2\sqrt{(x^2+3)}}.$$

39. Show that, if

$$I_n = \frac{1}{n!}\int_0^\infty x^n e^{-x}\cos x\,dx, \quad J_n = \frac{1}{n!}\int_0^\infty x^n e^{-x}\sin x\,dx,$$

then $2J_n = I_{n-2}$, $2I_n = -J_{n-2}$.

Hence, or otherwise, prove that

$$I_n = \frac{\cos\frac{1}{4}(n+1)\,\pi}{2^{\frac{1}{2}(n+1)}}.$$

40. Find a reduction formula for the integral

$$\int_0^{\frac{1}{4}\pi} \frac{dx}{\cos^n x},$$

and evaluate the integral for the cases $n = 1, 2$.

41. Prove that, if

$$I_n = \int \frac{dx}{(2x^2+1)^n\sqrt{(x^2+1)}} \quad (n \geqslant 0),$$

then $$(n+1)\,I_{n+2}-nI_n = \frac{x\sqrt{(x^2+1)}}{(2x^2+1)^{n+1}}.$$

Hence evaluate the integral

$$\int \frac{dx}{(2x^2+1)^4\sqrt{(x^2+1)}}.$$

42. Find a reduction formula for the integral

$$I_n = \int \frac{x^n\,dx}{\sqrt{(ax^2+2hx+b)}},$$

and use it to evaluate

$$\int \frac{x^3\,dx}{\sqrt{(x^2+2x+2)}}.$$

APPENDIX

First Steps in Partial Differentiation

The functions which we have considered in this volume have always involved a single variable. The work on functions of several variables belongs to a later stage, but it may be convenient to set down one or two of the most elementary properties—mainly definitions and first applications.

Consider, as an illustration, the expression

$$u \equiv x^4 y^3 z^2.$$

As x, y, z take various values, so also does u. For example,

if $\quad x = 1, \quad y = -2, \quad z = 3,$

then $\quad u = -72;$

if $\quad x = -1, \quad y = 0, \quad z = 3,$

then $\quad u = 0;$

if $\quad x = 2, \quad y = -2, \quad z = 1,$

then $\quad u = -128;$

and so on.

We say that u is then a *function of the three independent variables* x, y, z. To denote this functional dependence, we may use the notation

$$u(x, y, z) \equiv x^4 y^3 z^2.$$

The function u no longer has a unique differential coefficient. Each of the variables x, y, z is capable of its own independent variation, and each of these variations produces a differential coefficient of its own. More precisely, we use the notation

$\dfrac{\partial u}{\partial x} \equiv$ the differential coefficient of u with respect to x only, calculated on the assumption that y, z are constant;

$\dfrac{\partial u}{\partial y} \equiv$ the differential coefficient of u with respect to y only, calculated on the assumption that z, x are constant;

$\dfrac{\partial u}{\partial z} \equiv$ the differential coefficient of u with respect to z only, calculated on the assumption that x, y are constant.

Thus, if $\quad u \equiv x^4 y^3 z^2,$

then $$\frac{\partial u}{\partial x} = 4x^3 y^3 z^2, \quad \frac{\partial u}{\partial y} = 3x^4 y^2 z^2, \quad \frac{\partial u}{\partial z} = 2x^4 y^3 z.$$

As another illustration, suppose that

$$u \equiv \cos(ax + by^2)$$

is a function of the two variables x, y. Then

$$\frac{\partial u}{\partial x} = -a \sin(ax + by^2), \quad \frac{\partial u}{\partial y} = -2by \sin(ax + by^2).$$

The functions $\quad \dfrac{\partial u}{\partial x}, \quad \dfrac{\partial u}{\partial y}, \quad \dfrac{\partial u}{\partial z}$

are called the *partial differential coefficients* of u with respect to x, y, z respectively.

These partial differential coefficients are, in their turn, also functions of the three variables x, y, z, and have their own partial differential coefficients. We write

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\right) \equiv \frac{\partial^2 u}{\partial x^2},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial y}\right) \equiv \frac{\partial^2 u}{\partial y^2},$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial z}\right) \equiv \frac{\partial^2 u}{\partial z^2}.$$

The 'mixed' coefficients are a little more awkward. We write

$$\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial y}\right) \equiv \frac{\partial^2 u}{\partial x \partial y}, \quad \frac{\partial}{\partial x}\left(\frac{\partial u}{\partial z}\right) \equiv \frac{\partial^2 u}{\partial x \partial z},$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial z}\right) \equiv \frac{\partial^2 u}{\partial y \partial z}, \quad \frac{\partial}{\partial y}\left(\frac{\partial u}{\partial x}\right) \equiv \frac{\partial^2 u}{\partial y \partial x},$$

$$\frac{\partial}{\partial z}\left(\frac{\partial u}{\partial x}\right) \equiv \frac{\partial^2 u}{\partial z \partial x}, \quad \frac{\partial}{\partial z}\left(\frac{\partial u}{\partial y}\right) \equiv \frac{\partial^2 u}{\partial z \partial y}.$$

In practice, however, it may be proved that for 'ordinary' functions (a term which we do not attempt to make more precise) interchange of the order of partial differentiation leaves the result unaltered. Thus

$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial^2 u}{\partial z\,\partial y},$$

$$\frac{\partial^2 u}{\partial z\,\partial x} = \frac{\partial^2 u}{\partial x\,\partial z},$$

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial^2 u}{\partial y\,\partial x}.$$

For example, returning to our function

$$u \equiv x^4y^3z^2,$$

we have the relations

$$\frac{\partial^2 u}{\partial x\,\partial y} = \frac{\partial}{\partial x}(3x^4y^2z^2) = 12x^3y^2z^2,$$

$$\frac{\partial^2 u}{\partial y\,\partial x} = \frac{\partial}{\partial y}(4x^3y^2z^2) = 12x^3y^2z^2;$$

$$\frac{\partial^2 u}{\partial y\,\partial z} = \frac{\partial}{\partial y}(2x^4y^3z) = 6x^4y^2z,$$

$$\frac{\partial^2 u}{\partial z\,\partial y} = \frac{\partial}{\partial z}(3x^4y^2z^2) = 6x^4y^2z;$$

$$\frac{\partial^2 u}{\partial z\,\partial x} = \frac{\partial}{\partial z}(4x^3y^3z^2) = 8x^3y^3z,$$

$$\frac{\partial^2 u}{\partial x\,\partial z} = \frac{\partial}{\partial x}(2x^4y^3z) = 8x^3y^3z.$$

EXAMPLES I

Find $\frac{\partial u}{\partial x}, \frac{\partial u}{\partial y}, \frac{\partial^2 u}{\partial x^2}, \frac{\partial^2 u}{\partial x\,\partial y}, \frac{\partial^2 u}{\partial y\,\partial x}, \frac{\partial^2 u}{\partial y^2}$ for each of the following functions:

1. x^4y^3.	2. xy^2.	3. x^2+y^2.
4. $e^x \cos y$.	5. $\log(x+y)$.	6. $\log(xy)$.
7. $x^3 \sin^2 y$.	8. $e^{xy} \sin x$.	9. $x \tan^{-1} y$.
10. $\sec x + \sec y$.	11. $e^x \sin^2 2y$.	12. xe^y.

13. Prove that, if $f(x,y)$ is any *polynomial* in x, y, then

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial^2 f}{\partial y\,\partial x}.$$

14. Prove that, if $f(x,y,z)$ is any *polynomial* in x,y,z, then

$$\frac{\partial^2 f}{\partial y\,\partial z} = \frac{\partial^2 f}{\partial z\,\partial y}.$$

Illustration 1. *The 'homogeneous quadratic form'.*

Let $$u \equiv ax^2+by^2+cz^2+2fyz+2gzx+2hxy.$$

Then $$\frac{\partial u}{\partial x} = 2(ax+hy+gz),$$

$$\frac{\partial u}{\partial y} = 2(hx+by+fz),$$

$$\frac{\partial u}{\partial z} = 2(gx+fy+cz).$$

These expressions are probably familiar from analytical geometry.

To show how the partial differential coefficients are linked with the idea of gradient, we use an illustrative example.

Let OX, OY be the axes for a system of rectangular coordinates in a horizontal plane. This is illustrated in the diagram (Fig. 102), where the reader may regard himself as looking 'down' upon axes drawn in the usual position. The straight line OZ is drawn vertically upwards.

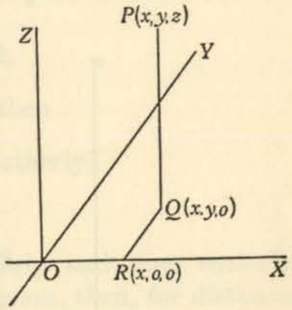


Fig. 102.

Given a point P in space, let the vertical line through it meet the plane XOY in Q; draw QR perpendicular to OX. Denote by x, y, z the lengths $\overrightarrow{OR}, \overrightarrow{RQ}, \overrightarrow{QP}$ respectively; then the triplet x, y, z may be used as coordinates for the point P in space, just as the pair x, y is used for a point in a plane. If P is the point (x, y, z), then Q is the point $(x, y, 0)$ in the horizontal plane; the coordinate z gives the height of P referred to the plane XOY as zero level. (Of course, P may be below the plane, in which case z is negative.)

In particular, if x, y, z are connected by the relation

$$z = f(x, y),$$

where we assume $f(x, y)$ to be a single-valued function defined for each pair of values of x, y, then, as x, y (and consequently z) vary, the point Q moves about the plane XOY, while P describes the surface whose height at any point is equal to the corresponding value of the function. We say that the surface *represents* the function $f(x, y)$.

For instance, it is an easy example on the theorem of Pythagoras to show that the function

$$z = +\sqrt{(1-x^2-y^2)}$$

is represented by the hemisphere of centre O and unit radius lying above the plane XOY.

We now assume, for convenience of language, that $\overrightarrow{OX}$ is due east and $\overrightarrow{OY}$ due north. We regard the surface

$$z = f(x, y)$$

as a hill, and P as the position of a climber on it.

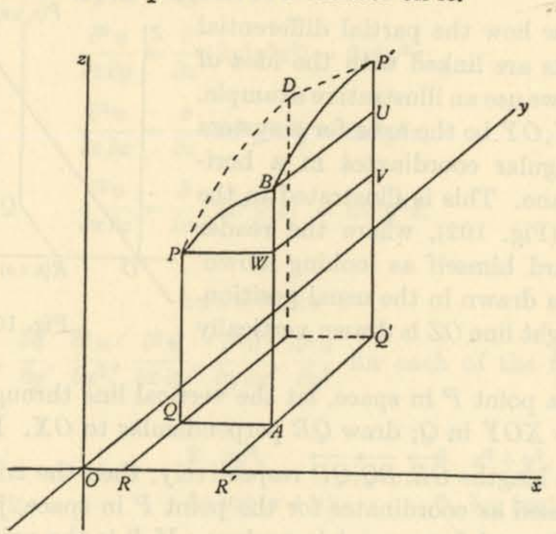


Fig. 103.

Suppose that the climber is at the point P (Fig. 103) defined by the values x, y of the easterly and northerly coordinates, and that he wishes to climb to the point P' defined by $x+h, y+k$. The crux of the difference between functions of one variable and functions

of two lies in the fact that, whereas for one variable motion along the CURVE representing the function is defined all the way, for two variables the SURFACE may be traversed by an innumerable choice of paths. Moreover, each way of leaving P will demand a gradient all of its own. The partial differential coefficients are the bases of the mathematical expressions for such gradients corresponding to the various paths.

From the mathematical point of view, the obvious way to pass from P to P' is firstly to move the distance h easterly, to B, and then to move the distance k northerly. The climber thus describes in succession the two arcs PB, BP' shown in the diagram.

Now suppose that P' is very close to P, so that the arcs PB, BP' are very small. The arc PB may be regarded as almost straight, so that the 'rise' between P and B is proportional to the length h, say

$$\delta z(\text{easterly}) = \alpha h.$$

Similarly BP' is almost straight, so that the 'rise' between B and P' is proportional to k, say

$$\delta z(\text{northerly}) = \beta k.$$

If δz is the total 'rise' between P, P', then

$$\begin{aligned}\delta z &= \delta z(\text{easterly}) + \delta z(\text{northerly}) \\ &= \alpha h + \beta k.\end{aligned}$$

If the climber had gone first northerly and then easterly, following the course PD, DP' in the diagram, then, for distances so small that the paths may be regarded as straight, $PBP'D$ is approximately a parallelogram, and so, once again,

$$\delta z = \alpha h + \beta k$$

for the SAME values of α, β.

Two simple observations complete the illustration. Geometrically, α, β are the gradients of those curves which are the sections of the hill in the easterly and northerly directions respectively. Analytically, we see by putting $k = 0$ that α is the ratio $\delta z \div h$ calculated on the assumption that y is constant; thus

$$\alpha = \frac{\partial z}{\partial x}$$

evaluated at P. Similarly we see by putting $h = 0$ that β is the ratio $\delta z \div k$ calculated on the assumption that x is constant; thus

$$\beta = \frac{\partial z}{\partial y}$$

calculated at P.

Hence *the partial differential coefficients* $\dfrac{\partial z}{\partial x}, \dfrac{\partial z}{\partial y}$ *are identified as the gradients of the surface* $z = f(x, y)$ *in the x- and y-directions respectively.*

EXAMPLES II

1. Show that the function

$$z = 1 - \sqrt{(2x - x^2 - y^2)},$$

with the z-axis measured vertically upwards, is represented by a hemisphere.

Prove also that the gradients in the x- and y-directions at the point (x, y, z) are in the ratio $(x-1) : y$, and that these gradients are equal only for points on a certain vertical diametral plane.

2. Prove that the function

$$z = x^2 + 4y^2,$$

with the z-axis measured vertically upwards, is represented by a 'bowl-shaped' surface whose horizontal sections are ellipses of eccentricity $\frac{1}{2}\sqrt{3}$.

Prove that the gradient in the x-direction at the point $(1, 2, 17)$ is 2, and that the gradient in the y-direction at the point $(3, 1, 13)$ is 8.

3. Find $\dfrac{\partial u}{\partial x}, \dfrac{\partial^2 u}{\partial y^2}, \dfrac{\partial^2 u}{\partial z\, \partial x}$ for each of the following functions:

(i) $e^{ax}\sin(by + cz)$, (ii) $xyze^{-x^2}$, (iii) $(y^2 + z^2)\log(ax + b)$.

4. Prove that, if

$$u \equiv ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy,$$

and if
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} = 0,$$

then
$$a + b + c = 0.$$

5. Prove that, if $r = \sqrt{(x^2 + y^2 + z^2)}$,

then
$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial^2 r}{\partial x^2} = \frac{1}{r} - \frac{x^2}{r^3}.$$

Deduce that
$$\frac{\partial^2 r}{\partial x^2} + \frac{\partial^2 r}{\partial y^2} + \frac{\partial^2 r}{\partial z^2} = \frac{2}{r}.$$

Prove also that

$$\frac{\partial^2}{\partial x^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial y^2}\left(\frac{1}{r}\right) + \frac{\partial^2}{\partial z^2}\left(\frac{1}{r}\right) = 0.$$

Finally, there is a point of notation which the reader may meet in physical applications. Consider, as an illustration, the transformation

$$x = r\cos\theta, \quad y = r\sin\theta$$

between the Cartesian and the polar coordinates of a point. Four variables are involved, of which two are independent—for example, r and θ. When we form the partial differential coefficient $\dfrac{\partial x}{\partial r}$, we naturally have in mind that θ is the other independent variable, and the relation

$$x = r\cos\theta$$

thus gives
$$\frac{\partial x}{\partial r} = \cos\theta = \frac{x}{r}.$$

But it is possible to express x in terms of r and y, in the form

$$x = \sqrt{(r^2 - y^2)},$$

and then
$$\frac{\partial x}{\partial r} = \frac{r}{\sqrt{(r^2 - y^2)}} = \frac{r}{x}.$$

These two values are quite different; they are, indeed, calculated under the quite different hypotheses θ = constant, y = constant respectively.

To make sure what is intended, we often use the notation

$$\left.\frac{\partial x}{\partial r}\right)_\theta$$

to denote the partial differential coefficient of x with respect to r when θ is the other independent variable (being kept constant during differentiation). With such notation, the two formulæ given above may be expressed in the form

$$\left.\frac{\partial x}{\partial r}\right)_\theta = \frac{x}{r},$$

$$\left.\frac{\partial x}{\partial r}\right)_y = \frac{r}{x}.$$

The following examples should serve to make the notation clear.

EXAMPLES III

Given the relations $x = r\cos\theta$, $y = r\sin\theta$, establish the following formulæ:

1. $\left.\frac{\partial x}{\partial r}\right)_\theta = \frac{x}{r}$.
2. $\left.\frac{\partial y}{\partial r}\right)_\theta = \frac{y}{r}$.
3. $\left.\frac{\partial x}{\partial \theta}\right)_r = -y$.
4. $\left.\frac{\partial y}{\partial \theta}\right)_r = x$.
5. $\left.\frac{\partial r}{\partial x}\right)_y = \frac{x}{r}$.
6. $\left.\frac{\partial r}{\partial y}\right)_x = \frac{y}{r}$.
7. $\left.\frac{\partial \theta}{\partial x}\right)_y = -\frac{y}{r^2}$.
8. $\left.\frac{\partial \theta}{\partial y}\right)_x = \frac{x}{r^2}$.
9. $\left.\frac{\partial x}{\partial r}\right)_y = \frac{r}{x}$.
10. $\left.\frac{\partial x}{\partial y}\right)_r = -\frac{y}{x}$.
11. $\left.\frac{\partial \theta}{\partial y}\right)_r = \frac{1}{x}$.
12. $\left.\frac{\partial \theta}{\partial r}\right)_x = \frac{x}{ry}$.

ANSWERS TO EXAMPLES

CHAPTER VII

Examples I:

1. $\log(x+1)$.
2. $\frac{1}{2}\log(2x+1)$.
3. $-\frac{1}{3}\log(2-3x)$.
4. $\frac{1}{2}x^2 + \log x$.
5. $\frac{1}{2}\log\frac{x-1}{x+1}$.
6. $\frac{1}{5}x^5 + 2\log x - \frac{1}{5}x^{-5}$.
7. $\log 2$.
8. $\log 2$.
9. $\frac{1}{3}\log\frac{5}{2}$.
10. $\frac{1}{4}\log 5$.
11. $\frac{1}{4}\log\frac{5}{3}$.
12. $\frac{1}{6}\log\frac{10}{7}$.
13. $\frac{3}{3x+2}$.
14. $2\,\mathrm{cosec}\,2x$.
15. $-\cot x$.
16. $x + 2x\log x$.
17. $x^{n-1} + nx^{n-1}\log x$.
18. $2x/(1+x^2)$.
19. $x\log x - x$.
20. $\frac{1}{2}(\log x)^2$.
21. $\log\sin x$.
22. $\frac{1}{2}x^2\log x - \frac{1}{4}x^2$.
23. $\frac{1}{2}\log\left(\frac{1-\cos x}{1+\cos x}\right)$.
24. $\cos x + \frac{1}{2}\log\left(\frac{1-\cos x}{1+\cos x}\right)$.
25. $\log(x^2+5x+12)$.
26. $\log(x^2-3x+7)$.
27. $\frac{7}{4}\tan^{-1}\left(\frac{x-1}{4}\right) + \log(x^2-2x+17)$.
28. $-12\tan^{-1}(x+3) + \log(x^2+6x+10)$.
29. $9\tan^{-1}\left(\frac{x-4}{3}\right) + \frac{5}{2}\log(x^2-8x+25)$.
30. $-\frac{37}{3}\tan^{-1}\left(\frac{x+5}{3}\right) + \frac{7}{2}\log(x^2+10x+34)$.

Examples II:

1. $\frac{1}{y}\frac{dy}{dx} = \frac{2}{1+x} + \frac{3}{1-x}$.
2. $\frac{1}{y}\frac{dy}{dx} = -2\tan x - \frac{2x}{1+x^2}$.
3. $\frac{1}{y}\frac{dy}{dx} = \frac{1}{x} + 2\cot x + \frac{6x^2}{1-2x^3}$.
4. $\frac{1}{y}\frac{dy}{dx} = \frac{2}{x} + \frac{2}{1+x} - \frac{8x^3}{1+x^4}$.

5. $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{1}{x} + \cot x - \dfrac{3}{1+x} + \dfrac{1}{1-x}$. 6. $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{4x}{1+x^2} - \dfrac{1}{x} + 2\tan x$.

7. $\dfrac{1}{y}\dfrac{dy}{dx} = \dfrac{4}{x} - \dfrac{3}{1-x} - 8\operatorname{cosec} 4x$.

8. $\dfrac{1}{y}\dfrac{dy}{dx} = -\dfrac{2\sin x}{1+\cos x} - \dfrac{1+2x}{1+x+x^2}$.

9. $\dfrac{1}{y}\dfrac{dy}{dx} = -\dfrac{1}{1-x} + \dfrac{4}{1+2x} + \dfrac{9}{1-3x} - \dfrac{16}{1+4x}$.

Examples III:

1. $\tfrac{1}{6}\log\left(\dfrac{x-3}{x+3}\right)$. 2. $x + 2\log(x-1) - 1/(x-1)$.

3. $\log\dfrac{(x-2)^2}{(x-1)}$. 4. $\tfrac{1}{3}x^3 + \tfrac{1}{2}x^2 + x + \log(x-1)$.

5. $\tfrac{1}{2}\tan^{-1}x + \tfrac{1}{4}\log\dfrac{(x+1)^2}{(x^2+1)}$. 6. $\tfrac{1}{3}\log(x+1) - \tfrac{16}{3}\log(x+4) + x$.

7. $\tfrac{1}{4}\log\left(\dfrac{x+1}{x-1}\right) - \dfrac{1}{2(x-1)}$. 8. $-\dfrac{1}{2(x-1)} - \tfrac{1}{2}\tan^{-1}x$.

9. $\tfrac{1}{21}\log(x-2) - \tfrac{1}{42}\log(x^2+4x+9) - \dfrac{4}{21\sqrt{5}}\tan^{-1}\left(\dfrac{x+2}{\sqrt{5}}\right)$.

10. $\log(x-2) - \dfrac{4}{(x-2)} - \dfrac{2}{(x-2)^2}$. 11. $\tfrac{1}{2}\log(x^2-4) - \dfrac{2}{x^2-4}$.

12. $\tfrac{11}{250}\tan^{-1}\tfrac{1}{2}x - \tfrac{1}{250}\log\dfrac{(x-1)^2}{(x^2+4)} + \dfrac{2}{25(x-1)} - \dfrac{1}{10(x-1)^2}$.

13. $\tfrac{1}{2}\log\dfrac{(x-1)(x-3)^3}{(x-2)^4}$.

14. $\dfrac{4}{x-2} + \tfrac{9}{2}\log(x-3) - 4\log(x-2) - \tfrac{1}{2}\log(x-1)$.

15. $\tfrac{5}{338}\tan^{-1}\left(\dfrac{x-3}{2}\right) + \tfrac{3}{169}\log\left(\dfrac{x^2}{x^2-6x+13}\right) - \dfrac{1}{13x}$.

16. $\tfrac{1}{16}\log\left(1+\dfrac{4}{x}\right) - \dfrac{1}{4x}$.

17. $\dfrac{1}{3x^2} - \dfrac{5}{9x} + \tfrac{5}{27}\log\left(1+\dfrac{3}{x}\right)$. 18. $\tfrac{1}{2}x + \tfrac{3}{8}\log(4x+7)$.

19. $x - 3\log(x^2+6x+25) - \tfrac{7}{4}\tan^{-1}\left(\dfrac{x+3}{4}\right)$.

20. $x + \log(x^2+1)$.

21. $\tfrac{1}{4}\log x + \tfrac{3}{8}\log(x^2+4) - \tan^{-1}\tfrac{1}{2}x$.

22. $\tfrac{6}{25}\log\left(\dfrac{x-2}{x+3}\right) - \dfrac{4}{5(x-2)}$. 23. $\tfrac{1}{4}\log\dfrac{(x^2+1)}{(x-1)^2} - \dfrac{1}{2(x-1)}$.

24. $x - \dfrac{16}{17(x+2)} - \tfrac{512}{289}\log(x+2) - \tfrac{611}{289}\log(x^2+2x+17)$

$$- \tfrac{495}{1156}\tan^{-1}\left(\dfrac{x+1}{4}\right).$$

Examples IV:

1. $2e^{2x}$. 2. $2xe^{x^2}$. 3. $e^{5x} + 5xe^{5x}$.

4. $e^x\cos x - e^x\sin x$. 5. $e^{\sin x}\cos x$. 6. $-e^{-x}(1-x)^2$.

7. $e^{2x} - e^{-2x}$. 8. $e^x\sin x + xe^x\sin x + xe^x\cos x$.

9. $\dfrac{e^x(1+2x-x^2)}{(1-x^2)^2}$. 10. $e^x\sin x + (1+e^x)\cos x$.

11. $e^{3x}(3\cos 4x - 4\sin 4x)$. 12. $e^x(\tan x + \sec^2 x)$.

13. $\tfrac{1}{2}e^{2x}$. 14. $-\tfrac{1}{5}e^{-5x}$. 15. e^{x^2}.

16. $-\tfrac{1}{3}e^{-x^3}$. 17. $(x-1)e^x$. 18. $e^x(x^2-2x+2)$.

19. $e^{\sin x}$. 20. $\tfrac{1}{2}e^{\sin^2 x}$. 21. $e^{\tan x}$.

22. $\tfrac{1}{2}e^x(\sin x + \cos x)$.

23. $\tfrac{1}{25}e^{3x}(3\cos 4x + 4\sin 4x)$.

24. $\tfrac{1}{4}e^{2x}(1+2x)$.

Examples V:

1. $aI_n = x^n e^{ax} - nI_{n-1}$.

2. $(a^2+n^2)I_n = e^{ax}\sin^{n-1}x(a\sin x - n\cos x) + n(n-1)I_{n-2}$.

3. $(a^2+b^2n^2)I_n$

$$= e^{ax}\cos^{n-1}bx(a\cos bx + nb\sin bx) + n(n-1)b^2 I_{n-2}.$$

4. $120 - 44e$. 5. $\tfrac{1}{85}(41e^{\frac{1}{2}\pi} - 24)$. 6. $-\tfrac{2}{5}(1+e^{\pi})$.

REVISION EXAMPLES III

1. (i) $(1+x)^{n-1}\{1+n\log(1+x)\}$. (ii) $\dfrac{1}{n}+\log(1+x)$.

2. $\dfrac{d^3z}{dx^3}-4\dfrac{dz}{dx}=0$.

3. $\left(\dfrac{1-2t}{1-t}\right)^2$, $\frac{2}{3}$.

4. $2\cot x$, $\dfrac{-2x}{(1-x^2)^{\frac{1}{2}}(1+x^2)^{\frac{3}{2}}}$.

5. $2\sin x-3\sin^3 x$, $2ae^{ax}\cos ax$, $\dfrac{1}{\sqrt{(x^2+1)}}$, $\dfrac{-2}{4+x^2}$.

6. $\dfrac{1}{2\sqrt{x}}$, 1, $\dfrac{-x}{(x^2+1)^{\frac{3}{2}}}$, $\dfrac{-2x}{\sqrt{(1-x^4)}}$, $\dfrac{1}{x\log_e 10}$.

7. $-\dfrac{1}{x^2}$, $2\cos 2x$, $\dfrac{2}{1+x^2}$, $2xe^{x^2}$.

8. $\dfrac{1}{(1-x)^2}$, $3\sin 6x$, $\log_e x$, $\dfrac{1}{1+x^2}$.

9. (i) $\dfrac{1}{(1-x^2)^{\frac{3}{2}}}$, (ii) $\sec x$, (iii) $\dfrac{-4}{5+3\cos x}$.

10. $-\dfrac{2}{x^3}$, $\dfrac{1-x}{(x+1)^3}$, $\dfrac{2x}{1+x^4}$. $4\operatorname{cosec} 4x$.

11. $1-\dfrac{1}{x^2}$, $-3x(a^2-x^2)^{\frac{1}{2}}$, $\dfrac{1}{(1+x)(1+2x)^{\frac{1}{2}}}$.

12. $\dfrac{12}{x^3}\left(1-\dfrac{3}{x^2}\right)$, $\dfrac{1}{(1-x)^{\frac{3}{2}}(1+x)^{\frac{1}{2}}}$, $\dfrac{2}{1+\sin 2x}$.

13. $\dfrac{1}{1+x^2}$, $\dfrac{-1}{1+x^2}$, $\dfrac{4x}{(1-x^2)^2}$, $\frac{1}{2}e^{-\frac{1}{2}x}(4\cos 2x-\sin 2x)$, $\tan x-\cot x$.

14. $\dfrac{x^2(3+2x)}{(1+x)^2}$, $2\sin x-7\sin^3 x+5\sin^5 x$, $\dfrac{1}{2\sqrt{(x-x^2)}}$.

15. $-\dfrac{4}{x^5}$, $\dfrac{1}{x^2(x^2-1)^{\frac{1}{2}}}$, $\dfrac{7}{(4\cos x+3\sin x)^2}$, $\dfrac{2x^2}{1+x^2}+\log(1+x^2)$.

16. Velocity e^{π}, acceleration $-e^{\pi}$.

18. Tangent: $x\tan t+y-a\sin t=0$.
Normal: $x\cos t-y\sin t-a\cos 2t=0$.

19. $\dfrac{ah\sin\theta}{h-a\cos\theta}$.

20. $\dfrac{3\sqrt{3}}{2}$.

21. $x=0, y=1$, minimum;
$x=1, y=2$, neither;
$x=-2, y=29$, maximum.

28. $(\sqrt{a}+\sqrt{b})^2$.

30. Area: $2+\frac{1}{2}\cot\theta+2\tan\theta$; angle: $\tan^{-1}\dfrac{1}{3\sqrt{2}}$.

31. $(-1,-1)$ minimum; $(1,1)$ maximum; $24x-25y+8=0$.

34. (i) $2\log(2x+3)-\log(x-1)$.
(ii) $\frac{1}{3}\cos^3 x-\cos x$.
(iii) $x^2\sin x+2x\cos x-2\sin x$.

35. (i) $\frac{1}{2}x+\frac{1}{8}\sin 4x$.
(ii) $\log(x-1)+\frac{1}{3}\log(3x+1)$.
(iii) $\frac{1}{4}x^4\log x-\frac{1}{16}x^4$.

36. (i) $\frac{1}{3}\sec^3 x-\sec x$.
(ii) $\frac{1}{2}(1+x^2)\tan^{-1}x-\frac{1}{2}x$.
(iii) $\log(4x-1)-\log(x+1)$.

37. (i) $2\log(2x-1)-2\log(x+2)$.
(ii) $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x$.
(iii) $-\dfrac{1}{(n-1)^2x^{n-1}}\{(n-1)\log x+1\}$.

38. (i) $3\log(x-3)-\frac{2}{3}\log(3x-2)$.
(ii) $\sin x-x\cos x$.
(iii) $\frac{3}{8}a^4\theta+\frac{1}{4}a^4\sin 2\theta+\frac{1}{32}a^4\sin 4\theta$, where $x=a\sin\theta$.

39. $-\sin x-\operatorname{cosec} x$, $\frac{1}{2}e^x(\sin x-\cos x)$, 2·24.

40. $\dfrac{\pi}{3}+\dfrac{\sqrt{3}}{2}$.

41. $\dfrac{1}{\sqrt{(1-x^2)}}$, $\frac{1}{2}x^2-4x+6\log(x+1)$, $2\log 2-\frac{3}{4}$.

42. 1.

43. $x^2\sin x+2x\cos x-2\sin x$, $\frac{2}{3}\log(1+3x)-\log(1-3x)$.

44. $\frac{1}{3}(1+x^2)^{\frac{3}{2}}$, $\frac{1}{2}x\sqrt{(1+x^2)}+\frac{1}{2}\log\{x+\sqrt{(1+x^2)}\}$,
$\frac{1}{2}x\sqrt{(1+x^2)}-\frac{1}{2}\log\{x+\sqrt{(1+x^2)}\}$, $\frac{1}{5}(3-\sqrt{2})$.

45. $\frac{1}{2}x\sqrt{(1-x^2)}+\frac{1}{2}\sin^{-1}x$, $\frac{1}{2}(\sin^{-1}x)^2$, $\dfrac{x}{\sqrt{(1-x^2)}}$, $e^x\cos x$.

46. $2\log(1+\sqrt{x})$, $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x$,
$x+37\log(x-6)-26\log(x-5)$.

47. $\dfrac{1}{\sqrt{2}}\tan^{-1}\left(\dfrac{1}{\sqrt{2}}\tan x\right)$,
$\frac{1}{3}\log(x+1)-\frac{1}{6}\log(x^2-x+1)+\dfrac{1}{\sqrt{3}}\tan^{-1}\left(\dfrac{2x-1}{\sqrt{3}}\right)$,
$x\sin^{-1}x+\sqrt{(1-x^2)}$.

48. $x-\log(1-x)$, $-\cos x+\frac{2}{3}\cos^3 x-\frac{1}{5}\cos^5 x$, $e^x(x-1)$.

50. $-\dfrac{1}{x}-\tan^{-1}x$, $\frac{1}{3}\tan^3 x-\tan x+x$, 1.

51. $x-\log(x+2)$, $x\tan x+\log\cos x$, $\frac{1}{2}$.

52. $\log(2x^2-x-3)$, $\frac{1}{3}\sin^3 x-\frac{1}{5}\sin^5 x$, $\pi-2$.

53. $x-\dfrac{6}{x}-\dfrac{3}{x^3}$, $\frac{2}{15}$, $\frac{1}{2}\log 2$.

54. $\frac{1}{2}\pi$, $16\log 2-\frac{15}{4}$, $\frac{1}{2}\pi^3-12\pi+24$.

55. (i) $\dfrac{7\sqrt{2}}{8}+\frac{3}{8}\log(1+\sqrt{2})$. (ii) 1, $\frac{2}{3}(4\sqrt{2}-5)$.

56. $(m+n)\int_0^{\frac{1}{2}\pi}\sin^m x\cos^n x\,dx=(n-1)\int_0^{\frac{1}{2}\pi}\sin^m x\cos^{n-2}x\,dx$, $\frac{1}{24}$, 0.

57. $\dfrac{e^{ax}}{a^2+c^2}(a\cos cx+c\sin cx)$,
$\dfrac{1}{2a}e^{ax}-\dfrac{e^{ax}}{2a^2+8b^2}(a\cos 2bx+2b\sin 2bx)$.

58. (i) $2(n+1)\int_0^1(1+x^2)^{n+\frac{1}{2}}dx=2^{n+\frac{1}{2}}+(2n+1)\int_0^1(1+x^2)^{n-\frac{1}{2}}dx$,
$\dfrac{67}{24\sqrt{2}}+\frac{5}{16}\log(1+\sqrt{2})$.

CHAPTER VIII

Examples II:

1. $x-\dfrac{x^3}{3!}+\dfrac{x^5}{5!}-\dots+(-1)^n\dfrac{x^{2n+1}}{(2n+1)!}+\dots$.

2. $1-\dfrac{x^2}{2!}+\dfrac{x^4}{4!}-\dots+(-1)^n\dfrac{x^{2n}}{(2n)!}+\dots$.

3. $1-x+x^2-x^3+\dots+(-1)^n x^n+\dots$.

Examples V:

2. $1+3x+\dfrac{(3x)^2}{2!}+\dfrac{(3x)^3}{3!}+\dots+\dfrac{(3x)^n}{n!}+\dots$.

3. $2x-\frac{1}{2}(2x)^2+\frac{1}{3}(2x)^3+\dots+\dfrac{(-1)^{n+1}}{n}(2x)^n+\dots$.

4. $2x-\dfrac{(2x)^3}{3!}+\dfrac{(2x)^5}{5!}-\dots+(-1)^n\dfrac{(2x)^{2n+1}}{(2n+1)!}+\dots$.

5. $1-x+x^2-x^3+\dots+(-1)^n x^n+\dots$.

6. $1+2x+3x^2+\dots+(n+1)x^n+\dots$.

7. $1-\dfrac{(4x)^2}{2!}+\dfrac{(4x)^4}{4!}-\dots+(-1)^n\dfrac{(4x)^{2n}}{(2n)!}+\dots$.

8. $1-2x+\dfrac{(2x)^2}{2!}-\dfrac{(2x)^3}{3!}+\dots+(-1)^n\dfrac{(2x)^n}{n!}+\dots$.

9. $1+x-\frac{1}{2}x^2+\dots+(-1)^{n+1}\dfrac{1.3.5\dots(2n-3)}{n!}x^n+\dots$.

10. $-3x-\frac{1}{2}(3x)^2-\dots-\dfrac{1}{n}(3x)^n-\dots$.

Examples VI:

1. 2·005. 2. 2·995.
3. 2·0017. 4. 2·999.
5. 1·9996. 6. 0·3328.

Examples VII:

1. $1680x^4 \sin x + 1344x^5 \cos x - 336x^6 \sin x - 32x^7 \cos x + x^8 \sin x$.

2. $x^2 \sin x - 8x \cos x - 12 \sin x$.

3. $e^{2x}(122 \cos 3x + 597 \sin 3x)$.

4. $3^4 e^{3x}(9x^3 + 54x^2 + 90x + 40)$.

5. $x^3 \cos\{\frac{1}{2}n\pi + x\} + 3nx^2 \cos\{(n-1)\frac{1}{2}\pi + x\}$
$+ 3n(n-1)x \cos\{(n-2)\frac{1}{2}\pi + x\}$
$+ n(n-1)(n-2)\cos\{(n-3)\frac{1}{2}\pi + x\}$.

6. $2^{n-3}e^{2x}\{8x^3 + 12nx^2 + 6n(n-1)x + n(n-1)(n-2)\}$.

7. $2^5.10.9.8(1-2x)^5(-132x^2 + 55x - 5)$.

8. $\dfrac{2.3^6.12!}{6!}(3x+1)^4(819x^2 + 312x + 28)$.

Examples VIII:

1. $x + \dfrac{1}{2}.\dfrac{x^3}{3} + \dfrac{1}{2}.\dfrac{3}{4}.\dfrac{x^5}{5} + \dfrac{1}{2}.\dfrac{3}{4}.\dfrac{5}{6}.\dfrac{x^7}{7} + \dots$

2. $x - \frac{1}{3}x^3 + \frac{1}{5}x^5 - \frac{1}{7}x^7 + \dots$

REVISION EXAMPLES IV

1. (i) $ax(1 + 2\log bx)e^{ax^2 \log bx}$. (ii) $\dfrac{a}{1 + a^2x^2}$.

$ab, \quad a^2b^2$.

2. $\dfrac{-1}{1+x^2}$, $\dfrac{-1}{(1+x)\sqrt{\{2x(1-x)\}}}$, $\sec x$, $\dfrac{2(-x^2-x+2)}{(x^2-8x-2)^2}$,

$x = 1$ maximum, $x = -2$ minimum.

4. $\sec^m x \tan^{n-1} x\{n + (m+n)\tan^2 x\}$.

$840 \tan^4 x + 640 \tan^2 x + 56$.

5. $\dfrac{x-1}{(x-2)^2(x-3)}$, $\tan^4 x$.

6. $4 \sin^3 x \cos x$, $12 \sin^2 x - 16 \sin^4 x$,

$24 \sin x \cos x - 64 \sin^3 x \cos x$, $256 \sin^4 x - 240 \sin^2 x + 24$,

$x = 0$ minimum, $x = \frac{1}{2}\pi$ maximum.

8. $n\left(x + \dfrac{1}{x}\right)^{n-1}\left(1 - \dfrac{1}{x^2}\right)$, $n \sin x \tan^2 x(\cos x + \sec x)^{n-1}$,

$\dfrac{(-1)^n(n+1)!}{x^{n+2}}$ when $n > 2$, $2 + \dfrac{6}{x^4}$ when $n = 2$,

$2x - \dfrac{2}{x^3}$ when $n = 1$.

10. (ii) $x^2\dfrac{d^2y}{dx^2} - n(n-1)y = 0$.

11. $\dfrac{1}{(x+1)^{\frac{3}{2}}(x-1)^{\frac{1}{2}}}$, $\tan x$, $\dfrac{2^n n!}{(1-2x)^{n+1}} - \dfrac{n!}{(1-x)^{n+1}}$.

12. (i) $\dfrac{(-1)^n n!}{x^{n+1}}$, $(-1)^n\dfrac{n!}{2}\left\{\dfrac{1}{(x-1)^{n+1}} - \dfrac{1}{(x+1)^{n+1}}\right\}$.

(ii) $\sin(x + \frac{1}{2}n\pi)$, $x \sin(x + \frac{1}{2}n\pi) + n \sin\{x + \frac{1}{2}(n-1)\pi\}$.

13. (i) $-2 \cos x$.

(ii) $\dfrac{\cos^3 x - \sin^3 x}{(\cos x + \sin x)^2}$, $-\dfrac{1 - x + 2x^2}{\sqrt{(1+x^2)}}$, $\dfrac{e^x - e^{-x}}{e^x + e^{-x}}$.

14. $-\dfrac{3}{x^4}$, $-\dfrac{3}{2x^5}$, $-\tan x$, $\dfrac{1-2x^2}{\sqrt{(1-x^2)}}$, $\dfrac{e^x(1-4x+x^2)}{(1+x^2)^3}$.

15. (i) $(-1)^n\dfrac{(n+1)!}{x^{n+2}}$. (ii) $2^n \sin(2x + \frac{1}{2}n\pi)$.

(iii) $2^n e^{2x}\Big\{\sin 2x + n \sin(2x + \frac{1}{2}\pi) + \dfrac{n(n-1)}{2!}\sin(2x + \pi) + \dots$
$\dots + \sin(2x + n.\frac{1}{2}\pi)\Big\}$.

$k = 0, 1, 2$.

16. $\dfrac{6x-8}{(1+3x)^3}$, $2 \tan x \sec^2 x\, e^{\tan^2 x}$, $\dfrac{-1}{1+x^2}$.

17. $\sec^2 x\, e^{\tan x}$, $\dfrac{-3}{2(x-2)^{\frac{3}{2}}(x+1)^{\frac{1}{2}}}$.

18. Velocity, $\frac{1}{2}ak(1-2\sin kt)$; acceleration, $-ak^2\cos kt$.

$$t=\frac{\pi}{6k},\quad x=\frac{a}{12}(\pi+6\sqrt{3}).\quad t=\frac{13\pi}{6k},\quad x=\frac{a}{12}(13\pi+6\sqrt{3}).$$

At rest when $t=\dfrac{5\pi}{6k}$, $x=\dfrac{a}{12}(5\pi-6\sqrt{3})$.

Total distance, $\dfrac{a}{3}(\pi+6\sqrt{3})$.

19. Velocity, $e^t\sqrt{2}$; acceleration, $2e^t$.

20. (i) $v^2=p^2(a^2\sin^2 pt+b^2\cos^2 pt)$.

(ii) $f^2=p^4(a^2\cos^2 pt+b^2\sin^2 pt)$.

21. Velocity, $t^2\cos t-4t\sin t-6\cos t$;

Minimum at $t=(4k+1)\frac{1}{2}\pi$; Maximum at $t=(4k-1)\frac{1}{2}\pi$.

22. Velocity, $-ap(\sin pt+\sin 2pt)$;

Acceleration, $-ap^2(\cos pt+2\cos 2pt)$;

$$x=-\frac{3a}{4},\quad -\frac{a}{2},\quad -\frac{3a}{4}.$$

24. $3x-y=0$ at $(1,3)$; $5x+y=0$ at $(-1,5)$.

25. $x-2y=0$.

27. $\cos^{-1}\left\{\dfrac{a(q-p)}{2pq}\right\}$.

29. Maximum.

32. (i) 0·857. (ii) 30·2.

33. (i) 2·004. (ii) 0·515.

34. 8·03.

35. 1·532.

39. $x-\frac{1}{2}x^2+\frac{1}{6}x^3-\frac{1}{12}x^4+\dots$.

40. $-\frac{1}{2}x^2-\frac{1}{12}x^4$.

43. $xy^{(n+2)}-(2x-1-n)y^{(n+1)}+(x-1-2n)y^{(n)}+ny^{(n-1)}=0$.

44. $c_1=b$, $c_2=2ab$, $c_3=3a^2b-b^3$.

45. $y=\alpha$, $y'=0$, $y''=\cot\alpha$, $45{\cdot}028°$.

46. $1+2x+2x^2+\frac{8}{3}x^3+\dots$, 0·9930.

47. 0·06285.

49. $y'=1$, $y''=1$, $y'''=2$, $y^{\text{iv}}=3$.

50. $y'''=2+8y^2+6y^4$,

$y^{\text{iv}}=16y+40y^3+24y^5$,

$y^{\text{v}}=16+136y^2+240y^4+120y^6$,

$x+\frac{1}{3}x^3+\frac{2}{15}x^5$.

52. $-\frac{1}{2}e^{-x^2}$, $\frac{1}{2}(\tan^{-1}x)^2$, $e^x(x^2-2x+2)$.

53. $1-\frac{1}{4}\pi$.

54. $\frac{1}{2}x^2\log x-\frac{1}{4}x^2$, $\log\tan\frac{1}{2}x$, $\frac{1}{4}\log\left(\dfrac{1+x}{1-x}\right)-\frac{1}{2}\tan^{-1}x$.

55. $\sin x-\frac{1}{3}\sin^3 x$, $\log\left(\dfrac{x}{x+1}\right)+\dfrac{1}{x}-\dfrac{1}{2x^2}$,

$x\{(\log x)^3-3(\log x)^2+6\log x-6\}$, $\dfrac{2}{\sqrt{5}}\tan^{-1}(\sqrt{5}\tan\frac{1}{2}x)$.

56. $12x+8\sin 2x+\sin 4x$, $x^3\{9(\log x)^2-6\log x+2\}$,

$\log\left(\dfrac{1+x}{1-x}\right)+2\tan^{-1}x-2x$, $\dfrac{a^x}{\log a}$.

57. $4\log x-2\log(1+x)-\log(1+x^2)-2\tan^{-1}x$,

$2\log 2-1$, $\frac{1}{2}\tan^{-1}\frac{1}{2}$.

58. $\frac{1}{6}\log\left(\dfrac{1+x^3}{1-x^3}\right)$, $\sin^{-1}(x-1)$,

$x\tan x+\log\cos x$, $\frac{1}{3}\tan^3 x-\tan x+x$.

59. $\dfrac{1}{6(1-3x)^2}$, $\frac{1}{2}\tan 2x-x$, $(1+x^2)\tan^{-1}x-x$, $\tan^{-1}(\sin x)$.

60. $\frac{8}{15}$, $\frac{3}{8}\pi$, $\frac{2}{5}(e^{2\pi}-1)$.

61. (i) $\dfrac{1}{1+x}+\log\left(\dfrac{x}{1+x}\right)$, $\log(x^2+2x+2)+\tan^{-1}(x+1)$.

62. $\frac{1}{3}\sin 3x-\frac{1}{9}\sin^3 3x$, $\dfrac{2}{\sqrt{5}}\tan^{-1}(\sqrt{5}\tan\frac{1}{2}x)$,

$-x^3\cos x+3x^2\sin x+6x\cos x-6\sin x$,

$(1-x^2)y-4x^2+Ax+B$.

64. $\frac{3}{2}\pi a^2$, $(\frac{5}{6}a, 0)$.

65. $x\sec\xi + y - \sin\xi - \xi\sec\xi = 0$.

67. $\frac{4}{3}\pi a^2 b$.

68. $\frac{a}{3}\cdot\frac{(2ac+3b)}{(ac+2b)}$.

69. $y = 2+3x-x^3$; $(1,4), (-1,0)$; $\frac{14}{15}$.

71. $\frac{1}{2}x - \frac{1}{4}\sin 2x$, $\frac{1}{3}\cos^3 x - \cos x$, $x^2\sin x + 2x\cos x - 2\sin x$.

72. $\frac{4}{3}a^2$, $\frac{8}{3}\pi a^2(2\sqrt{2}-1)$.

73. $\frac{16}{3}$, $(\frac{21}{5}, -\frac{2}{5})$.

75. $\frac{18}{5}a^2$, $\frac{81}{160}a^2$.

77. $\frac{2555}{558}a^2$.

78. $\left(\frac{128\sqrt{2}}{105}\frac{a}{\pi}, 0\right)$; $\frac{32\sqrt{2}}{105}\pi a^3$.

79. $(\frac{4}{7}, 0)$; $\frac{32\sqrt{2}}{105}\pi$.

80. $\frac{4}{3}a^2$, $(\frac{3}{5}a, \frac{3}{4}a)$; $(\frac{2}{3}a, 0)$, $\frac{8}{3}\pi\rho a^5$.

83. $I_1 = 1$, $I_2 = \frac{1}{4}\pi$, $I_3 = \frac{2}{3}$, $I_4 = \frac{3}{16}\pi$.

84. $\sqrt{\left(\frac{8\pi a^2}{21}\right)}$.

85. $\frac{3}{2}\pi a^2$, $\frac{8}{3}\pi a^3$.

86. -1 and 0, 0 and 1, 2 and 3, $2{\cdot}88$.

87. $2{\cdot}426$.

88. $-1{\cdot}844$.

89. $\frac{2^n}{n!}\sin\frac{1}{3}n\pi$.

90. $x - \frac{3}{2}x^2 + \frac{11}{6}x^3 - \frac{25}{12}x^4 + \dots$.

CHAPTER IX

Examples I:

1. $3\cosh 3x$.
2. $4\cosh 2x\sinh 2x$.
3. $\tanh x + x\,\mathrm{sech}^2 x$.
4. $4\cosh(2x+1)\sinh(2x+1)$.
5. $\cosh x\cos x - \sinh x\sin x$.
6. $2\,\mathrm{sech}\,x\sin x\cos x - \mathrm{sech}\,x\tanh x\sin^2 x$.
7. $3(1+x)^2\cosh^3 3x + 9(1+x)^3\cosh^2 3x\sinh 3x$.
8. $2x\tanh^2 4x + 8x^2\tanh 4x\,\mathrm{sech}^2 4x$.
9. $\coth x$.
10. 1.
11. $\cosh x e^{\sinh x}$.
12. $e^{-\tanh x}(1 - x\,\mathrm{sech}^2 x)$.

13. $\frac{1}{4}\cosh 4x$.
14. $\frac{1}{4}\sinh 2x - \frac{1}{2}x$.
15. $\frac{1}{4}\sinh 2x + \frac{1}{2}x$.
16. $x\cosh x - \sinh x$.
17. $\frac{1}{4}e^{2x} + \frac{1}{2}x$.
18. $\frac{1}{8}\cosh 4x + \frac{1}{4}\cosh 2x$.
19. $\frac{1}{4}x\sinh 2x - \frac{1}{8}\cosh 2x - \frac{1}{4}x^2$.
20. $\sinh x + \frac{1}{3}\sinh^3 x$.
21. $x - \tanh x$.
22. $x^2\sinh x - 2x\cosh x + 2\sinh x$.
23. $\frac{1}{14}e^{7x} + \frac{1}{6}e^{-3x}$.
24. $\frac{1}{2}\tanh^2 x$.

Examples II:

1. $\cosh^{-1}x \pm \frac{x}{\sqrt{(x^2-1)}}$.
2. $\frac{2x}{\sqrt{(x^4+2x^2+2)}}$.
3. $\frac{1}{1-x^2}$.
4. $\frac{\pm 1}{x\sqrt{(1-x^2)}}$
5. $\frac{-1}{x\sqrt{(1+x^2)}}$.
6. $\frac{\pm 1}{\sqrt{(x^2-1)}\cosh^{-1}x}$.
7. $\cosh^{-1}(x^2+1) \pm \frac{2x}{\sqrt{(x^2+2)}}$
8. $\pm\frac{2\cosh^{-1}x}{\sqrt{(x^2-1)}}$.
9. $\frac{-1}{\sqrt{(x^2+1)}(\sinh^{-1}x)^2}$.
10. $\cosh^{-1}\frac{1}{2}x$ $(x > 2)$.
11. $\frac{1}{3}\sinh^{-1}3x$.
12. $\frac{1}{2}\cosh^{-1}\frac{2}{3}x$ $(x > \frac{3}{2})$.
13. $\cosh^{-1}\left(\frac{x+1}{2}\right)$ $(x > 1)$.
14. $\sinh^{-1}\left(\frac{x+1}{2}\right)$.
15. $\frac{1}{2}\cosh^{-1}\left(\frac{2x-1}{4}\right)$ $(x > \frac{5}{2})$.

CHAPTER X

Examples II:

1. $x^2 - y^2 = a^2$.
2. $x^2 - y^2 = a^2$.
3. $x^2 - y^2 = a^2$.
4. $x^2 + y^2 = a^2$.
5. $4(x-2a)^3 - 27ay^2 = 0$.
6. $x = \frac{1}{2}c\left(3t + \frac{1}{t^3}\right)$, $y = \frac{1}{2}c\left(\frac{3}{t} + t^3\right)$.

Examples III:

1. $x = a(2+3t^2), \quad y = -2at^3.$

2. $x = \dfrac{1}{a}(a^2-b^2)\cos^3 t, \quad y = -\dfrac{1}{b}(a^2-b^2)\sin^3 t.$

3. $x = 2a\sec^3 t, \quad y = -2a\tan^3 t.$

REVISION EXAMPLES V

1. Tangent, $x\sin\psi - y\cos\psi - 2a\psi\sin\psi = 0$;

   normal, $x\cos\psi + y\sin\psi - 2a\psi\cos\psi - 2a\sin\psi = 0.$

2. $t = \sinh^{-1}\left(\dfrac{s}{\sqrt{2}}\right)$ 3. $\dfrac{2}{\sqrt{\lambda}}(c+\lambda)^{\frac{3}{2}}.$

4. $(1, \frac{1}{4}).$

6. $\dot{x} = -2a\sin 2t - 2a\sin t, \quad \dot{y} = 2a\cos 2t + 2a\cos t;$

   speed, $4a\cos\frac{1}{2}t$ (numerical value).

9. $\dfrac{t(2+t)}{1+t}.$ 10. $\frac{3}{8}+\frac{1}{2}\cosh 2\theta + \frac{1}{8}\cosh 4\theta.$

13. $2\pi a\left(x\sinh\dfrac{x}{a} - a\cosh\dfrac{x}{a} + a\right).$ 14. $8a.$

15. $\frac{27}{2}.$ 16. $1, \quad (0, 2).$

17. $(b^2/2a).$ 18. $\dfrac{1}{2a}.$

20. $\dfrac{13^{\frac{3}{2}}a}{6}.$ 21. $\dfrac{13^{\frac{3}{2}}a}{4}.$

22. $\sqrt{2}(1+2x+2x^2)^{\frac{3}{2}}, \quad (-1, -1).$

24. $\frac{1}{12}\pi a^2.$ 26. $-\frac{2}{9}\sqrt{3}$ at $x = \frac{1}{2}\sqrt{2}.$

REVISION EXAMPLES VI

6. $f_{n+2}(x) = f_n''(x) + 4xf_n'(x) + 2(1+2x^2)f_n(x).$

7. $y = \displaystyle\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}(2n-1)^2(2n-3)^2\ldots 3^2 x^{2n+1}.$

13. $f(x), f'(x), f''(x)$ continuous everywhere in the range;

    $f'''(x)$ has a discontinuity at $x = 1$;

    Maximum.

16. $\frac{1}{4}.$ 17. $0{\cdot}55.$

18. $y_{n+2} + a\left\{y_1 y_{n+1} + ny_2 y_n + \dfrac{n(n-1)}{2!}y_3 y_{n-1} + \ldots + y_{n+1}y_1\right\} = 0.$

19. $1 + \sqrt{3}x + x^2 - \frac{1}{3}x^4 - \frac{2}{15}\sqrt{3}x^5 - \frac{4}{45}x^6 + \ldots + \left(\dfrac{2^r}{r!}\cos\dfrac{r\pi}{6}\right)x^r + \ldots;$

    $2^{-n}e^{x\sqrt{3}}\cos\left(x - \dfrac{n\pi}{6}\right)$ + arbitrary polynomial of degree $n-1$.

21. $1{\cdot}8, \quad 4{\cdot}5$ (radians).

23. $r^k = a^k \sin k\theta.$

26. $-\dfrac{1}{27} \pm \dfrac{1}{\sqrt{10}}.$

27. $\dfrac{dy}{dx} = \dfrac{dy}{dt}\bigg/\dfrac{dx}{dt}, \quad \dfrac{d^2y}{dx^2} = \dfrac{\dfrac{dx}{dt}\dfrac{d^2y}{dt^2} - \dfrac{d^2x}{dt^2}\dfrac{dy}{dt}}{\left(\dfrac{dx}{dt}\right)^3},$

    $\dfrac{dx}{dy} = \dfrac{dx}{dt}\bigg/\dfrac{dy}{dt}, \quad \dfrac{d^2x}{dy^2} = \dfrac{\dfrac{d^2x}{dt^2}\dfrac{dy}{dt} - \dfrac{dx}{dt}\dfrac{d^2y}{dt^2}}{\left(\dfrac{dy}{dt}\right)^3}.$

28. $\frac{1}{3}\{b + \sqrt{(3a^2+b^2)}\}.$ 29. Equality when $e^{x-1} = y$.

30. (ii) Minimum. 35. $\frac{1}{3}\pi.$

37. $\frac{8}{3}\pi.$

38. $x = -\frac{11}{12}\pi, \quad y = -0{\cdot}65$, maximum;

    $x = -\frac{13}{20}\pi, \quad y = -1{\cdot}19$, minimum;

    $x = -\frac{1}{4}\pi, \quad y = 0$, inflexion;

    $x = \frac{3}{20}\pi, \quad y = 1{\cdot}19$, maximum;

    $x = \frac{5}{12}\pi, \quad y = 0{\cdot}65$, minimum;

    $x = \frac{11}{20}\pi, \quad y = 0{\cdot}73$, maximum;

    $x = \frac{19}{20}\pi, \quad y = 0{\cdot}44$, minimum.

40. $(1, 2)$, $\left(\frac{1}{\sqrt{2}}, 3-\frac{1}{\sqrt{2}}\right)$; $\sqrt{2}-1$.

41. (i) Maximum, $y = \frac{1}{2}(\sqrt{2}-1)$; minimum, $y = -\frac{1}{2}(\sqrt{2}+1)$.

(ii) Inflexions, $(1, 0)$, $\{-2+\sqrt{3}, -\frac{1}{4}(3-\sqrt{3})\}$,
and $\{-2-\sqrt{3}, -\frac{1}{4}(3+\sqrt{3})\}$.

42. $A = \dfrac{p}{\sqrt{(p^2+k^2)}}$.

47. $$\frac{dy}{dx} = \frac{\tan\alpha + \dfrac{d\eta}{d\xi}}{1-\dfrac{d\eta}{d\xi}\tan\alpha}, \quad \frac{d^2y}{dx^2} = \frac{\dfrac{d^2\eta}{d\xi^2}}{\left(\cos\alpha - \dfrac{d\eta}{d\xi}\sin\alpha\right)^3}.$$

49. $\dfrac{5\sqrt{5}}{6}$.

50. $x = 0$, maximum, $\kappa = -16$;
$x = \frac{4}{5}$, minimum, $\kappa = \frac{144}{25}$.

51. $3a\cos t\sin t$ (numerical value of).

52. $(0, 0)$, $(2, 2)$.

53. $\frac{1}{2}$, $2\sqrt{2}$.

56. Minimum.

58. $\frac{1}{16}\log\left(\dfrac{x^2+2x+2}{x^2-2x+2}\right)+\frac{1}{8}\tan^{-1}(x+1)+\frac{1}{8}\tan^{-1}(x-1)$,

$\dfrac{e^{ax}}{a^2+b^2}(a\cos bx + b\sin bx)$,

$\frac{1}{2}x^2 - \frac{1}{2}x\sqrt{(x^2-1)}+\frac{1}{2}\cosh^{-1}x$.

60. (b) $\dfrac{A}{x}$, A an arbitrary constant.

61. $a^2(\alpha+3\sin\alpha)$, where $\cos\alpha = -\frac{1}{3}$ $(\frac{1}{2}\pi<\alpha<\pi)$.

62. (i) $\frac{8}{15}\log\sin x - \frac{1}{15}x\cos x(3\operatorname{cosec}^5 x + 4\operatorname{cosec}^3 x + 8\operatorname{cosec} x)$
$-\frac{1}{20}\operatorname{cosec}^4 x - \frac{2}{15}\operatorname{cosec}^2 x$.

(ii) $\dfrac{b^5}{a^6}\log\left(\dfrac{a+bx}{x}\right)-\dfrac{b^4}{a^5x}+\dfrac{b^3}{2a^4x^2}-\dfrac{b^2}{3a^3x^3}+\dfrac{b}{4a^2x^4}-\dfrac{1}{5ax^5}$.

63. $\frac{1}{2}at\sqrt{(1+t^2)}+\frac{1}{2}a\sinh^{-1}t$.

64. (i) $n = 4k, \lambda = 1$; $n = 4k+1, \lambda = n$;
$n = 4k+2, \lambda = -1$; $n = 4k+3, \lambda = -n$.

(ii) $(\frac{1}{3}+\frac{1}{2}\pi)a^2$.

65. $(\tan^{-1}x)^2 - 2x\tan^{-1}x + x\tan^{-1}x\log(1+x^2)$
$+\log(1+x^2) - \frac{1}{4}\{\log(1+x^2)\}^2$.

68. $\phi(x) = (1+x)(3+x)$.
$f(x) = (1+x)^2$, $g(x) = 3(1+x)^2$.

71. $A = \frac{3}{8}$, $B = \frac{2}{3}$, $C = \frac{5}{8}$, $D = \frac{3}{8}$.

72. $A = \frac{1}{16}$, $B = \frac{1}{6}$, $C = -\frac{1}{16}$, $D = \frac{1}{16}$.

76. $a^2 I_n = a(1-x^2)^n\sinh ax + 2nx(1-x^2)^{n-1}\cosh ax$
$-2n(2n-1)I_{n-1}+4n(n-1)I_{n-2}$ $(n \geqslant 2)$.

$I_3 = \dfrac{888}{e} - 120e$.

77. $\cos^{q-1}x\{p\sin^{p-1}x - (p+q+k^2+pk^2)\sin^{p+1}x$
$+k^2(p+q+1)\sin^{p+3}x\}$.
$k^2(m-1)I_m - (1+k^2)(m-2)I_{m-2}+(m-3)I_{m-4} = 0$.

79. Volume $5\pi^2a^3$, area $\frac{64}{3}\pi a^2$.

81. $x = a(\cos t\cos 3t + 3\sin t\sin 3t)$,
$y = a(\cos t\sin 3t - 3\sin t\cos 3t)$.

84. $\dfrac{1}{x^2}+\dfrac{1}{y^2} = \dfrac{1}{a^2}$; $\frac{2}{3}a$.

85. $P(-8at^3, -6at)$;
Normal, $4t^2x + y + 6at + 32at^5 = 0$;
Inflexion at origin.

86. Normal, $2x + 3ty - 3at^4 - 2at^2 = 0$;
Centre of curvature, $(-\frac{9}{2}at^4 - at^2, 4at^3 + \frac{4}{3}at)$;
Radius $\frac{1}{6}at(4+9t^2)^{\frac{3}{2}}$.

87. Envelope, $x = -f'(t)$, $y = f(t) - tf'(t)$;
$\rho = f''(t)(1+t^2)^{\frac{3}{2}}$.

88. Envelope, $x = a\sin t(3-2\sin^2 t)$,
$y = a\cos t(3-2\cos^2 t)$.

89. Tangent, $x\sin t - y\cos t + \cos 2t = 0$;
Normal, $x\cos t + y\sin t - 2\sin 2t = 0$.

90. Tangent, $t(3+t^2)x-2y-t^3=0$;
Normal, $2x+t(3+t^2)y-t^2(2+t^2)=0$.

92. $\pi(ab-cdk)$. 93. $\frac{16}{5}ab$.

94. $\dfrac{1}{8e^2}$. 95. $\frac{6}{5}\sqrt{3}$.

98. $2-\frac{1}{2}\pi$. 102. $2c\sinh\dfrac{a}{c}$; $\pi c\left(2a+c\sinh\dfrac{2a}{c}\right)$.

103. $\frac{64}{3}\pi a^2$. 105. Volume, $2\pi^2a^2h$; area $4\pi^2ah$.

106. $\dfrac{4\sqrt{2}}{\pi}a$, $\dfrac{2}{\pi a}\log(1+\sqrt{2})$. 107. $f+\dfrac{a^2}{3f}$.

108. $\dfrac{1}{a(a^2-f^2)}$ if $f<a$, $\dfrac{1}{f(f^2-a^2)}$ if $f>a$. 109. $\frac{6}{5}a$.

CHAPTER XI

Examples I:

1. 1 and 3; $2\pm\sqrt{3}$; $2\pm i$.
2. -5 and -3; $-4\pm\sqrt{5}$; $-4\pm 2i$.
3. -1 and 3; $1\pm\sqrt{5}$; $1\pm 3i$.

Examples II:

1. $10+3i$. 2. $-8+8i$. 3. $34+22i$. 4. $-16-3i$. 5. $2+7i$.
6. $0+0i$. 7. $0+2i$. 8. $2+11i$. 9. $-2-16i$. 10. $-10+0i$.

Examples III:

1. $-7+22i$. 2. $26+2i$.
3. $7-i$. 4. a^2+b^2.
5. $-3+4i$. 6. $\cos(A+B)+i\sin(A+B)$.
7. 10. 8. $-46+9i$.

Examples IV:

1. i. 2. $\frac{24}{25}+\frac{7}{25}i$. 3. $\frac{5}{7}-\frac{6}{7}i$.
4. $\frac{5}{17}-\frac{14}{17}i$. 5. $-\frac{27}{37}+\frac{23}{37}i$. 6. $\cos\theta+i\sin\theta$.

Examples V:

1. $5-2i$, $-1+8i$, $21-i$, $-\frac{9}{34}+\frac{19}{34}i$.
2. $-7+9i$, $-1-5i$, $-2-34i$, $\frac{13}{29}+\frac{11}{29}i$.
3. $4+2i$, $4-2i$, $8i$, $-2i$. 4. $3+i$, $3+3i$, $2-3i$, $-2+3i$.
5. 4, $6i$, 13, $-\frac{5}{13}+\frac{12}{13}i$. 6. -6, $8i$, 25, $-\frac{7}{25}-\frac{24}{25}i$.
7. $\frac{3}{25}-\frac{4}{25}i$. 8. $-\frac{5}{169}-\frac{12}{169}i$.
9. $-\frac{1}{6}i$. 10. $-\frac{3}{50}+\frac{2}{25}i$.
11. $-2\pm 3i$. 12. $1\pm i$.
13. $-3\pm i$. 14. $\pm\frac{3}{2}i$.
15. $4\pm 3i$. 16. $-2\pm i$.

Examples VII:

3. 2, $-30°$; 5, $53°\ 7'$; 13, $112°\ 36'$; 3, $0°$; 10, $-53°\ 7'$; 2, $-90°$.
5. Straight line, $4x+10y-21=0$.

Examples VIII:

1. (*a*) (3, 2); (*b*) (2, 1); (*c*) (4, 7).
2. (i) (*a*) (1, 0); (*b*) (3, -5); (*c*) (3, -5).
(ii) (*a*) (2, -1); (*b*) (2, 1); (*c*) (5, 0).

Examples IX:

1. (i) $-0{\cdot}5+0{\cdot}866i$.
(ii) $\pm(0{\cdot}866+0{\cdot}5i)$.
(iii) $0{\cdot}940+0{\cdot}342i$, $-0{\cdot}766+0{\cdot}643i$, $-0{\cdot}174-0{\cdot}985i$.
(iv) $\pm(0{\cdot}966+0{\cdot}259i)$, $\pm(0{\cdot}259-0{\cdot}966i)$.

2. (i) $7+24i$.

(ii) $\pm(2{\cdot}121+0{\cdot}707i)$.

(iii) $1{\cdot}671+0{\cdot}364i,\ -1{\cdot}151+1{\cdot}265i,\ -0{\cdot}520-1{\cdot}629i$.

(iv) $\pm(1{\cdot}476+0{\cdot}239i),\ \pm(0{\cdot}239-1{\cdot}476i)$.

3. (i) $119-120i$.

(ii) $\pm(3{\cdot}535+0{\cdot}708i)$.

(iii) $2{\cdot}331+0{\cdot}308i,\ -1{\cdot}432+1{\cdot}865i,\ -0{\cdot}899-2{\cdot}173i$.

(iv) $\pm(1{\cdot}890+0{\cdot}187i),\ \pm(0{\cdot}187-1{\cdot}890i)$.

Examples X:

1. $4(\cos\pi+i\sin\pi)$.

2. $\dfrac{1}{4\sqrt{2}}\left(\cos\dfrac{5\pi}{4}+i\sin\dfrac{5\pi}{4}\right)$.

3. $64(\cos 2\pi+i\sin 2\pi)$.

4. $2^{\frac{1}{3}}\left\{\cos\dfrac{\pi}{18}(12k-1)+i\sin\dfrac{\pi}{18}(12k-1)\right\}$.

5. $2^{\frac{1}{12}}\left\{\cos\dfrac{\pi}{24}(8k-1)+i\sin\dfrac{\pi}{24}(8k-1)\right\}$.

6. $2^{\frac{1}{4}}\left\{\cos\dfrac{7\pi}{24}(12k+1)+i\sin\dfrac{7\pi}{24}(12k+1)\right\}$.

7. (i) $1,\ \cos\dfrac{2\pi}{5}\pm i\sin\dfrac{2\pi}{5},\ \cos\dfrac{4\pi}{5}\pm i\sin\dfrac{4\pi}{5}$.

(ii) $\pm1,\ \pm\frac{1}{2}(1\pm i\sqrt{3})$.

Examples XI:

1. $e^{-(\frac{1}{2}\pi+2k\pi)}\{\cos(\frac{1}{2}\log 2)+i\sin(\frac{1}{2}\log 2)\}$.
2. $-2e^{(2k\pi-\frac{1}{4}\pi)}\{\sin(\frac{1}{2}\log 2)+i\cos(\frac{1}{2}\log 2)\}$.
3. $e^{-(\frac{1}{12}\pi+k\pi)}\{\cos(\frac{1}{2}\log 2)+i\sin(\frac{1}{2}\log 2)\}$.
4. $-8e^{-(4k\pi-\frac{3}{2}\pi)}\{\cos(\log 4)+i\sin(\log 4)\}$.
5. $2e^{-(k\pi-\frac{1}{12}\pi)}\{\cos(\frac{1}{2}\log 2-\frac{1}{6}\pi)+i\sin(\frac{1}{2}\log 2-\frac{1}{6}\pi)\}$.
6. $-4e^{-(k\pi+\frac{1}{8}\pi)}\{\cos(\frac{1}{4}\log 2)+i\sin(\frac{1}{4}\log 2)\}$.

Examples XII:

1. $\dfrac{1-\cos\theta+\cos n\theta-\cos(n+1)\theta}{2(1-\cos\theta)}$.
2. $\dfrac{\sin\theta+(-1)^{n+1}x^n\{x\sin n\theta+\sin(n+1)\theta\}}{1+2x\cos\theta+x^2}$.
3. $\cos x+x\sin x$.
4. $\frac{1}{2}e^x(\sin x-\cos x)$.
5. $\frac{1}{25}e^{4x}(4\sin 3x-3\cos 3x)$.
6. $\frac{1}{5}xe^{2x}(2\sin x-\cos x)+\frac{1}{25}e^{2x}(4\cos x-3\sin x)$.
7. $\frac{1}{9}\sin 3x-\frac{1}{3}x\cos 3x$.
8. $-\frac{1}{25}xe^{-4x}(4\cos 3x-3\sin 3x)-\frac{1}{625}e^{-4x}(7\cos 3x-24\sin 3x)$.

REVISION EXAMPLES VII

3. $A=\dfrac{\sin a\cos a}{\cos^2 a+\sinh^2 b},\quad B=\dfrac{-\sinh b\cosh b}{\cos^2 a+\sinh^2 b}$.

5. $-\tan\dfrac{\pi}{4n}(4k-1),\ k=1,\dots,n$.

7. $\cos^2 x+\sinh^2 y$.

8. $X=\dfrac{(x_1^2+y_1^2)x_2+(x_2^2+y_2^2)x_1-(x_1+x_2)}{(x_1^2+y_1^2)(x_2^2+y_2^2)-2(x_1x_2-y_1y_2)+1}$.

$Y=-\dfrac{(x_1^2+y_1^2)y_2+(x_2^2+y_2^2)y_1+(y_1+y_2)}{(x_1^2+y_1^2)(x_2^2+y_2^2)-2(x_1x_2-y_1y_2)+1}$.

10. $x-iy,\ r(\cos\theta-i\sin\theta);\ \pm1\pm3i$.

11. $z_1=2(\cos\frac{1}{4}\pi+i\sin\frac{1}{4}\pi),\ z_2=8(\cos\frac{3}{4}\pi+i\sin\frac{3}{4}\pi)$;

$2(\cos\frac{11}{12}\pi+i\sin\frac{11}{12}\pi),\ 2(\cos\frac{19}{12}\pi+i\sin\frac{19}{12}\pi)$.

16. (i) $(-1,0),\ (3,0)$;

(ii) circle, centre $(2,0)$, radius 3;

(iii) ellipse, foci $(1,0),\ (2,0)$, eccentricity $\frac{1}{3}$.

17. $1,\omega,\omega^2,\omega^3,\omega^4$ where $\omega=\cos\frac{2}{5}\pi+i\sin\frac{2}{5}\pi$;

$x=\frac{1}{4}(1+i\cot\frac{1}{5}k\pi),\ k=1,2,3,4$.

19. $(1,\sqrt{3})$, distance 2.

20. $z_3 = 7+2\sqrt{3}+i(4+3\sqrt{3})$ or $7-2\sqrt{3}+i(4-3\sqrt{3})$.
Vertices $(4+4\sin\frac{1}{3}k\pi+6\cos\frac{1}{3}k\pi,\ 6+6\sin\frac{1}{3}k\pi-4\cos\frac{1}{3}k\pi)$,
$k = 0, 1, \ldots, 5$.

21. $\cos\frac{2}{5}k\pi+i\sin\frac{2}{5}k\pi$; $k = 0, 1, \ldots, 4$.

22. $\dfrac{4-5\cos\frac{2}{5}k\pi-3i\sin\frac{2}{5}k\pi}{5-4\cos\frac{2}{5}k\pi}$ $(k = 0, 1, 2, 3, 4)$.

23. $(\sqrt{3}+i)(a+ib)$ and $a+ib$ subtend an angle $\frac{1}{6}\pi$ at the origin, and the distance of $(\sqrt{3}+i)(a+ib)$ from the origin is twice that of $a+ib$ from the origin. C' is at $\pm 2\sqrt{3}+2i$.

24. $\pm(3-2i)$; $1+i$, $\sqrt{2}\cos\frac{11}{12}\pi+i\sqrt{2}\sin\frac{11}{12}\pi$,
$\sqrt{2}\cos\frac{19}{12}\pi+i\sqrt{2}\sin\frac{19}{12}\pi$.

26. $-2300-2100i$.

27. (i) $(1, 1)$, $(1, -1)$,
(ii) all points with $x \geqslant 0$,
(iii) interior of circle, centre $(-\frac{5}{3}, 0)$, radius $\frac{4}{3}$,
(iv) interior of ellipse, foci $(\pm 1, 0)$, eccentricity $\frac{1}{2}$.

29. (i) $\frac{1}{5}$, $-\frac{2}{5}$; $\dfrac{1}{\sqrt{5}}$.
(ii) $\dfrac{2ab}{a^2+b^2}$, $\dfrac{b^2-a^2}{a^2+b^2}$; 1,
(iii) $\dfrac{\cos\frac{1}{2}\alpha\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}\beta}$, $\dfrac{\cos\frac{1}{2}\alpha\sin\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}\beta}$; $\left|\dfrac{\cos\frac{1}{2}\alpha}{\cos\frac{1}{2}\beta}\right|$.

30. $i\pi$, $\frac{1}{2}\log_e 5-i\tan^{-1}\frac{1}{2}$, $(x+iy)\log_e 10$.

32. $1{\cdot}67-0{\cdot}36i$, $-0{\cdot}52+1{\cdot}63i$, $-1{\cdot}15-1{\cdot}27i$.

33. $\frac{1}{2}(z_1+z_2)$; $(2+2\sqrt{3})i$.

34. $\dfrac{1}{2}\left(z+\dfrac{1}{z}\right)$, $\dfrac{1}{2i}\left(z-\dfrac{1}{z}\right)$, $\dfrac{1}{2}\left(z^n+\dfrac{1}{z^n}\right)$, $\dfrac{1}{2i}\left(z^n-\dfrac{1}{z^n}\right)$;
$\dfrac{-1}{2^6}(\sin 7\theta-7\sin 5\theta+21\sin 3\theta-35\sin\theta)$.

37. Circle, centre $(5, -1)$, radius 3; 8, 2.

38. $\dfrac{\sqrt{(10+6y)}}{|10-6y|}$.

39. Amplitude increases from $-\pi$ to π.

40. $2^{\frac{4}{3}}\cos\frac{1}{9}\pi$, $-2^{\frac{4}{3}}\cos\frac{2}{9}\pi$, $-2^{\frac{4}{3}}\cos\frac{4}{9}\pi$.

41. $\dfrac{\cos x\cosh y+i\sin x\sinh y}{\cos^2 x+\sinh^2 y}$.

42. $(a+ib)(a-ib)$, $(c+id)(c-id)$.

CHAPTER XIII

Examples I:

1. 1. 2. $\sqrt{2}$. 3. 2. 4. $\frac{1}{4}\pi$.
5. $\frac{3}{4}\pi$. 6. $\frac{1}{2}$. 7. $-1/a$ $(a<0)$.

REVISION EXAMPLES VIII

1. $\frac{1}{2}\log\dfrac{x-3}{x-1}$; $x+\frac{1}{2}\log\dfrac{(x-3)^9}{x-1}$; $\cos^{-1}(2-x)$.

2. $x+\log\dfrac{(x-3)^5}{x-1}$; $\sqrt{(x^2+2x+2)}-\sinh^{-1}(x+1)$.

3. $\frac{1}{2}(a+b)\pi$.

4. $x+\frac{1}{5}\log\{(x-2)^8(x^2+1)\}-\frac{1}{5}\tan^{-1}x$;
$\frac{3}{7}(x+b)^{\frac{7}{3}}+\frac{3}{4}(a-b)(x+b)^{\frac{4}{3}}$; $\frac{2}{3}\cos^3 x-\cos x-\frac{1}{5}\cos^5 x$.

5. $\frac{1}{4}\log\dfrac{1+x^2}{(1-x)^2}-\frac{1}{2}\tan^{-1}x$;
$\sqrt{(x^2+4x+5)}-2\log\{x+2+\sqrt{(x^2+4x+5)}\}$;
$\frac{1}{2}(a^2+x^2)-a^2\log(a^2+x^2)-\frac{1}{2}a^4/(a^2+x^2)$.

6. $\frac{1}{6}\log\dfrac{(x-1)^{10}}{x^9(x+2)}$; $\sin^{-1}\left(\dfrac{x-1}{x\sqrt{2}}\right)$; $e^x/(x+1)$.

7. $\frac{1}{30}\log\dfrac{(x-3)^8}{x^5(x+2)^3}$; $\frac{3}{8}\sinh^{-1}\left(\dfrac{2x+1}{\sqrt{3}}\right)+\frac{1}{4}(2x+1)\sqrt{(x^2+x+1)}$;
$x\{(\log x)^2-2\log x+2\}$.

8. $u_n = \dfrac{2n+1}{2n+2}u_{n-1}+\dfrac{1}{n+1}.2^{n-\frac{1}{2}}$; $\frac{67}{48}\sqrt{2}+\frac{5}{16}\log(1+\sqrt{2})$.

REVISION EXAMPLES IX

3. (ii) $\dfrac{1}{\sqrt{3}}\tan^{-1}\left\{\dfrac{t\sqrt{3}}{\sqrt{(t^2+4)}}\right\}-\dfrac{1}{2\sqrt{3}}\log\left\{\dfrac{\sqrt{(t^2+4)}+\sqrt{3}}{\sqrt{(t^2+4)}-\sqrt{3}}\right\}$.

4. $\dfrac{\pi^2}{72}+\dfrac{\pi\sqrt3}{6}-1$; $2\log(2+\sqrt3)-\frac{2}{3}\pi$.

6. $\dfrac{2}{\sqrt5}\log\dfrac{\sqrt5+1}{\sqrt5-1}$; $\frac{3}{8}\pi$.

7. $\frac{1}{8}\pi$; $\dfrac{\pi}{2a(a+b)}$; $\dfrac{1}{\sqrt2}\log\dfrac{\sqrt2+\sqrt{(1+x)}}{\sqrt2-\sqrt{(1+x)}}$.

8. $\frac{1}{4}\pi$; $-x/\sqrt{(x^2-1)}$; $\frac{1}{8}x^4-\frac{1}{8}(2x^3-3x)\sin 2x-\frac{3}{16}(2x^2-1)\cos 2x$.

9. $I_n = 2\pi$.

10. $a-\dfrac{a}{\sqrt2}\log(1+\sqrt2)$; $\dfrac{\sqrt2}{16}\left\{7\tan^{-1}\left(\dfrac{1}{\sqrt2}\right)+\sqrt2\right\}$.

11. $1-\frac{1}{4}\pi$; $\frac{3}{4}\pi$; $-ae^a/(1-e^a)$.

12. $\frac{1}{8}(\frac{4}{5}-\log 3)$.

13. $\Sigma\dfrac{\pi}{2a(a^2-b^2)(a^2-c^2)}$.

14. $\frac{1}{4}\pi\sec\frac{1}{2}\alpha$, $\frac{1}{2}\pi\sec\frac{1}{2}\alpha$.

16. $\sqrt2\log\dfrac{\sqrt2-1}{2-\sqrt3}$; $\frac{1}{3}\pi-\frac{1}{3}\log 2-\frac{1}{6}$.

17. $$\begin{cases} 2\tan^{-1}\left\{\left(\dfrac{1+r}{1-r}\right)\tan\dfrac{\delta}{2}\right\}+\delta,\ r<1, \\ -2\tan^{-1}\left\{\left(\dfrac{r+1}{r-1}\right)\tan\dfrac{\delta}{2}\right\}+\delta,\ r>1. \end{cases}$$

Limits $\pi+\delta, r<1$; $-\pi+\delta, r>1$; $I=\delta$ when $r=1$.

18. $x\cos\alpha+\sin\alpha\log\sin(x-\alpha)$;

$\frac{1}{2}\log\dfrac{t}{t-1}-\dfrac{1}{\sqrt7}\tan^{-1}\dfrac{2t+1}{\sqrt7}$, where $t=\tan\frac{1}{2}x$.

19. $\log\left\{x+\dfrac{1}{x}+\sqrt{(x^2+1+x^{-2})}\right\}$.

20. $\dfrac{1}{\sqrt{(1-b^2)}}\log\left\{\dfrac{x\sqrt{(1-b^2)}+\sqrt{(x^2+b^2)}}{\sqrt{(x^2+1)}}\right\}$; $\dfrac{3(b-a)^4\pi}{128}$; $\dfrac{\pi}{2a}\log\dfrac{1+a}{1-a}$.

21. $2+\frac{3}{2}\log 3$; $\log(2+\sqrt3)-\frac{1}{2}\sqrt3$; $\sqrt3-\frac{1}{3}\pi$.

22. $\frac{1}{7}\log\dfrac{x-2}{x+2}+\dfrac{\sqrt3}{7}\tan^{-1}\left(\dfrac{x}{\sqrt3}\right)$; $e^x(x\log x-1)$; $\dfrac{\pi}{4\sqrt2}$.

23. $\frac{1}{4}\pi$.

24. $\frac{1}{2}\log\dfrac{1+t^2}{2+t^2}+\dfrac{1}{\sqrt2}\tan^{-1}\left(\dfrac{t}{\sqrt2}\right)-\tan^{-1}c+\dfrac{1}{2\sqrt2}\log\dfrac{\sqrt2+s}{\sqrt2-s}$;

$\tan\theta-\sec\theta\equiv-\dfrac{2}{1+\tan\frac{1}{2}\theta}$, differing by a constant.

25. $\dfrac{4}{x+1+\sqrt{(x^2-1)}}+\log\{x+\sqrt{(x^2-1)}\}$.

26. $9(n-1)I_n-5(2n-3)I_{n-1}+(n-2)I_{n-2}=-\dfrac{4\sin x}{(5+4\cos x)^{n-1}}$.

27. $(n-2)u_{n-2}-(n-1)u_n=0$, $n>2$;
$u_1=\frac{1}{2}\pi$, $u_2=1$.

28. $\dfrac{8}{\alpha(\alpha+2)(\alpha+4)}$.

31. If the given integral is u_{n+1}, then
$$2n(ab-h^2)u_{n+1}-(2n-1)au_n=\frac{ax+h}{(ax^2+2hx+b)^n}.$$

33. $2(n+1)u_{n+2}-(2n+1)u_{n+1}+nu_n=-\dfrac{(1+x^2)^{\frac12}}{(1+x)^{n+1}}$.

34. (i) $\frac{1}{12}\sqrt3\log\dfrac{2\sqrt{(1+x^2)}+x\sqrt3}{2\sqrt{(1+x^2)}-x\sqrt3}$.

(ii) $$u_n-\frac{2n(2n-1)}{a^2}u_{n-1}+\frac{2n(2n-2)}{a^2}u_{n-2} = \frac{1}{a}e^{ax}(1+x^2)^n-\frac{2nx}{a^2}e^{ax}(1+x^2)^{n-1}.$$

35. $\pi/\sqrt{(a^2-1)}$; $u_0=\log(1+\operatorname{cosec}\frac{1}{2}\alpha)$, $u_1=u_0\cos\alpha-1+2\sin\frac{1}{2}\alpha$.

36. $u_n=nu_{n-1}$, $u_n=n!$; $u_n+\dfrac{n(n-1)}{4}u_{n-2}=\dfrac{\pi^{n+1}}{2(n+1)}$.

38. $\dfrac{1}{6}-\dfrac{\sqrt2}{8}\tan^{-1}\left(\dfrac{\sqrt2}{4}\right)$.

40. $(n-1)u_n=(n-2)u_{n-2}+2^{(n-2)/2}$; $u_1=\log(1+\sqrt2)$, $u_2=1$.

41. $\dfrac{x(8x^4+8x^2+3)\sqrt{(x^2+1)}}{3(2x^2+1)^3}$.

42. Formula: see p. 208.

$I_3=\frac{1}{2}\log\{x+1+\sqrt{(x^2+2x+2)}\}+\frac{1}{6}(2x^2-5x+7)\sqrt{(x^2+2x+2)}$.

APPENDIX

Examples I:

1. $4x^3y^3,\ 3x^4y^2,\ 12x^2y^3,\ 12x^3y^2,\ 12x^3y^2,\ 6x^4y.$
2. $y^2,\ 2xy,\ 0,\ 2y,\ 2y,\ 2x.$
3. $2x,\ 2y,\ 2,\ 0,\ 0,\ 2.$
4. $e^x\cos y,\ -e^x\sin y,\ e^x\cos y,\ -e^x\sin y,\ -e^x\sin y,\ -e^x\cos y.$
5. $\dfrac{1}{x+y},\ \dfrac{1}{x+y},\ -\dfrac{1}{(x+y)^2},\ -\dfrac{1}{(x+y)^2},\ -\dfrac{1}{(x+y)^2},\ -\dfrac{1}{(x+y)^2}.$
6. $\dfrac{1}{x},\ \dfrac{1}{y},\ -\dfrac{1}{x^2},\ 0,\ 0,\ -\dfrac{1}{y^2}.$
7. $3x^2\sin^2 y,\ 2x^3\sin y\cos y,\ 6x\sin^2 y,\ 6x^2\sin y\cos y,$
   $6x^2\sin y\cos y,\ 2x^3(\cos^2 y-\sin^2 y).$
8. $ye^{xy}\sin x+e^{xy}\cos x,\ xe^{xy}\sin x,$
   $y^2e^{xy}\sin x+2ye^{xy}\cos x-e^{xy}\sin x,$
   $e^{xy}\sin x+xye^{xy}\sin x+xe^{xy}\cos x,$
   $e^{xy}\sin x+xye^{xy}\sin x+xe^{xy}\cos x,\ x^2e^{xy}\sin x.$
9. $\tan^{-1}y,\ \dfrac{x}{1+y^2},\ 0,\ \dfrac{1}{1+y^2},\ \dfrac{1}{1+y^2},\ -\dfrac{2xy}{(1+y^2)^2}.$
10. $\sec x\tan x,\ \sec y\tan y,\ \sec x\tan^2 x+\sec^3 x,\ 0,\ 0,$
    $\sec y\tan^2 y+\sec^3 y.$
11. $e^x\sin^2 2y,\ 4e^x\sin 2y\cos 2y,\ e^x\sin^2 2y,\ 4e^x\sin 2y\cos 2y,$
    $4e^x\sin 2y\cos 2y,\ 8e^x\cos^2 2y-8e^x\sin^2 2y.$
12. $e^y,\ xe^y,\ 0,\ e^y,\ e^y,\ xe^y.$

Examples II:

3. (i) $ae^{ax}\sin(by+cz),\ -b^2e^{ax}\sin(by+cz),\ ac\,e^{ax}\cos(by+cz);$

   (ii) $yze^{-x^2}-2x^2yze^{-x^2},\ 0,\ ye^{-x^2}-2x^2ye^{-x^2};$

   (iii) $\dfrac{a(y^2+z^2)}{ax+b},\ 2\log(ax+b),\ \dfrac{2az}{ax+b}.$

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