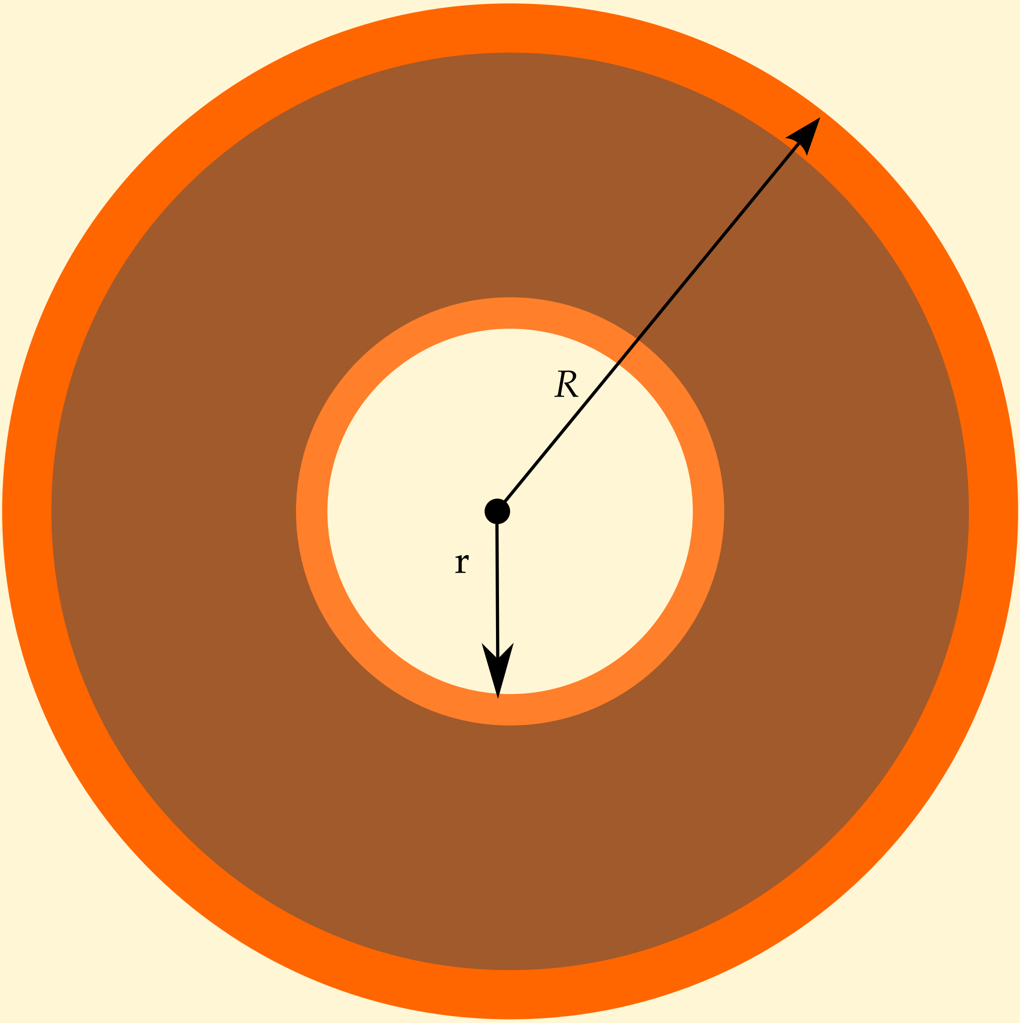


# THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

A. G. SVESHNIKOV. A. N. TIKHONOV

MIR PUBLISHERS

ИЗДАТЕЛЬСТВО • МИР •

А. Г. СВЕШНИКОВ, А. Н. ТИХОНОВ

# ТЕОРИЯ ФУНКЦИЙ КОМПЛЕКСНОЙ ПЕРЕМЕННОЙ

ИЗДАТЕЛЬСТВО «НАУКА»

A. G. SVESHNIKOV and A. N. TIKHONOV

# THE THEORY OF FUNCTIONS OF A COMPLEX VARIABLE

Translated from the Russian
by GEORGE YANKOVSKY

MIR PUBLISHERS · MOSCOW

First published 1971
Second printing 1973
Second edition 1978
Second edition, second printing 1982

*На английском языке*

# CONTENTS

# INTRODUCTION

The concept of a complex number arose primarily from the need to automatize calculations. Even the most elementary algebraic operations involving real numbers take us beyond the domain of real numbers. It will be recalled that not every algebraic equation can be solved in terms of real numbers. It is therefore necessary either to give up routine methods of solution and each time carry out a detailed investigation of the possibility of their application or extend the domain of real numbers so that basic algebraic operations can always be employed. Complex numbers are just such an extension of the domain of real numbers. A remarkable property of complex numbers is that the basic mathematical operations involving them do not take one outside the domain of complex numbers.

The introduction of complex numbers and functions of a complex variable is likewise convenient when integrating elementary functions, when solving differential equations, and in other cases when one frequently has to move into the domain of complex numbers. The complex notation is also convenient in the mathematical formulation of many physical propositions (for example, in electrical engineering, radio engineering, electrodynamics, and so forth).

One of the principal classes of functions of a complex variable—analytic functions—is closely connected with solving the Laplace equation, to which numerous problems of mechanics and physics reduce. For this reason, the methods of the theory of functions of a complex variable have found extensive and effective use in solving a broad range of problems in hydrodynamics and aerodynamics, the theory of elasticity, electrodynamics and other natural sciences.

# CHAPTER 1

# THE COMPLEX VARIABLE AND FUNCTIONS OF A COMPLEX VARIABLE

## 1.1. Complex Numbers and Operations on Complex Numbers

### *a. The concept of a complex number*

We assume that the reader is acquainted with the concept of a complex number and with the definition of arithmetical operations involving complex numbers. A brief résumé is given below.

*A complex number z is characterized by a pair of real numbers* $(a, b)$ *having an established sequential order of the numbers a and b.* This is stated succinctly in the notation $z = (a, b)$. The first number $a$ of the pair $(a, b)$ is called the *real part* of the complex number $z$ and is denoted by the symbol $a = \operatorname{Re} z$; the second number $b$ of the pair $(a, b)$ is called the *imaginary part* of the complex number $z$ and is symbolized by $b = \operatorname{Im} z$.

Two complex numbers $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ are equal only when both the real and imaginary parts are equal, that is, $z_1 = z_2$ only when $a_1 = a_2$ and $b_1 = b_2$.

### *b. Operations on complex numbers*

Let us now define algebraic operations involving complex numbers.

The *sum* of two complex numbers $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ is a complex number $z = (a, b)$, where $a = a_1 + a_2$, $b = b_1 + b_2$. It will readily be seen that in this definition the commutative and associative laws for addition, $z_1 + z_2 = z_2 + z_1$ and $z_1 + (z_2 + z_3) = (z_1 + z_2) + z_3$, hold true. As in the domain of real numbers, zero is a complex number 0 such that the sum of it and any complex number $z$ is equal to $z$, that is, $z + 0 = z$. There is obviously a unique complex number $0 = (0, 0)$ that possesses this property.

The *product* of the complex numbers $z_1 = (a_1, b_1)$ and $z_2 = (a_2, b_2)$ is a complex number $z = (a, b)$ such that $a = a_1a_2 - b_1b_2$, $b = a_1b_2 + a_2b_1$. In this definition of a product, we find that the commutative $[z_1z_2 = z_2z_1]$, associative $[z_1 (z_2 \cdot z_3) = (z_1 \cdot z_2) z_3]$ and distributive $[(z_1 + z_2) z_3 = z_1z_3 + z_2z_3]$ laws hold.

Let us include the real numbers in the set of complex numbers and regard the real number $a$ as the complex number $a = (a, 0)$. Then, as follows from the definition of the operations of addition and multiplication, the familiar rules involving real numbers hold true for complex numbers as well. Thus, the set of complex numbers is regarded as an extension of the set of real numbers.* Note that multiplication by a real unit (1, 0) does not change a complex number: $z \cdot 1 = z$.

A complex number of the form $z = (0, b)$ is called a *pure imaginary* number and is symbolized as $z = ib$. The pure imaginary number $(0, b) = ib$ may be regarded as the product of the imaginary unit (0, 1) and a real number $(b, 0)$. The unit imaginary number is ordinarily denoted by the symbol $(0, 1) = i$. By virtue of the definition of a product of complex numbers, the following relation holds true: $i \cdot i = i^2 = -1$. It enables one to attribute a direct algebraic meaning to the *real-imaginary form* of a complex number:

$$z = (a, b) = a + ib \tag{1-1}$$

and perform operations of addition and multiplication of complex numbers in accordance with the usual rules of the algebra of polynomials.

The complex number $\bar{z} = a - ib$ is said to be the *complex conjugate number* of $z = a + ib$.

The operation of subtraction of complex numbers is defined as the inverse operation of addition. A complex number $z = a + ib$ is termed the *difference* between the complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2$ if $a = a_1 - a_2$, $b = b_1 - b_2$.

The operation of dividing complex numbers is defined as the inverse operation of multiplication. A complex number $z = a + ib$ is called the *quotient* of the complex numbers $z_1 = a_1 + ib_1$ and $z_2 = a_2 + ib_2 \neq 0$ if $z_1 = z \cdot z_2$, whence it follows that the real part $a$ and the imaginary part $b$ of the quotient $z$ are found from the linear system of algebraic equations

$$a_2 a - b_2 b = a_1$$

$$b_2 a + a_2 b = b_1$$

with the determinant $a_2^2 + b_2^2$ different from zero. Solving this system, we get

$$z = \frac{z_1}{z_2} = \frac{a_1 a_2 + b_1 b_2}{a_2^2 + b_2^2} + i\,\frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2} \tag{1-2}$$

* As will follow from subsequent analysis, the set of complex numbers, unlike the set of real numbers, does not possess the property of ordering since there does not exist a rational system for comparing complex numbers.

### *c. The geometric interpretation of complex numbers*

The study of complex numbers is greatly facilitated by interpreting them geometrically. Insofar as a complex number is defined as a pair of real numbers, it is natural to depict the complex number $z = a + ib$ as a point in the $x$, $y$-plane with Cartesian coordinates $x = a$ and $y = b$. The number $z = 0$ corresponds to the origin of the plane. We shall henceforward call this the *complex plane*; the axis of abscissas is the *real* axis, the axis of ordinates is the *imaginary* axis of the complex plane. We have thus obviously established a reciprocal one-to-one correspondence between the set of all complex numbers and the set of points of the complex plane, and also between the set of all complex numbers $z = a + ib$ and the set of free vectors, the projections $x$ and $y$ of which on the axis of abscissas and the axis of ordinates are, respectively, equal to $a$ and $b$.

There is another extremely important form of representing complex numbers. It is possible to define the position of a point in the plane by means of polar coordinates $(\rho, \varphi)$, where $\rho$ is the distance of the point from the coordinate origin, and $\varphi$ is the angle which the radius vector of the given point makes with the positive direction of the axis of abscissas. The positive direction of the variation of the angle $\varphi$ is the counterclockwise direction $(-\infty < \varphi < \infty)$. Taking advantage of the relationship between Cartesian and polar coordinates $x = \rho \cos \varphi$, $y = \rho \sin \varphi$, we get the so-called *trigonometric form* (or polar form) of a complex number:

$$z = \rho (\cos \varphi + i \sin \varphi) \tag{1-3}$$

Here, $\rho$ is usually called the *modulus* (or absolute value) and $\varphi$ the *argument* (amplitude) of the complex number and $\rho = |z|$, $\varphi = \operatorname{Arg} z$. These formulas express the real and imaginary parts of the complex number in terms of its modulus and argument. It is easy to express the modulus and argument of a complex number in terms of its real and imaginary parts: $\rho = \sqrt{a^2 + b^2}$, $\tan \varphi = \frac{b}{a}$ (when choosing the value of $\varphi$ in the latter equation, take into account the signs of $a$ and $b$). Note that the argument of the complex number is not defined uniquely, but to within an additive multiple of $2\pi$. In a number of cases, it is convenient to denote, in terms of $\arg z$, the value of the argument contained within the range $\varphi_0 \leqslant \arg z < 2\pi + \varphi_0$, where $\varphi_0$ is an arbitrary fixed number (say, $\varphi_0 = 0$ or $\varphi_0 = -\pi$). Then $\operatorname{Arg} z = \arg z + 2k\pi$ $(k = 0, \pm 1, \pm 2, \ldots)$. The argument of the complex number $z = 0$ is not defined and its modulus is zero. Two nonzero complex numbers are equal if and only if their moduli are equal and the values of their arguments are either equal or differ by a multiple of $2\pi$. Complex conjugate numbers have the same modulus while the values of their arguments (given an appropriate choice of their ranges) differ in sign.

Finally, taking advantage of a familiar formula due to Euler,* $e^{i\varphi} = \cos\varphi + i \sin\varphi$, we obtain the so-called *exponential form* of a complex number:

$$z = \rho e^{i\varphi}. \tag{1-4}$$

The earlier noted correspondence between the set of all complex numbers and the plane vectors enables us to identify the operations

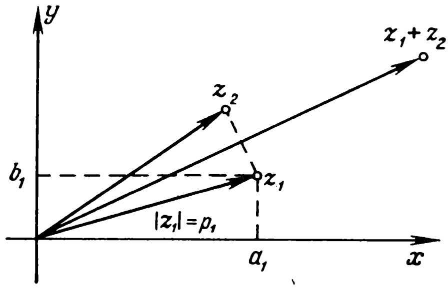


**Fig. 1.1**

of addition and subtraction of complex numbers with the corresponding operations involving vectors (Fig. 1.1). We thus readily establish the triangle inequalities

$$|z_1 + z_2| \leqslant |z_1| + |z_2|, \quad |z_1 - z_2| \geqslant |z_1| - |z_2| \tag{1-5}$$

The modulus (absolute value) of the difference of two complex numbers is geometrically interpreted as the distance between the corresponding points in the complex plane. Note also the obvious inequalities $|z| \geqslant a$, $|z| \geqslant b$.

In performing operations of multiplication it is convenient to make use of the trigonometric form of representing complex numbers. By the rules of multiplication, we get**

$$\begin{aligned} z = \rho(\cos\varphi + i\sin\varphi) &= z_1 \cdot z_2 \\ &= \rho_1(\cos\varphi_1 + i\sin\varphi_1)\,\rho_2(\cos\varphi_2 + i\sin\varphi_2) \\ &= \rho_1\rho_2(\cos\varphi_1\cos\varphi_2 - \sin\varphi_1\sin\varphi_2) \\ &+ i\rho_1\rho_2(\sin\varphi_1\cos\varphi_2 + \cos\varphi_1\sin\varphi_2) \\ &= \rho_1\rho_2[\cos(\varphi_1+\varphi_2) + i\sin(\varphi_1+\varphi_2)] = \rho_1 \cdot \rho_2 \cdot e^{i(\varphi_1+\varphi_2)} \end{aligned}$$

* For the time being we shall regard this expression as an abridged form of representing the complex number $z = \cos\varphi + i\sin\varphi$. The full meaning of this notation will be established later on.

** This formula demonstrates that the earlier introduced function $e^{i\varphi}$ possesses the property $e^{i\varphi_1} \cdot e^{i\varphi_2} = e^{i(\varphi_1+\varphi_2)}$.

Whence $\rho = \rho_1 \cdot \rho_2$, $\varphi = \varphi_1 + \varphi_2$, that is the modulus of the product is equal to the product of the moduli, and the argument is equal to the sum of the arguments of the factors. A similar relation

$$\frac{z_1}{z_2} = \frac{\rho_1}{\rho_2} e^{i(\varphi_1 - \varphi_2)}$$

holds true in the case of division of complex numbers when $\rho_2 \neq 0$.

### *d. Extracting the root of a complex number*

The trigonometric and exponential forms of representing complex numbers are convenient for considering the algebraic operations of raising a complex number to a positive integral power and of extracting the root of a complex number. Thus, if $z = z_1^n$, then $\rho = \rho_1^n$ and $\varphi = n\varphi_1$.

The complex number $z_1 = \sqrt[n]{z}$ is called the *n*th *root* of the complex number $z$ if $z = z_1^n$. From this definition it follows that $\rho_1 = \sqrt[n]{\rho}$ and $\varphi_1 = \frac{\varphi}{n}$. As has been pointed out, the argument of a complex number is not defined uniquely, but to within an additive multiple of $2\pi$. For this reason, from the expression for the argument of a complex number $z_1$,

$$\varphi_k = \frac{\varphi}{n} + \frac{2\pi k}{n}$$

where $\varphi$ is one of the values of the argument of the complex number $z$, we get that there exist different complex numbers which, when raised to the $n$th power, are equal to one and the same complex number $z$. The moduli of these complex numbers are the same and are equal to $\sqrt[n]{\rho}$, while the arguments differ by a multiple of $\frac{2\pi}{n}$. The number of distinct values of the $n$th root of the complex number $z$ is $n$. The points in the complex plane that correspond to different values of the $n$th root of the complex number $z$ are situated on the vertices of a regular $n$-gon inscribed in a circle of radius $\sqrt[n]{\rho}$ centred at the point $z = 0$. The appropriate values of $\varphi_k$ are obtained as $k$ takes the values $k = 0, 1, \ldots, n - 1$.

Classical analysis posed the problem of extending the set of real numbers so that not only the elementary algebraic operations of addition and multiplication but also the operation of extraction of roots does not require going outside the extended set. We thus see that the introduction of complex numbers solves this problem.

*Example 1*. Find all the values of $\sqrt{i}$. Writing the complex number in exponential form as $z = i = e^{i\frac{\pi}{2}}$, we find the following expressions for the values of the square root of this complex number $z_k = e^{i\frac{\pi}{4}+i\frac{2\pi k}{2}}$, $k = 0, 1$ (Fig. 1.2), whence

$$z_0 = e^{i\frac{\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2}(1+i)$$

$$z_1 = e^{i\frac{5\pi}{4}} = -e^{i\frac{\pi}{4}} = -\frac{\sqrt{2}}{2}(1+i)$$

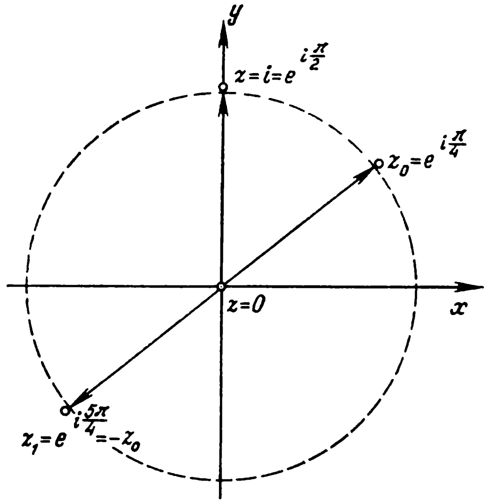


Fig. 1.2

*Example 2.* Find all the values of $\sqrt[p]{1}$, where $p > 0$ is an integer. Taking advantage of the representation $1 = e^{i0}$, we get $z_k = e^{i\frac{2\pi}{p}k}$ as in the preceding example, $k = 0, \ldots, p-1$, whence

$$z_0 = e^{i0} = 1,\ z_1 = e^{i\frac{2\pi}{p}} = \cos\frac{2\pi}{p} + i\sin\frac{2\pi}{p},\ \ldots,\ z_{p-1}$$

$$= e^{i\frac{2\pi}{p}(p-1)} = e^{-i\frac{2\pi}{p}} = \cos\frac{2\pi}{p} - i\sin\frac{2\pi}{p}$$

Thus, the $p$th root of 1 has exactly $p$ distinct values. These complex numbers correspond to the vertices of a regular $p$-gon inscribed in a circle of unit radius centred at the point $z = 0$, one of the vertices lying at the point $z = 1$.

*Example 3*. Find all the values of $\sqrt{1-i\sqrt{3}}$. Since $z = 1 - i\sqrt{3} = 2e^{-i\frac{\pi}{3}}$, it follows that for values of the square root of this complex number we get the expressions $z_k = \sqrt{2}\, e^{-i\frac{\pi}{6}+i\frac{2\pi k}{2}}$, $k = 0, 1$, whence

$$z_0 = \sqrt{2}\, e^{-i\frac{\pi}{6}} = \sqrt{2}\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right) = \frac{\sqrt{3}-i}{\sqrt{2}}$$

$$z_1 = \sqrt{2}\, e^{i\frac{5\pi}{6}} = -\frac{\sqrt{3}-i}{\sqrt{2}} = -z_0$$

Thus, to extract the $n$th root of a complex number, one has to represent it in exponential form, extract the $n$th root of the modulus of the given complex number (take the arithmetic—real and positive—values of the root), and divide the argument of the given complex number by $n$. (Bear in mind the multivaluedness of the argument when obtaining all the values of the root.)

## 1.2. The Limit of a Sequence of Complex Numbers

### *a. The definition of a convergent sequence*

In building the theory of functions of a complex variable, it is of great importance to carry the principal ideas of analysis into the complex domain. One of the fundamental concepts of analysis is that of a limit and, in particular, the concept of a convergent sequence of numbers. A similar role is played by the respective concepts in the domain of complex numbers. Here, many of the definitions associated with passage to the limit fully repeat the appropriate definitions in the theory of functions of a real variable.

*A sequence of complex numbers is a consecutive infinite set of complex numbers*. We will denote a sequence of complex numbers by the symbol $\{z_n\}$. The complex numbers $z_n$ that form the sequence $\{z_n\}$ are called its elements.*

*The number $z$ is called the limit of the sequence $\{z_n\}$ if for any positive number $\varepsilon$ it is possible to indicate an $N(\varepsilon)$ such that all subsequent elements $z_n$ of the sequence satisfy the inequality*

$$|z - z_n| < \varepsilon \quad \text{for} \quad n \geqslant N(\varepsilon) \tag{1-6}$$

* The definition of a sequence does not exclude the possibility of repeating elements and the particular case of all elements of a sequence coinciding.

*The sequence* $\{z_n\}$ *which has a limit* $z$ *is called a sequence convergent to the number* $z$ *and is symbolized as* $\lim\limits_{n\to\infty} z_n = z$.

For a geometric interpretation of the limit process in the complex domain it is convenient to use the concept of the ε-neighbourhood of a point in the complex plane.

*The set of points* $z$ *of the complex plane which lie inside a circle of radius* $\varepsilon$ *centred in the point* $z_0$ ($|z - z_0| < \varepsilon$) *is termed the ε-neighbourhood of the point* $z_0$.

From this definition it follows that the point $z$ is the limit of the convergent sequence $\{z_n\}$ if in any ε-neighbourhood of the point $z$ there lie all the elements of the sequence from a certain number onward which is dependent on ε.

Since every complex number $z_n = a_n + ib_n$ is characterized by the pair of real numbers $a_n$ and $b_n$, to the sequence of complex numbers $\{z_n\}$ correspond two sequences of real numbers $\{a_n\}$ and $\{b_n\}$, which are respectively made up of the real and imaginary parts of the elements $z_n$ of the sequence $\{z_n\}$.

The following assertion holds.

***Theorem 1.1.*** *A necessary and sufficient condition for the convergence of a sequence* $\{z_n\}$ *is the convergence of the sequences of real numbers* $\{a_n\}$ *and* $\{b_n\}$ ($z_n = a_n + ib_n$).

*Proof.* Indeed, if the sequence $\{z_n\}$ converges to the number $z = a + ib$, then for any $\varepsilon > 0$ $|a_n - a| \leqslant |z_n - z| < \varepsilon$ and $|b_n - b| < \varepsilon$ for $n \geqslant N(\varepsilon)$. This proves the convergence of the sequences $\{a_n\}$ and $\{b_n\}$ to $a$ and $b$, respectively. The converse assertion follows from the relation $|z_n - z| = \sqrt{(a_n - a)^2 + (b_n - b)^2}$. where $a$ and $b$ are the limits of the sequences $\{a_n\}$ and $\{b_n\}$ and $z = a + ib$.

*A sequence* $\{z_n\}$ *is called bounded if there exists a positive number* $M$ *such that for all elements* $z_n$ *of the sequence the inequality* $|z_n| < M$ *holds.* The basic property of a bounded sequence is characterized by the following theorem.

***Theorem 1.2.*** ***From any bounded sequence it is possible to extract a convergent subsequence.***

*Proof.* Since the sequence $\{z_n\}$ is bounded, it is clear that the real sequences $\{a_n\}$ and $\{b_n\}$ corresponding to it are likewise bounded. Let us consider the sequence $\{a_n\}$. Since this sequence is bounded, we can extract a convergent subsequence $\{a_{n_i}\}$ the limit of which we denote by $a$. To the sequence $\{a_{n_i}\}$ there corresponds a sequence $\{b_{n_i}\}$, which is also bounded. We can therefore extract from it a convergent subsequence $\{b_{n_k}\}$ with limit $b$. And the corresponding sequence $\{a_{n_k}\}$ converges to $a$ as before. From this it follows that the sequence of complex numbers $\{z_{n_k}\} = \{a_{n_k} + ib_{n_k}\}$ is likewise convergent, and $\lim\limits_{n_k\to\infty} z_{n_k} = z = a + ib$. The theorem is proved.

### *b. Cauchy's test*

When investigating the convergence of a sequence, it is often convenient to use the necessary and sufficient condition for the convergence of a sequence called Cauchy's test.

***Cauchy's test.*** *A sequence* $\{z_n\}$ *converges if and only if for every* $\varepsilon > 0$ *there is an* $N(\varepsilon)$ *such that*

$$|z_n - z_{n+m}| < \varepsilon \tag{1-7}$$

*for* $n \geqslant N(\varepsilon)$ *and for every integer* $m \geqslant 0$.

*Proof.* To prove Cauchy's test we again take advantage of the equivalence of convergences of the sequence $\{z_n\}$ and the sequences of the real numbers $\{a_n\}$ and $\{b_n\}$, and also of the fact that the Cauchy test is a necessary and sufficient condition for the convergence of a sequence of real numbers. We begin by proving the necessity of Cauchy's test. Since the sequence $\{z_n\}$ converges, it follows that the sequences of the real numbers $\{a_n\}$ and $\{b_n\}$ also converge. Whence it follows that for every $\varepsilon > 0$ and for every integer $m > 0$ $|a_n - a_{n+m}| < \frac{\varepsilon}{2}$ for $n \geqslant N_1(\varepsilon)$ and $|b_n - b_{n+m}| < \frac{\varepsilon}{2}$ for $n \geqslant N_2(\varepsilon)$. Choosing for the $N(\varepsilon)$ the greater of $N_1$ and $N_2$, we get $|z_n - z_{n+m}| < \varepsilon$ for $n > N(\varepsilon)$, by virtue of the triangle inequality.

Now let us take up the sufficiency of Cauchy's test. For $n \geqslant N$ there follow from the relation (1-7) the inequalities $|a_n - a_{n+m}| \leqslant \leqslant |z_n - z_{n+m}| < \varepsilon$ and $|b_n - b_{n+m}| \leqslant |z_n - z_{n+m}| < \varepsilon$ which are sufficient conditions for the convergence of the sequences $\{a_n\}$ and $\{b_n\}$, that is, for the convergence of the sequence $\{z_n\}$. We have thus proved that fulfilment of Cauchy's test is a necessary and sufficient condition for the convergence of a sequence $\{z_n\}$ with complex elements.

### *c. Point at infinity*

We introduce the concept of the point at infinity in the complex plane, which will be needed later on. Let there be a sequence of complex numbers $\{z_n\}$ such that for every positive number $R$ there exists an integer $N$, beginning with which the terms of the sequence satisfy the condition $|z_n| > R$ for $n \geqslant N$. We call such a sequence indefinitely increasing. According to the definitions introduced earlier the given sequence, like any subsequence of it, has no limit. This special instance of a sequence increasing without bound gives rise to a number of inconveniences. We get around them by introducing the complex number $z = \infty$, and we assume that *every sequence increasing without bound converges to this number*, to which, in the complex plane, there corresponds the point at infinity. We introduce the concept of the *extended complex plane* consisting of the ordinary complex plane and a single infinitely distant element—the point at infinity $z = \infty$.* If we illustrate this geometrically by associating the elements of an indefinitely increasing sequence $\{z_n\}$ with the points of the complex plane, we will see that the points of this sequence lie (as their number increases) outside concentric circles centred at the coordinate origin, the radii of the circles being arbitrarily large. Note that the points of the given sequence tend to the point $\infty$ *irrespective of the direction* in the extended complex plane.

In connection with the concepts just introduced it is natural to use the term neighbourhood of the point at infinity for the set of points $z$ of the extended complex plane that satisfy the condition $|z| > R$, where $R$ is a sufficiently large positive number.

Let us determine the algebraic properties of the number $z = \infty$. From the elements of an indefinitely increasing sequence $\{z_n\}$, form the sequence $\left\{\frac{1}{z_n}\right\}$. This sequence converges to the point $z = 0$. Indeed, from earlier considerations it follows that for every $\varepsilon > 0$ there exists a number $N$ such that $\left|\frac{1}{z_n}\right| < \varepsilon$ for $n \geqslant N$. The converse is obvious, i.e., if a sequence $\{\zeta_n\}$ converges to zero and consists of nonzero elements, then the sequence $\left\{\frac{1}{\zeta_n}\right\}$ converges to the point at infinity.

We thus assume $\frac{1}{\infty} = 0$ and $\frac{1}{0} = \infty$. Generally, the following relations are established for the point at infinity: $z \cdot \infty = \infty$ for $z \neq 0$, and $z + \infty = \infty$, $\frac{z}{\infty} = 0$ for $z \neq \infty$, which are natural from the viewpoint of the limit process in operations of addition and multiplication. From this point of view, the operation $\frac{\infty}{\infty}$ is naturally indeterminate.

## 1.3. The Concept of a Function of a Complex Variable. Continuity

### *a. Basic definitions*

We now introduce the concept of a function of a complex variable—in the same way as that of a function of a real variable. We say that on a set $E$ of the complex plane there is specified a function of a complex variable if a law is given that puts every point of $E$ in a one-to-one correspondence with a certain complex number. The set $E$ is called the set of values of the independent variable.

---

* Note that the argument of the complex number $\infty$ is not defined, just as its real and imaginary parts are not defined.

The structure of this set may be extremely complicated and diversified; however, in the theory of functions of a complex variable we consider sets of a special structure. Certain auxiliary notions will be needed in the sequel.

A point $z$ is called an *interior point of the set $E$* if there exists an ε-neighbourhood of $z$, all the points of which belong to $E$. For instance, the point $z$ of the set $|z| \leqslant 1$ is an interior point if $|z| < 1$; the point $z = 1$ is not an interior point of the given set.

*The set $E$ is called a domain if the following conditions are fulfilled:* (1) *every point of the set $E$ is an interior point of the set*; (2) *any two points of the set $E$ may be connected by a polygonal line, all the points of which belong to $E$.*

In this definition of a domain, the second requirement is the connectivity requirement of a domain. For example, the set of points

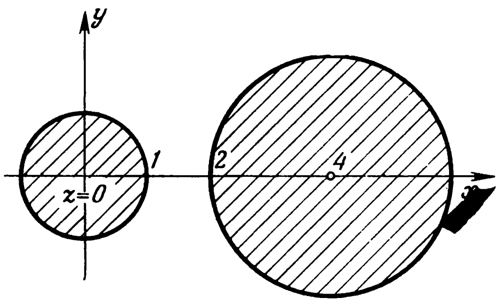


Fig. 1.3

$|z| < 1$ forms a domain. In exactly the same way, the ε-neighbourhood of the point $z_0$ ($|z - z_0| < \varepsilon$) forms a domain. The set of points $|z| \leqslant 1$ is not a domain, since not all its points are interior points. Neither are the set of points $|z| \neq 1$ and the set $|z| < 1$, $|z - 4| < 2$ (Fig. 1.3) domains, since they are not connected. The letters $\mathfrak{G}$, $G$, $D$ are ordinarily used to denote domains.

The point $z$ is an *exterior point of the domain* $\mathfrak{G}$ if there exists an ε-neighbourhood of $z$ such that none of its points lie in the domain $\mathfrak{G}$.

The point $z$ is a *boundary point of the domain* $\mathfrak{G}$ if any ε-neighbourhood of it contains both the points that belong to the domain $\mathfrak{G}$ and the points that do not belong to $\mathfrak{G}$. For example, $z = 1$ is a boundary point of the domain $|z| < 1$. *The collection of all boundary points form the boundary of the domain.* In the future we will ordinarily use the letters $\gamma$, $\Gamma$, $C$ to denote the boundary of a domain. The simplest instance of a boundary of a domain is, obviously, a curve; however, the boundary of a domain can also consist of a discrete set of points. For example, the point set $|z| \neq 0$ forms in the complex plane a domain whose boundary is the point $z = 0$.

*The set obtained by adjoining to a domain all its boundary points is called a closed domain.* We will denote a closed domain by a bar over the symbol of the domain ($\bar{\mathfrak{G}}$, $\bar{G}$, $\bar{D}$).

In the future we will consider those cases when the boundary of a domain is one or several piecewise smooth curves, which, in particular cases, can degenerate into individual points. Both singly connected and multiply connected domains will be considered. For example, the domain $|z - i| < 2$ is a singly connected domain whose boundary is the circle $|z - i| = 2$; the annulus $1 < |z| < 2$ (Fig. 1.4) is a doubly connected domain; the point set $z \neq 0$ is a singly connected domain, and so forth.

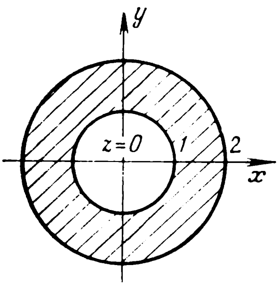


Fig. 1.4

If the domain $\mathfrak{G}$ lies entirely inside some circle of finite radius, it is called *bounded*, otherwise it is *unbounded*.

We will mainly be concerned with cases when the set $E$ of values of a complex variable represents a domain $\mathfrak{G}$ or a closed domain $\bar{\mathfrak{G}}$ of the complex plane. Then, *a single-valued function of a complex variable z specified in the domain $\mathfrak{G}$ is defined by a law that associates with every value of z in the domain $\mathfrak{G}$ a definite complex number w.* This correspondence will be written symbolically as follows:

$$w = f(z) \tag{1-8}$$

The set of complex numbers $w$ that correspond to all $z \in \mathfrak{G}$ is called the set of values of the function $f(z)$. Since every complex number is characterized by a pair of real numbers, the specification of a complex function $w = u + iv$ of the complex variable $z = x + iy$ is equivalent to the specification of two real functions of two real variables; this may be written as

$$w(z) = u(x, y) + iv(x, y) \tag{1-9}$$

The functions $u(x, y)$ and $v(x, y)$ are defined in the domain $\mathfrak{G}$ of the plane of real variables $x$, $y$, corresponding to the domain $\mathfrak{G}$ of the complex $z$-plane. The function $u(x, y)$ is called the *real* part, and the function $v(x, y)$ is called the *imaginary* part of the function $w = f(z)$. In the future, unless otherwise stated, we will always use the representation (1-9), denoting the real part of the function $f(z)$ by the symbol $u$ and the imaginary part by the symbol $v$.

One is frequently concerned with multiple-valued functions of a complex variable when every value $z \in \mathfrak{G}$ is associated with several complex numbers. In the present chapter we will only consider single-valued functions of a complex variable. A detailed analysis of multiple-valued functions will be given later.

The set of values $w$ of the function $f(z)$ in the complex $w$-plane can have a highly diversified structure. In particular, it may be a domain $G$ or a closed domain $\bar{G}$. We will consider only such cases in the future. The geometrical interpretation of the concept of the function $f(z)$ of a complex variable consists in the fact that the equality $w = f(z)$ establishes a law of correspondence between the points of the domain $\mathfrak{G}$ of the complex $z$-plane and the points of the domain $G$ of the complex $w$-plane. The converse correspondence is also established: with every point $w \in G$ there is associated one or several points $z$ of the domain $\mathfrak{G}$. This signifies that in the domain $G$ there is specified a (single-valued or multiple-valued) function of the complex variable $w$:

$$z = \varphi(w) \tag{1-10}$$

This function is called the inverse function of $f(z)$. The domain $G$ of specification of the function $\varphi(w)$ is obviously the domain of values of the function $f(z)$. If the function $\varphi(w)$, which is the inverse of the single-valued function $f(z)$ specified in the domain $\mathfrak{G}$, is a single-valued function in the domain $G$, then a one-to-one correspondence is established between the domains $\mathfrak{G}$ and $G$.

*The function $f(z)$ is called a univalent function in the domain $\mathfrak{G}$ if at distinct points $z$ of this domain it assumes distinct values.*

From this definition it follows that a function which is the inverse of a univalent function is single-valued.

### b. Continuity

Let us discuss the continuity of a function of a complex variable. Let a function $f(z)$ be defined on a certain set $E$. We consider various sequences $\{z_n\}$ of points of this set which converge to some point $z_0$ and consist of points $z_n$ that differ* from the point $z_0$ $(z_n \neq z_0)$, and the associated sequences $\{f(z_n)\}$ of values of the function. *If, irrespective of the choice of the sequence $\{z_n\}$, there exists a unique limit, $\lim\limits_{z_n \to z_0} f(z_n) = w_0$, then this limit is called the limiting value, or the limit, of the function $f(z)$ at the point $z_0$. This is written as*

$$\lim_{z \to z_0} f(z) = w_0 \tag{1-11}$$

* Here it is assumed that the point $z_0$ is the condensation point of the set $E$, i.e., there exist sequences $\{z_n\}$ of points of this set which converge to the point $z_0$.

A different definition * of the concept of limiting value (or limit) of a function is frequently used.

*A number $w_0$ is called the limiting value of a function $f(z)$ at the point $z_0$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all points $z \in E$ and satisfying the condition $0 < |z - z_0| < \delta$, the inequality $|f(z) - w_0| < \varepsilon$ holds.*

We shall demonstrate the equivalence of these definitions. Let the function $f(z)$ satisfy the second definition. Take an arbitrary positive number $\varepsilon$ and choose for it a corresponding $\delta(\varepsilon)$. Consider an arbitrary sequence $\{z_n\} \to z_0$. We find $N[\delta(\varepsilon)] = N(\varepsilon)$, for which henceforth $0 < |z_n - z_0| < \delta$. Then, by hypothesis, $|f(z_n) - w_0| < \varepsilon$ for $n \geqslant N(\varepsilon)$, and since $\varepsilon$ is an arbitrary number greater than zero, this means, by virtue of the arbitrary choice of the sequence $\{z_n\}$, that $\lim\limits_{z_n \to z_0} f(z_n) = w_0$; that is, the function $f(z)$ satisfies the first definition as well. Hence, the first definition follows from the second.

Now let us prove that the second definition follows from the first. Assume that this is not so. Then there is an $\varepsilon_0 > 0$ such that for every $\delta_n > 0$ there is a point $z_n \in E$ such that for $0 < |z_n - z_0| < \delta_n$ the inequality $|f(z_n) - w_0| > \varepsilon_0$ will be fulfilled. Choose a sequence $\{\delta_n\} \to 0$ and a corresponding sequence $\{z_n\}$ of points satisfying the foregoing inequalities. Obviously, $\{z_n\} \to z_0$ and the sequence $\{f(z_n)\}$ does not converge to the number $w_0$, since all terms of this sequence differ from $w_0$ by more than $\varepsilon_0$. But the result thus obtained contradicts the first definition. Hence, the assumption does not hold and the second definition follows from the first. The equivalence of both definitions is proved.

As in the case of a real variable, an important role is played by the concept of continuity of a function. Let us begin with the concept of continuity at a point. We will consider that the point $z_0$ at which this concept is defined must belong to the set $E$ over which the function is specified.

*A function $f(z)$, specified on a set $E$, is called continuous at a point $z_0 \in E$ if the limiting value of the function at the point $z_0$ exists, is finite and coincides with the value $f(z_0)$ of $f(z)$ at the point $z_0$; i.e.,* $\lim\limits_{z \to z_0} f(z) = f(z_0)$.

This definition of continuity is extended both to interior points and boundary points of the set.**

* Note that this definition, unlike the first one, is meaningful only for finite values of $z_0$ and $w_0$.

** If a point $z_0$ is an isolated point of the set $E$ (i.e., there exists an $\varepsilon$-neighbourhood of $z_0$ in which there are no other points of the set $E$), then, by definition the function $f(z)$ is considered continuous at the point $z_0$.

If the function $f(z)$, which is specified on a set $E$, is continuous at all points of the set, then we say that $f(z)$ is continuous on the set $E$. In particular, we will consider functions which are continuous in a domain, in a closed domain and on a curve. We once again stress the point that by virtue of the definitions given above one should regard the limiting values of the function $f(z)$ only on sequences of points that belong to the given set (in the latter cases, a closed domain, a curve, etc.).

With the aid of the $\varepsilon$-$\delta$ definition of a limiting value, the conditions of continuity of the function $f(z)$ at a point $z_0$ may also be formulated as follows. *The function $f(z)$ is continuous at a point $z_0$ if for every $\varepsilon > 0$ there is a $\delta > 0$ such that for all points $z \in E$ which satisfy the inequality $|z - z_0| < \delta$, the inequality $|f(z) - f(z_0)| < \varepsilon$ holds.* Geometrically, this signifies that a function of a complex variable which is continuous at a certain point* $z_0$ associates with every point of a $\delta$-neighbourhood of the point $z_0$ a certain point belonging to the $\varepsilon$-neighbourhood of the point $w_0 = f(z_0)$.

From the continuity of the function of a complex variable $f(z) = u(x, y) + iv(x, y)$ there follows the continuity of its real part $u(x, y)$ and imaginary part $v(x, y)$ in the variables $x$, $y$ taken together. The converse assertion is also true; i.e. if $u(x, y)$ and $v(x, y)$ are continuous functions of the variables $x$, $y$ taken together, at some point $(x_0, y_0)$, then $f(z) = u(x, y) + iv(x, y)$ is a function of the complex variable $z = x + iy$ continuous at the point $z_0 = x_0 + iy_0$. These assertions follow from the fact that the necessary and sufficient condition for the convergence of a sequence of complex numbers is the convergence of the sequences of their real and imaginary parts.

This enables us to extend to functions of a complex variable the basic properties of continuous functions of two real variables. Thus, the sum and product of two functions $f_1(z)$ and $f_2(z)$ of a complex variable which are continuous in the domain $\mathfrak{G}$ are also continuous functions in that domain; the function $\varphi(z) = \frac{f_1(z)}{f_2(z)}$ is continuous at those points of the domain $\mathfrak{G}$ where $f_2(z) \neq 0$, the function $f(z)$ continuous on a closed set $\overline{E}$ is bounded in modulus on $\overline{E}$, etc.

---

* Note that the given definitions of the concept of continuity of a function $f(z)$ at a point $z_0$ hold true not only in the case of the finite point $z_0$, but also in the case of the point at infinity $z_0 = \infty$. Then, by virtue of the definition given on page 24, the limiting value of the function $f(z)$ at the point $\infty$ should be understood as the limit of the sequence $\{f(z_n)\}$, where $\{z_n\}$ is any indefinitely increasing sequence. In the second definition of continuity, the condition $|z - z_0| < \delta$ has to be replaced by the condition $|z| > R$.

### c. Examples

Let us consider some elementary examples.

1. Consider the following linear function as an instance of a function of a complex variable:

$$f(z) = w = az + b \tag{1-12}$$

Here, $a$ and $b$ are specified complex constants. We assume that $a \neq 0$, otherwise the function (1-12) associates with all points $z$ of the complex plane one and the same complex number $b$. The function (1-12) is defined for all values of the independent variable $z$. Its domain of definition is the entire* complex $z$-plane. To every value of $z$ there corresponds only one value of $w$; i.e. $f(z)$ is a single-valued function of $z$. Quite obviously, the inverse function $\varphi(w) = z = \frac{1}{a} w - \frac{b}{a} = a_1 w + b_1$ possesses the same properties that $f(z)$ does. Thus, $f(z)$ is a univalent function of $z$ in the entire complex plane establishing a one-to-one correspondence between the planes $z$ and $w$. By virtue of the continuity of the real and imaginary parts of $f(z)$ with respect to the variables $x$, $y$ taken together, this function is continuous over the entire complex plane (for all finite values of $x$, $y$). To clarify the geometrical meaning of this correspondence, consider the auxiliary function $\zeta = az$. By the rule of multiplication of complex numbers we have

$$\zeta = |a| \cdot |z| \cdot \{\cos(\arg a + \arg z) + i \sin(\arg a + \arg z)\}$$

Whence it follows that $|\zeta| = |a| \cdot |z|$, $\arg \zeta = \arg z + \arg a$. That is to say, the function $\zeta = az$ associates with every complex number $z$ a complex number $\zeta$, the modulus of which is $|a|$ times the modulus of $z$, and the argument is obtained from the argument of $z$ by adding a constant term—the argument of the complex number $a$. The geometrical meaning of this transformation is obvious: a stretching of the $z$-plane by a factor of $|a|$ and a rotation of the plane as a whole around the point $z = 0$ through the angle $\arg a$.

Returning to the function (1-12), which can now be written as $w = \zeta + b$, we see that the geometrical meaning of the latter transformation consists in a translation of the $z$-plane characterized by the vector $b$.

Thus, a linear function transforms the complex $z$-plane into the complex $w$-plane by means of a stretching, a rotation, and a translation.

* In future we will say that the function of a complex variable $f(z)$ is defined in the entire complex plane if it is defined for all values of the complex argument $z$ bounded in absolute value, and we will say that $f(z)$ is defined in the extended complex plane if it is specified for $z = \infty$ as well. In our example, $f(\infty) = \infty$.

2. As a second example, consider the function

$$w = f(z) = \frac{1}{z} \tag{1-13}$$

This function is likewise defined in the extended complex plane; $f(0) = \infty$ and $f(\infty) = 0$. As in the first example, we establish that $f(z)$ is a single-valued and univalent function of $z$ mapping the entire $z$-plane onto the entire $w$-plane. It is readily found that the

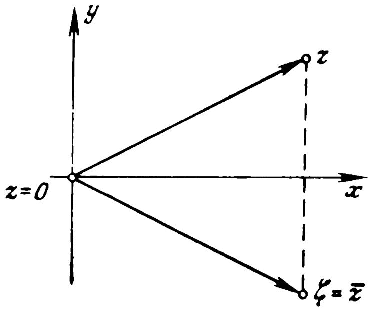


Fig. 1.5

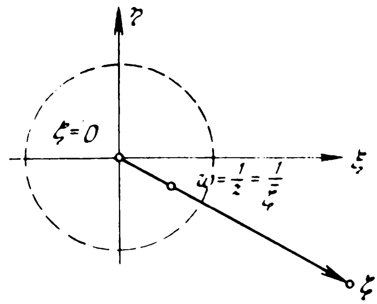


Fig. 1.6

function $f(z)$ is continuous in the entire complex plane, with the exception of the point $z = 0$. For a geometrical interpretation of this function, we take advantage of the exponential form of representing complex numbers: $w = re^{i\psi} = \frac{1}{\rho} e^{-i\varphi}$ $(z = \rho e^{i\varphi})$. This equality states that $\arg w = -\arg z$, $|w| = \frac{1}{|z|}$. The relations obtained permit regarding the mapping, by the given function, as a combination of two mappings: $\zeta = \zeta(z)$, where $|\zeta| = |z|$, $\arg \zeta = -\arg z$, and $w = w(\zeta)$, where $|w| = \frac{1}{|\zeta|}$, $\arg w = \arg \zeta$. The first mapping has the geometrical meaning of a mirror reflection about the real axis, in which the point $z$ is carried into the point $\bar{z}$, and the second mapping has the meaning of an inversion* in the unit circle, which inversion carries the point $\bar{z}$ into the point $w = \frac{1}{z}$ (Figs. 1.5, 1.6). In this case, the points of the $z$-plane that lie outside the unit circle are carried into points lying inside the unit circle of the $w$-plane, and vice versa.

---

* *Inversion* (or transformation of inverse radii) in a circle of radius $a$ is a transformation in which with every point inside (or outside) the circle there is associated a point outside (inside) the circle lying on the ray drawn from the centre of the circle to the given point so that the product of the distances of these points from the centre of the circle is equal to the square of the radius of the circle.

3. We consider the function

$$w = f(z) = z^2 \tag{1-14}$$

This function is a single-valued function of the complex variable $z$ defined on the extended complex $z$-plane. By representing the complex numbers in exponential form ($z = \rho e^{i\varphi}$, $w = re^{i\psi} = \rho^2 e^{i2\varphi}$) it will readily be concluded that the points of the $z$-plane lying on the ray forming the angle $\varphi$ with the positive direction of the real axis go into points of the $w$-plane lying on the ray which forms an angle $2\varphi$ with the positive direction of the real axis. Therefore, to the points $z$ and $-z$, the arguments of which differ by $\pi$ and the moduli are the same, there corresponds one and the same value of $w$ ($e^{i2\pi} = \cos 2\pi + i \sin 2\pi = 1$). Thus, the inverse function turns out to be multiple-valued. Consider the mapping by the function $w = z^2$. The upper half of the $z$-plane goes into the extended $w$-plane together with the real axis. For the sake of definiteness, suppose that in the upper half-plane the argument of $z$ lies within the range $0 < \varphi < \pi$. Then to different points of the range $0 < \varphi < \pi$ there correspond distinct values of $w$. Such a range of an independent variable, to various points of which there correspond distinct values of the function, is called the domain of univalence of the function. In the previous examples the univalence domain was the entire domain of definition of the function; in the given case, the half-plane is the univalence domain for the function $w = z^2$, the domain of definition of the function being the extended complex $z$-plane. Note that in the case under consideration, the boundaries of the univalence domain—the rays $\varphi = 0$ and $\varphi = \pi$—go into one and the same straight line, which is the positive part of the real axis of the $w$-plane. Continuing our examination, it is easy to demonstrate that the function $w = z^2$ also maps the lower half of the $z$-plane, together with the real axis, onto the extended $w$-plane. Thus, the inverse function

$$z = \sqrt{w} \tag{1-15}$$

defined over the extended $w$-plane is no longer a single-valued function: one point of the $w$-plane is associated with two distinct points of the $z$-plane, one in the upper and the other in the lower half-plane.

In order to study the mapping generated by the given function, let us again take advantage of the exponential form of representing a complex number: $w = re^{i\psi}$. Then, by the law of extracting the root of a complex number, we get two distinct values of the function $z(w)$: $z_k = \sqrt{r}\, e^{\frac{i}{2}(\psi + 2\pi k)}$ ($k = 0, 1$) (note that $\arg z_1 - \arg z_0 = \pi$). In the $w$-plane, consider a certain closed curve $C$ without self-intersections. Specify on it a point $w_0$, to which we assign a definite value of the argument $\psi_0$; we find $z_0(w_0)$, $z_1(w_0)$ and note the variation of the functions $z_0(w)$ and $z_1(w)$ as the point $w$ moves continuously along the curve $C$. The argument of the point $w$ on the curve $C$ varies continuously. Therefore, as is readily seen, the functions $z_0(w)$ and $z_1(w)$ are continuous functions of $w$ on the curve $C$. Here, two different cases are possible. In the first case, the curve $C$ does not contain the point $w = 0$ inside it. Then, after traversing the curve $C$, the argument of the point $w_0$ will return to the original value $\arg w_0 = \psi_0$. Hence, the values of the functions $z_0(w)$ and $z_1(w)$ will also be equal to their original values at the point $w = w_0$ after traversing the curve $C$. Thus, in this case, two distinct single-valued functions of the complex variable $w$ are defined on $C$: $z_0 = \sqrt{r}e^{\frac{i\psi}{2}}$ and $z_1 = \sqrt{r}e^{\frac{i}{2}(\psi+2\pi)}$ ($\psi$ varies continuously on $C$ beginning with the value $\psi_0$ at the point $w_0$). Obviously, if the domain $D$ of the $w$-plane has the property that any closed curve in the domain does not contain the point $w = 0$, then two distinct single-valued continuous functions $z_0(w)$ and $z_1(w)$ are defined in $D$. The functions $z_0(w)$ and $z_1(w)$ are called *branches of the multiple-valued function* $z(w) = \sqrt{w}$.

In the second case, the curve $C$ contains the point $w = 0$ inside. Then, after traversing $C$ in the positive direction, the value of the argument of the point $w_0$ will not return to the original value $\psi_0$ and will change by $2\pi$: $\arg w_0 = \psi_0 + 2\pi$. Therefore, the values of the functions $z_0(w)$ and $z_1(w)$ at the point $w_0$, as a result of their continuous variation after traversing the curve $C$, will no longer be equal to the original values. More exactly, we get $\tilde{z}_0(w_0) = z_0(w_0)e^{i\pi}$, $\tilde{z}_1(w_0) = z_1(w_0)e^{i\pi}$. That is, the function $z_0(w)$ goes into the function $z_1(w)$, and vice versa.

If for the point $z_0$ it is possible to indicate an $\varepsilon$-neighbourhood such that for a single circuit about the point $z_0$ along any closed contour lying entirely in the $\varepsilon$-neighbourhood one branch of the multiple-valued function goes into another, then the point $z_0$ is called a *branch point* of the given multiple-valued function. In the neighbourhood of a branch point, the individual branches of a multiple-valued function can no longer be regarded as distinct single-valued functions, since in successive circuits about the branch point their values vary. In the example at hand, the branch point is $w = 0$.

Note that traversal of a circle $|z| = R$ of arbitrarily large radius corresponds to going round the point $\zeta = 0$ in the plane $\zeta = \frac{1}{z}$ along the circle $|\zeta| = \rho = \frac{1}{R}$. According to Subsection 1.2.c we have the relation $\frac{1}{0} = \infty$. And so we will consider that the circuit of a circle of infinitely large radius ($R \to \infty$) is the circuit of the point at infinity $z = \infty$. As is readily seen, in the case at hand, one branch of the function $z = \sqrt{w}$ goes into another as a circuit is made about the point $w = \infty$. Hence, $w = \infty$ is the second branch point of the function $z = \sqrt{w}$ in the complex $w$-plane. The domain $D$, in which are defined single-valued branches of the function $z = \sqrt{w}$, is any domain of the $w$-plane in which encircling the branch points $w = 0$ and $w = \infty$ along a closed contour is impossible. Such a domain is, for instance, the entire $w$-plane with a cut (branch cut) along the positive real axis. Here, the edges of the branch cut are the boundary of the given domain, so that in the case of continuous motion inside the domain we cannot intersect the branch cut (the boundary of the domain).

If it is assumed that the argument of the points $w$ for the first branch varies over the range $0 < \arg w < 2\pi$ and for the second, over the range $2\pi < \arg w < 4\pi$, then the first branch of the function $z = \sqrt{w}$ maps the plane with the branch cut onto the upper half of the $z$-plane and the second branch of the given function maps the same domain onto the lower half of the $z$-plane.

In similar fashion it may readily be shown that the function $w = z^n$ ($n > 0$ is an integer) maps any sector $\frac{2\pi k}{n} < \arg z < \frac{2\pi(k+1)}{n}$ ($k = 0, 1, \ldots, n-1$) of the $z$-plane onto the entire $w$-plane cut along the positive real axis. These sectors are thus domains of univalence of the given function. The inverse function $z = \sqrt[n]{w}$ is multiple-valued, and the points $w = 0$ and $w = \infty$ are its branch points.

## 1.4. Differentiating the Function of a Complex Variable

### *a. Definition. Cauchy-Riemann conditions*

Up to this point, the theory of functions of a complex variable has been built up in complete analogy with the theory of functions of a real variable. However, the concept of a differentiable function of a complex variable, which was introduced by analogy with the corresponding concept in the theory of functions of a real variable, leads to essential differences.

Let us give a definition of the derivative of a function of a complex variable. Let there be given a function $f(z)$ in the domain $\mathfrak{G}$ in the complex $z$-plane. *If for the point $z_0 \in \mathfrak{G}$ there exists, as $\Delta z \to 0$, a limit (limiting value) of the difference quotient*

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$$

*then this limit is called the derivative of the function $f(z)$ with respect to the complex variable $z$ at the point $z_0$ and it is denoted by $f'(z_0)$; i.e.*

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} \tag{1-16}$$

Then $f(z)$ is called a *differentiable function* at the point $z_0$. We stress once again that if the limit (1-16) exists, it does not depend on the manner in which $\Delta z$ tends to zero; that is, on the manner in which the point $z = z_0 + \Delta z$ approaches the point $z_0$. The requirement of differentiability of a function of a complex variable at a point $z_0$ imposes extremely important conditions on the behaviour of the real and imaginary parts of this function in the neighbourhood of the point $(x_0, y_0)$. These conditions are known as the Cauchy-Riemann conditions. They can be stated in the form of the following theorems.

***Theorem 1.3.*** *If a function $f(z) = u(x, y) + iv(x, y)$ is differentiable at a point $z_0 = x_0 + iy_0$, then at the point $(x_0, y_0)$ there exist partial derivatives of the functions $u(x, y)$ and $v(x, y)$ with respect to the variables $x$, $y$, and the following relations* hold:*

$$\frac{\partial u(x_0, y_0)}{\partial x} = \frac{\partial v(x_0, y_0)}{\partial y}, \quad \frac{\partial u(x_0, y_0)}{\partial y} = -\frac{\partial v(x_0, y_0)}{\partial x} \tag{1-17}$$

*Proof.* By hypothesis, there exists the limit (1-16) that is independent of the manner in which $\Delta z$ approaches zero. Put $\Delta z = \Delta x$ and consider the expression

$$f'(z_0) = \lim_{\Delta x \to 0} \frac{u(x_0 + \Delta x, y_0) - u(x_0, y_0)}{\Delta x} + \\ + i \lim_{\Delta x \to 0} \frac{v(x_0 + \Delta x, y_0) - v(x_0, y_0)}{\Delta x}$$

From the existence of the limit of a complex expression there follows the existence of the limits of its real and imaginary parts. Therefore, at the point $x_0$, $y_0$ there exist partial derivatives with respect to $x$ of the functions $u(x, y)$ and $v(x, y)$ and we have the formula

$$f'(z_0) = u_x(x_0, y_0) + iv_x(x_0, y_0)$$

Putting $\Delta z = i\,\Delta y$, we find

$$f'(z_0) = -i \lim_{\Delta y \to 0} \frac{u(x_0, y_0 + \Delta y) - u(x_0, y_0)}{\Delta y} + \\ + \lim_{\Delta y \to 0} \frac{v(x_0, y_0 + \Delta y) - v(x_0, y_0)}{\Delta y} = -iu_y(x_0, y_0) + v_y(x_0, y_0)$$

Comparing these two formulas, we are convinced of the validity of the relations (1-17).

---

* The relations (1-17) are ordinarily called Cauchy-Riemann relations.

***Theorem 1.4.*** *If the functions* $u(x, y)$ *and* $v(x, y)$ *are differentiable at the point* $(x_0, y_0)$, *and their partial derivatives are connected by the relations* (1-17), *then the function* $f(z) = u(x, y) + iv(x, y)$ *is a differentiable function of the complex variable* $z$ *at the point* $z_0 = x_0 + iy_0$.

*Proof*. By the definition of differentiability, the increments of the functions $u(x, y)$ and $v(x, y)$ in the neighbourhood of the point $(x_0, y_0)$ may be written as

$$u(x_0 + \Delta x, y_0 + \Delta y) - u(x_0, y_0) = u_x(x_0, y_0)\,\Delta x + u_y(x_0, y_0)\,\Delta y + \xi(x, y)$$

$$v(x_0 + \Delta x, y_0 + \Delta y) - v(x_0, y_0) = v_x(x_0, y_0)\,\Delta x + v_y(x_0, y_0)\,\Delta y + \eta(x, y) \quad (1\text{-}18)$$

where the functions $\xi(x, y)$ and $\eta(x, y)$ approach zero, as $x \to x_0$, $y \to y_0$, faster than $\Delta x$ and $\Delta y$ $\left(\lim\limits_{|\Delta z| \to 0} \frac{\xi(x, y)}{|\Delta z|} = 0, \lim\limits_{|\Delta z| \to 0} \frac{\eta(x, y)}{|\Delta z|} = 0, |\Delta z| = \sqrt{(\Delta x)^2 + (\Delta y)^2}\right)$. Now let us form the difference quotient $\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$, where $\Delta z = \Delta x + i\Delta y$ and, utilizing (1-18) and (1-17), transform it to the form

$$\frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = u_x(x_0, y_0)\frac{\Delta x + i\,\Delta y}{\Delta x + i\,\Delta y} + v_x(x_0, y_0)\frac{i\,\Delta x - \Delta y}{\Delta x + i\,\Delta y} + \frac{\xi(x, y) + i\eta(x, y)}{\Delta x + i\,\Delta y} = u_x(x_0, y_0) + iv_x(x_0, y_0) + \frac{\zeta(z)}{\Delta z}$$

$$[\zeta(z) = \xi(x, y) + i\eta(x, y)]$$

Note that the last term of this formula approaches zero as $\Delta z$ tends to zero, and the first terms remain unchanged. Therefore, there is a limit, $\lim\limits_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z} = f'(z_0)$, and this proves the differentiability of the function $f(z)$ at the point $z_0$.

*If a function* $f(z)$ *is differentiable at all points of some domain* $\mathfrak{G}$, *and its derivative is continuous in this domain, then the function* $f(z)$ *is called an analytic function* in the domain* $\mathfrak{G}$.

---

* The definition of an analytic function given here differs from that usually given in the literature by the additional requirement that the derivative be continuous. This is done in order to facilitate subsequent proofs. As follows from a more detailed investigation, the mathematical content of the concept of an analytic function is not thereby altered. For one thing, it may be shown that if we further require that the function $f(z)$ in the domain $\mathfrak{G}$ be continuous, fulfilment of the Cauchy-Riemann conditions (1-17) everywhere in the domain is a necessary and sufficient condition for the analyticity of $f(z)$ and the continuity of all its derivatives in $\mathfrak{G}$. (See [10].)

The continuity of partial derivatives, it will be recalled, is a sufficient condition for the existence of the first differential (differentiability) of a function of many variables. It therefore follows from Theorems 1.3 and 1.4 that a *necessary and sufficient condition for the analyticity of a function* $f(z) = u(x, y) + iv(x, y)$ *in a domain* $\mathfrak{G}$ *is the existence, in that domain, of continuous partial derivatives of the functions* $u(x, y)$ *and* $v(x, y)$ *connected by the Cauchy-Riemann relations* (1-17).

The concept of the analytic function is basic to the theory of functions of a complex variable by virtue of the specific role played by the class of analytic functions both in the solution of numerous purely mathematical problems and in various applications of functions of a complex variable in cognate fields of the natural sciences.

The Cauchy-Riemann relations are frequently employed in studying various properties of analytic functions. The equalities (1-17) are not the only possible form of the Cauchy-Riemann relations. As the reader himself can establish, the real and imaginary parts of the analytic function $f(z) = u(\rho, \varphi) + iv(\rho, \varphi)$ of the complex variable $z = \rho e^{i\varphi}$ are connected by the relations

$$\frac{\partial u}{\partial \rho} = \frac{1}{\rho}\frac{\partial v}{\partial \varphi}, \quad \frac{1}{\rho}\frac{\partial u}{\partial \varphi} = -\frac{\partial v}{\partial \rho} \tag{1-19}$$

where $\rho$ and $\varphi$ are the polar coordinates of the point $(x, y)$. In similar fashion, it is easy to establish that the modulus and argument of the analytic function $f(z) = R(x, y)\, e^{i\Phi(x, y)}$ are connected by the relations

$$\frac{\partial R}{\partial x} = R\frac{\partial \Phi}{\partial y}, \quad \frac{\partial R}{\partial y} = -R\frac{\partial \Phi}{\partial x} \tag{1-20}$$

We will also note that the relations (1-17) permit obtaining various expressions for the derivative of a function of a complex variable:

$$\begin{aligned} f'(z) &= u_x(x, y) + iv_x(x, y) = v_y(x, y) + iv_x(x, y) \\ &= u_x(x, y) - iu_y(x, y) = v_y(x, y) - iu_y(x, y) \end{aligned} \tag{1-21}$$

Here, the derivative $f'(z)$ is each time expressed in terms of the partial derivatives of the functions $u(x, y)$ and $v(x, y)$.

### *b. Properties of analytic functions*

The definition of a derivative (1-16) permits extending to analytic functions of a complex variable a number of properties of differentiable functions of a real variable.

1. If the function $f(z)$ is analytic in the domain $\mathfrak{G}$, then it is continuous in this domain.

2. If $f_1(z)$ and $f_2(z)$ are analytic functions in the domain $\mathfrak{G}$, then their sum and product are also analytic functions in $\mathfrak{G}$ and the function $\varphi(z) = \frac{f_1(z)}{f_2(z)}$ is an analytic function wherever $f_2(z) \neq 0$.

3. If $w = f(z)$ is an analytic function in the domain $\mathfrak{G}$ of the plane of the complex variable $z$, and an analytic function $\zeta = \varphi(w)$ is defined in the range $G$ of its values in the $w$-plane, then the function $F(z) = \varphi[f(z)]$ is an analytic function of the complex variable $z$ in the domain $\mathfrak{G}$.

4. If $w = f(z)$ is an analytic function in the domain $\mathfrak{G}$, and $|f'(z)| \neq 0$ in the neighbourhood of a point $z_0 \in \mathfrak{G}$, then in the neighbourhood of the point $w_0 = f(z_0)$ of the domain $G$ of values of the function $f(z)$ is defined an inverse function $z = \varphi(w)$, which is an analytic function of the complex variable $w$. We then have the relation $f'(z_0) = \frac{1}{\varphi'(w_0)}$.

*Proof*. For the existence of an inverse function, it is necessary that the equations $u = u(x, y)$ and $v = v(x, y)$ be solvable for $x$, $y$ in the neighbourhood of point $w_0$. For this purpose it is sufficient that in the neighbourhood of point $z_0$ the following condition be fulfilled:

$$\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = u_x v_y - u_y v_x \neq 0$$

By virtue of the relations (1-17) this condition may be rewritten as $u_x^2 + v_x^2 \neq 0$. But when $|f'(z)| \neq 0$, this holds true. Thus, the existence of the inverse function $z = \varphi(w)$ is proved. Forming the difference quotient $\frac{\Delta z}{\Delta w} = \frac{1}{\frac{\Delta w}{\Delta z}}$, it is easy to prove the existence and the continuity of the derivative $\varphi'(w_0)$, provided that $|f'(z_0)| \neq 0$.

5. Let the function $u(x, y)$, which is the real part of the analytic function $f(z)$, be given in the domain $\mathfrak{G}$ of the $x$, $y$-plane. Then the imaginary part of this function is defined to within an additive constant. Indeed, by virtue of the Cauchy-Riemann conditions, the total differential of the unknown function $v(x, y)$ is determined uniquely from the given function $u(x, y)$:

$$dv = v_x\,dx + v_y\,dy = -u_y\,dx + u_x\,dy$$

This proves the assertion.

6. Let the function $f(z)$ be analytic in the domain $\mathfrak{G}$. Consider in the corresponding domain of the $x$, $y$-plane, families of the curves $u(x, y) = C$ and $v(x, y) = C$, which are level lines of the real and imaginary parts of the function $f(z)$. By means of the relations

(1-17) it is easy to demonstrate that at all points of the given domain, grad $u \cdot$grad $v = u_x v_x + u_y v_y = -u_x u_y + u_y u_x = 0$. Since a gradient is orthogonal to a level line, it follows that the families of curves $u(x, y) = C$ and $v(x, y) = C$ are mutually orthogonal.

### c. *The geometric meaning of the derivative of a function of a complex variable*

Let $f(z)$ be an analytic function in some domain $\mathfrak{G}$. Choose some point $z_0 \in \mathfrak{G}$ and draw through it an arbitrary* curve $\gamma_1$ lying entirely in $\mathfrak{G}$. The function $f(z)$ maps the domain $\mathfrak{G}$ of the complex $z$-plane onto some domain $G$ of the complex $w$-plane. Let the point $z_0$ go into the point $w_0$ and the curve $\gamma_1$ into the curve $\Gamma_1$ that passes

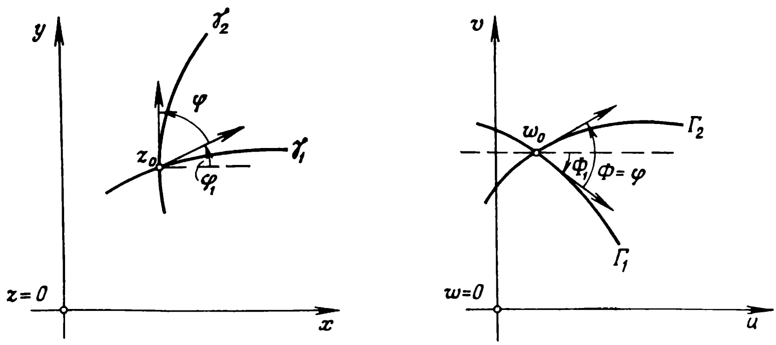


**Fig. 1.7**

through $w_0$ (Fig. 1.7). It is given that there exists a derivative $f'(z)$ of the function $w = f(z)$ at the point $z_0$. Suppose that $f'(z_0) \neq 0$ and represent the complex number $f'(z_0)$ in exponential form**

$$f'(z_0) = \lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = ke^{i\alpha} \tag{1-22}$$

We choose $\Delta z$ to approach zero in such a manner that the points $z = z_0 + \Delta z$ should lie on the curve $\gamma_1$. Obviously, the points $w = w_0 + \Delta w$ corresponding to them lie on the curve $\Gamma_1$. The complex numbers $\Delta z$ and $\Delta w$ are depicted by the vectors of transversals to the curves $\gamma_1$ and $\Gamma_1$, respectively. Note that arg $\Delta z$ and arg $\Delta w$ have the geometric meaning of angles of the appropriate vectors with positive directions of the axes $x$ and $u$, and $|\Delta z|$ and $|\Delta w$

* Here and henceforward, unless otherwise stated, an arbitrary curve is to be understood as a smooth curve.

** The condition $f'(z_0) \neq 0$ is necessary for such a representation to be possible.

are the lengths of these vectors. As $\Delta z \to 0$, the vectors of the transversals pass into vectors of tangents to the corresponding curves. From (1-22) it follows that

$$\alpha = \arg f'(z_0) = \lim_{\Delta z \to 0} \arg \Delta w - \lim_{\Delta z \to 0} \arg \Delta z = \Phi_1 - \varphi_1 \quad (1\text{-}23)$$

i.e., the argument $\alpha$ of the derivative has the geometric meaning of the difference of the angle $\Phi_1$ of the vector of the tangent to the curve $\Gamma_1$ at the point $w_0$ with the $u$-axis and the angle $\varphi_1$ of the vector of the tangent to the curve $\gamma_1$ at the point $z_0$ with the $x$-axis (Fig. 1.7). Since the derivative $f'(z_0)$ does not depend on the manner in which the limit is approached, this difference will be the same for any other curve passing through the point $z_0$ (though the values of the angles $\Phi_1$ and $\varphi_1$ themselves may change). Whence it follows that in a mapping accomplished by the analytic function $f(z)$ satisfying the condition $f'(z_0) \neq 0$, the angle $\varphi = \varphi_2 - \varphi_1$ between any curves $\gamma_2$, $\gamma_1$ intersecting in the point $z_0$ is equal to the angle $\Phi = \Phi_2 - \Phi_1$ between their images (the curves $\Gamma_2$ and $\Gamma_1$) intersecting in the point $w_0 = f(z_0)$. Observe that in the process, not only the absolute value of the angles between the curves $\gamma_2$, $\gamma_1$ and their images is preserved, but the directions of the angles are preserved as well. This property of the given mapping is called the *angle-preserving property*.

Analogously, from the relation (1-22) we get

$$k = |f'(z_0)| = \lim_{\Delta z \to 0} \frac{|\Delta w|}{|\Delta z|} \quad (1\text{-}24)$$

That is, to within higher-order infinitesimals, we have the equality $|\Delta w| = k\,|\Delta z|$. Observe that this relation too is independent of the choice of the curve $\gamma_1$. The geometric meaning of this relation consists in the fact that in the case of a mapping accomplished by an analytic function satisfying the condition $f'(z_0) \neq 0$, infinitesimal line elements are transformed in a similar fashion, and $|f'(z_0)|$ defines the coefficient of magnification. This property of the given mapping is called the *property of invariance of stretching*.

*The mapping of a neighbourhood of a point $z_0$ onto the neighbourhood of a point $w_0$ accomplished by an analytic function $w = f(z)$ and possessing at the point $z_0$ the angle-preserving property and invariance of stretching is called a conformal mapping.* In conformal mapping of the neighbourhood of the point $z_0$ onto the neighbourhood of the point $w_0$, infinitely small triangles with vertex at the point $z_0$ are transformed into similar infinitely small triangles with vertex at the point $w_0$. The fundamentals of the theory of conformal mapping will be given in more detail in Chapter 6.

### d. Examples

To conclude this section, we note that, as may readily be verified, the linear function and the function $w = z^2$ which were introduced at the beginning of the section are analytic functions over the entire complex plane; the function $w = \frac{1}{z}$ is analytic everywhere with the exception of the point $z = 0$. Since the definition of the derivative (1-16) is similar to the definition of a derivative of a function of one real variable, it follows that for the derivatives of the given functions of a complex variable we have the expressions

$$(az+b)' = a, \quad (z^2)' = 2z, \quad \left(\frac{1}{z}\right)' = -\frac{1}{z^2} \tag{1-25}$$

Let us consider the function of the complex variable $w = e^z$, which is widely employed in a variety of applications. We define this function by specifying the analytic expressions of its real and imaginary parts:

$$u(x, y) = e^x \cos y, \quad v(x, y) = e^x \sin y \tag{1-26}$$

On the real axis this function coincides with the real function $e^x$ of the real argument $x$ and, as will be demonstrated later on, it preserves the basic properties of an exponential function in the complex domain. It is therefore natural to maintain the notation

$$e^z = e^x (\cos y + i \sin y) = e^x \cdot e^{iy} \tag{1-27}$$

We shall show that $e^z$ is an analytic function throughout the entire complex $z$-plane. To do this, verify the fulfilment of the Cauchy-Riemann conditions (1-17):

$$\frac{\partial u}{\partial x} = e^x \cos y = \frac{\partial v}{\partial y}, \quad \frac{\partial u}{\partial y} = -e^x \sin y = -\frac{\partial v}{\partial x}$$

and note that all the derivatives in these equalities are continuous with respect to the collection of arguments throughout the entire $x$, $y$-plane. Computing the derivative of $e^z$ by the formulas (1-21), we get

$$(e^z)' = u_x + iv_x = e^x (\cos y + i \sin y) = e^z$$

Analogously,

$$(e^{\alpha z})' = \alpha e^{\alpha z} \tag{1-28}$$

where $\alpha$ is an arbitrary complex constant.

Let us consider two more functions $f_1(z)$ and $f_2(z)$ defined by means of the relations

$$f_1(z) = \frac{1}{2}(e^{iz} + e^{-iz}), \quad f_2(z) = \frac{1}{2i}(e^{iz} - e^{-iz}) \tag{1-29}$$

As is readily seen, for the real values of the complex variable $z = x$, these functions coincide with cos $x$ and sin $x$; therefore, it is natural to preserve the earlier notation for them. Later on we will make a detailed study of the properties of these functions; for the present we need only observe that, as complex functions of an analytic function, cos $z$ and sin $z$ are analytic over the entire complex plane. By direct verification it will readily be seen that $(\cos z)' = -\sin z$. Indeed, by means of (1-28) we get

$$f_1'(z) = \frac{i}{2}(e^{iz} - e^{-iz}) = -f_2(z) \tag{1-30}$$

Similarly, direct computation yields

$$f_1^2(z) + f_2^2(z) \equiv 1 \tag{1-31}$$

since, by the rule of raising a complex number to an integral power, from (1-27) we get

$$(e^{\alpha z})^2 = e^{2\alpha z} \tag{1-32}$$

## 1.5. An Integral with Respect to a Complex Variable

### *a. Basic properties*

Let a piecewise smooth curve $C$ of finite length $L$ be given in the complex $z$-plane. Utilizing the parametric representation of the curve $C$, specify the coordinates $\xi$, $\eta$ of each of its points by the equations $\xi = \xi(t)$, $\eta = \eta(t)$, where $\xi(t)$ and $\eta(t)$ are piecewise smooth functions of the real parameter $t$ varying over the range $\alpha \leqslant t \leqslant \beta$ ($\alpha$ and $\beta$ can respectively take the values $\pm\infty$), which functions satisfy the condition $[\xi'(t)]^2 + [\eta'(t)]^2 \neq 0$. Specifying the coordinates $\xi$, $\eta$ of this curve $C$ is equivalent to specifying the complex function $\zeta(t) = \xi(t) + i\eta(t)$ of the real variable $t$.

Let the value of the function $f(\zeta)$ be defined at every point $\zeta$ of the curve $C$. An important concept in the theory of functions of a complex variable is that of the integral of a function $f(\zeta)$ over the curve $C$. This concept is introduced as follows. Partition the curve $C$ into $n$ arcs by the division points $\zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_n$, which correspond to increasing values of the parameter $t$ ($t_{i+1} > t_i$). Denote $\Delta\zeta_i = \zeta_i - \zeta_{i-1}$ and form the sum

$$S(\zeta_i, \zeta_i^*) = \sum_{i=1}^{n} f(\zeta_i^*)\Delta\zeta_i \tag{1-33}$$

where $\zeta_i^*$ is an arbitrary point of the $i$th arc.

*If as* max $|\Delta\zeta_i| \to 0$ *there exists a limit of the sum* (1-33) *that is independent of the manner of partitioning C and of the choice of* *points $\zeta_i^*$, then this limit is called the integral of the function $f(\zeta)$ over the curve $C$ and is denoted as*

$$\int_C f(\zeta)\,d\zeta \tag{1-34}$$

The question of the existence of the integral (1-34) reduces to the question of the existence of certain line integrals of the real part $u$ and the imaginary part $v$ of the function $f(z)$. Indeed, writing $f(\zeta_i^*) = u(P_i^*) + iv(P_i^*)$, $\Delta\zeta_i = \Delta\xi_i + i\,\Delta\eta_i$, where $P_i(\xi_i^*, \eta_i^*)$ is a point of the curve $C$ on the $x$, $y$-plane, we can represent the expression (1-33) as

$$S(\zeta_i, \zeta_i^*) = \sum_{i=1}^{n} \{u(P_i^*)\,\Delta\xi_i - v(P_i^*)\,\Delta\eta_i\} + i\sum_{i=1}^{n} \{u(P_i^*)\,\Delta\eta_i + v(P_i^*)\,\Delta\xi_i\}$$

The real and imaginary parts of $S(\zeta_i, \zeta_i^*)$ are integral sums of the line integrals of the second kind

$$\int_C u\,d\xi - v\,d\eta \quad \text{and} \quad \int_C u\,d\eta + v\,d\xi \tag{1-35}$$

respectively, whence the assertion follows. We stress that for the existence of the line integrals (1-35) and thus of the integral (1-34) with respect to a complex variable, it suffices that the functions $u$ and $v$ of real variables be piecewise continuous. This means that the integral (1-34) also exists when the function $f(z)$ is nonanalytic if the function is piecewise continuous.

Thus, we represent the integral (1-34) in the form

$$\int_C f(\zeta)\,d\zeta = \int_C u\,d\xi - v\,d\eta + i\int_C u\,d\eta + v\,d\xi \tag{1-36}$$

This relation can by itself serve as a definition of the integral of a function $f(z)$ over the curve $C$. There follow from it a number of properties which are an obvious consequence of the respective properties of line integrals:

1. $$\int_{AB} f(\zeta)\,d\zeta = -\int_{BA} f(\zeta)\,d\zeta \tag{1-37}$$

2. $$\int_{C_1} f(\zeta)\,d\zeta + \int_{C_2} f(\zeta)\,d\zeta = \int_{C_1+C_2} f(\zeta)\,d\zeta \tag{1-38}$$

3. If $a$ is a complex constant, then

$$\int_C af(\zeta)\,d\zeta = a\int_C f(\zeta)\,d\zeta \tag{1-39}$$

4. $$\int_C \{f_1(\zeta)+f_2(\zeta)\}\,d\zeta = \int_C f_1(\zeta)\,d\zeta + \int_C f_2(\zeta)\,d\zeta \tag{1-40}$$

5. $$\left|\int_C f(\zeta)\,d\zeta\right| \leqslant \int_C |f(\zeta)|\,ds \tag{1-41}$$

where $ds$ is the differential of the arc length of the curve $C$, and the integral on the right is a line integral of the first kind. Indeed, by virtue of the triangle inequality we have

$$\left|\int_C f(\zeta)\,d\zeta\right| = \left|\lim_{\max|\Delta\zeta_i|\to 0}\sum_{i=1}^{n} f(\zeta_i^*)\,\Delta\zeta_i\right|$$
$$\leqslant \lim_{\max|\Delta\zeta_i|\to 0}\sum_{i=1}^{n} |f(\zeta_i^*)|\,|\Delta\zeta_i| = \int_C |f(\zeta)|\,ds$$

If $\max\limits_{\zeta\in C} |f(\zeta)| = M$ and $L$ is the arc length of the curve $C$, then

$$\left|\int_C f(\zeta)\,d\zeta\right| \leqslant M\cdot L \tag{1-42}$$

6. The following formula for changing the integration variable holds:

$$\int_C f(z)\,dz = \int_\Gamma f[\varphi(\zeta)]\,\varphi'(\zeta)\,d\zeta \tag{1-43}$$

where $z = \varphi(\zeta)$ is an analytic function of $\zeta$ which establishes a one-to-one correspondence between the curves $C$ and $\Gamma$. In particular,

$$\int_C f(z)\,dz = \int_\alpha^\beta f[z(t)]\,z'(t)\,dt \tag{1-44}$$

where $z = z(t)$ is the parametric representation of the curve $C$, and $z(\alpha)$ and $z(\beta)$ are the initial and terminal points of the latter.

*Example*. As an example (which will be essential for what follows) of computing an integral over a complex variable, we consider the integral

$$I = \int_{C_\rho} \frac{d\zeta}{\zeta - z_0} \tag{1-45}$$

where the curve $C_\rho$ is a circle of radius $\rho$ centred in the point $z_0$, which circle may be traversed counterclockwise. Taking advantage of the parametric form of representing the curve $C_\rho$, $\zeta = z_0 + \rho e^{i\varphi}$ $(0 \leqslant \varphi \leqslant 2\pi)$, we get

$$I = \int_0^{2\pi} \frac{i\rho e^{i\varphi}\, d\varphi}{\rho e^{i\varphi}} = i \int_0^{2\pi} d\varphi = 2\pi i \tag{1-46}$$

Whence it follows that the integral (1-45) does not depend either on $\rho$ or on $z_0$.

*Note.* Formula (1-36), by virtue of which an integral with respect to a complex variable is a complex number, the real and imaginary parts of which are line integrals of the second kind, and also the relation (1-44) permit carrying the concept of an improper integral from a function of a real variable directly to the case of a complex variable. In this course we deal mainly with improper integrals of the first kind, which are integrals over an infinite curve $C$. An improper integral of the first kind over an infinite curve $C$ is said to be convergent if the limit exists of a sequence of integrals $\int_{C_n} f(\zeta)\, d\zeta$ over any sequence of finite curves $C_n$ that constitute a part of $C$, when $C_n$ tend to $C$; this limit does not depend on the choice of the sequence $\{C_n\}$. Only if for a certain choice of the sequence $\{C_n\}$ there exists a limit of the sequence of integrals $\int_{C_n} f(\zeta)\, d\zeta$ is the improper integral said to be convergent in the principal-value sense.

In the future we will consider integrals of functions that are analytic in a certain bounded domain; we will be interested in the case when the boundary of the domain is a piecewise smooth closed curve without self-intersections. *A piecewise smooth closed curve without points of self-intersection will be called a closed contour.* If a function $z(t)$ $(\alpha \leqslant t \leqslant \beta)$ represents parametrically a closed contour, then it satisfies the condition $z(t_i) \neq z(t_k)$ for $t_i \neq t_k$, except for $t_i = \alpha$, $t_k = \beta$. The integral (1-34) along a closed contour is often called a contour integral.

### *b. Cauchy's Theorem*

Since the value of a contour integral depends on the sense of integration, let us agree to take for the *positive sense of traversing* a contour a direction such that the interior domain bounded by the given closed contour remains *on the left* of the direction of motion. We will denote integration in the positive sense by the symbol

$\int\limits_{C+} f(z)\,dz$ or simply $\int\limits_{C} f(z)\,dz$ and integration in the negative sense by the symbol $\int\limits_{C-} f(z)\,dz$.

The properties of integrals, along a closed contour, of functions that are analytic inside the domain bounded by the given contour are largely determined by the familiar properties of line integrals of the second kind.* Let us recall** that for line integrals along a closed contour the following assertion holds: *if the functions $P(x, y)$ and $Q(x, y)$ are continuous in a closed domain $\overline{\mathfrak{G}}$ bounded by a piecewise smooth contour $C$, and their partial derivatives of the first order are continuous in $\mathfrak{G}$, then*

$$\int\limits_{C} P\,dx + Q\,dy = \iint\limits_{\mathfrak{G}} \left\{ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right\} dx\,dy \qquad (1\text{-}47)$$

Let us now take up the proof of the basic proposition of this section.

***Theorem 1.5 (Cauchy's theorem).*** *Let a single-valued analytic function $f(z)$ be given in a singly connected domain $\mathfrak{G}$. Then the integral of this function $f(z)$ along any closed contour $\Gamma$ lying entirely within $\mathfrak{G}$ is equal to zero.*

*Proof.* According to formula (1-36)

$$\int\limits_{\Gamma} f(\zeta)\,d\zeta = \int\limits_{\Gamma} u\,dx - v\,dy + i \int\limits_{\Gamma} v\,dx + u\,dy$$

Since the function $f(z)$ is analytic everywhere inside the contour $\Gamma$, it follows that the functions $u(x, y)$ and $v(x, y)$ possess continuous partial derivatives of the first order in the domain bounded by this contour. We can therefore apply formula (1-47) to the line integrals on the right-hand side of the last equation. Besides, the partial derivatives of the functions $u(x, y)$ and $v(x, y)$ are connected by the Cauchy-Riemann relations. Therefore,

$$\int\limits_{\Gamma} u\,dx - v\,dy = \iint\limits_{\mathfrak{G}} \left\{ -\frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \right\} dx\,dy = 0$$

---

* By our definition, the contours of integration are always piecewise smooth curves.

** Elsewhere this theorem has been proved with the supplementary condition of boundedness of the partial derivatives of the functions $P$ and $Q$ in the domain $\mathfrak{G}$, this condition being introduced to simplify the proof. In the case of a piecewise smooth boundary, this condition can be removed with the aid of an additional passage to the limit. Here we will not give the detailed proof and will confine ourselves to the remark just made.

and

$$\int_{\Gamma} v\,dx + u\,dy = \iint_{\mathfrak{G}} \left\{\frac{\partial u}{\partial x} - \frac{\partial v}{\partial y}\right\} dx\,dy = 0$$

This proves the assertion of the theorem.

Thus, Theorem 1.5 establishes the fact that the integral of an analytic function over any closed contour lying entirely in the singly connected domain of its analyticity is zero. Given the supplementary condition of continuity of the function in the closed domain, the given assertion holds true also for a closed contour which is the boundary of the domain of analyticity. This latter assertion is actually a slightly modified formulation of the Cauchy theorem. Because of its importance in practical applications, however, we will state it as a separate theorem.

***Theorem 1.6 (Second formulation of Cauchy's theorem).*** *If a function $f(z)$ is an analytic function in a singly connected domain $\mathfrak{G}$ bounded by a piecewise smooth contour $C$ and is continuous in the closed domain $\bar{\mathfrak{G}}$, then the integral of the function $f(z)$ along the boundary $C$ of the domain $\mathfrak{G}$ is equal to zero:*

$$\int_{C} f(\zeta)\,d\zeta = 0 \tag{1-48}$$

Cauchy's theorem establishes one of the basic properties of an analytic function of a complex variable. Its fundamental significance will be evident from what follows. For the present, we confine ourselves to the following remark.

The theorem was stated for a singly connected domain, but it can readily be generalized to the case of a multiply connected domain. Then the total boundary of the domain consists of several closed contours: the exterior contour $C_0$ and the interior contours $C_1, C_2, \ldots, C_n$. *The positive sense of traversal of the total boundary* of a multiply connected domain will be that sense for which *the domain is always on the left.* The exterior contour is traversed in the positive sense and the interior contours are traversed in the negative sense.

***Theorem 1.7.*** *Let $f(z)$ be an analytic function in a multiply connected domain $\mathfrak{G}$ bounded from without by the contour $C_0$ and from within by the contours $C_1, C_2, \ldots, C_n$, and let $f(z)$ be continuous in the closed domain $\bar{\mathfrak{G}}$. Then $\int_C f(\zeta)\,d\zeta = 0$, where $C$ is the total boundary of the domain $\mathfrak{G}$ consisting of the contours $C_0, C_1, \ldots, C_n$; the boundary $C$ is traversed in the positive sense.*

*Proof*. Draw smooth curves $\gamma_1, \ldots, \gamma_n$ connecting the contour $C_0$ with the contours $C_1$, $C_2$ and so forth (Fig. 1.8). Then the domain bounded by the curves $C_0, C_1, \ldots, C_n$ and the curves $\gamma_1, \gamma_2, \ldots$ $\ldots, \gamma_n$, which are traversed twice in opposite directions, proves

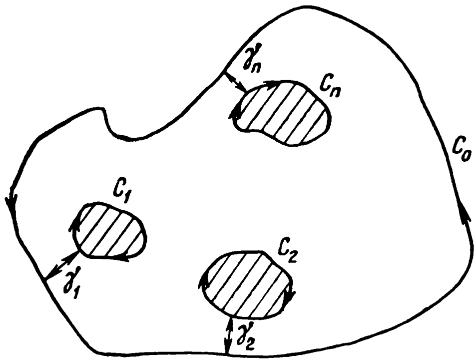


**Fig. 1.8**

to be singly connected*. By virtue of Theorem 1.6, the integral around the boundary of this domain is zero. But the integrals along the auxiliary curves $\gamma_1, \ldots, \gamma_n$ are taken twice in opposite directions and so cancel when summed. Therefore, we have the equality

$$\int_{C_0^+} f(\zeta)\, d\zeta + \int_{C_1^-} f(\zeta)\, d\zeta + \ldots + \int_{C_n^-} f(\zeta)\, d\zeta = 0 \tag{1-49}$$

(the plus and minus signs on $C_i$ indicate the sense of traversal).

### *c. Indefinite integral*

The following proposition is an important consequence of Cauchy's theorem. Let a function $f(z)$ be an analytic function in a singly connected domain $\mathfrak{G}$. In this domain, fix some point $z_0$ and denote by $\int_{z_0}^{z} f(\zeta)\, d\zeta$ the integral along a certain curve lying entirely in $\mathfrak{G}$ and connecting the points $z$ and $z_0$. By virtue of Cauchy's theorem, this integral is independent of the choice of the integration curve

* It is easy to see that the curves $\gamma_1, \ldots, \gamma_n$ can always be chosen so that they do not intersect; that is, we obtain a singly connected domain.

in the domain $\mathfrak{G}$ and is a single-valued function of $z$:

$$\int_{z_0}^{z} f(\zeta)\,d\zeta = \Phi(z) \tag{1-50}$$

***Theorem 1.8.*** *Let a function $f(z)$ be defined and continuous in some singly connected domain $\mathfrak{G}$ and let the integral of this function around any closed contour $\Gamma$ lying entirely in the given domain be zero. Then the function $\Phi(z) = \int_{z_0}^{z} f(\zeta)\,d\zeta$ $(z, z_0 \in \mathfrak{G})$ is an analytic function in the domain $\mathfrak{G}$ and $\Phi'(z) = f(z)$.*

*Proof.* Form the difference quotient

$$\frac{\Phi(z+\Delta z)-\Phi(z)}{\Delta z} = \frac{1}{\Delta z}\left\{\int_{z_0}^{z+\Delta z} f(\zeta)\,d\zeta - \int_{z_0}^{z} f(\zeta)\,d\zeta\right\} = \frac{1}{\Delta z}\int_{z}^{z+\Delta z} f(\zeta)\,d\zeta$$

The latter equality holds true due to the fact that the value of the integral defining the function $\Phi(z)$ is independent of the path of integration and (1-38). Let us choose the straight line connecting the points $z$ and $z+\Delta z$ as the integration path of the last integral. This integration path is convenient since we have the obvious relation $\int_{z}^{z+\Delta z} d\zeta = \Delta z$. Let us evaluate the expression

$$\left|\frac{\Phi(z+\Delta z)-\Phi(z)}{\Delta z} - f(z)\right| = \frac{1}{|\Delta z|}\left|\int_{z}^{z+\Delta z}\{f(\zeta)-f(z)\}\,d\zeta\right|$$

$$\leqslant \frac{1}{|\Delta z|}\max_{\zeta\in[z,\, z+\Delta z]}|f(\zeta)-f(z)|\cdot|\Delta z| = \max_{\zeta\in[z,\, z+\Delta z]}|f(\zeta)-f(z)|$$

By virtue of the continuity of the function $f(z)$ at the point $z$, for any positive number $\varepsilon$ there is a value of $\delta > 0$ such that for $|\Delta z| < \delta$ $\max_{\zeta\in[z,\, z+\Delta z]}|f(\zeta)-f(z)| < \varepsilon$, i.e. for any $\varepsilon > 0$ there is a $\delta > 0$ such that

$$\left|\frac{\Phi(z+\Delta z)-\Phi(z)}{\Delta z} - f(z)\right| < \varepsilon \text{ for } 0 < |\Delta z| < \delta$$

This means that there exists

$$\lim_{\Delta z\to 0}\frac{\Phi(z+\Delta z)-\Phi(z)}{\Delta z} = \Phi'(z) = f(z) \tag{1-51}$$

Thus, the function $\Phi(z)$ defined by the integral (1-50) at all points of the domain $\mathfrak{G}$ has a continuous derivative [by hypothesis, the function $f(z)$ is continuous in $\mathfrak{G}$]. Thus, $\Phi(z)$ is an analytic function in the domain $\mathfrak{G}$.

The foregoing theorem permits us to introduce the concept of the indefinite integral of a function of a complex variable. The analytic function $\Phi(z)$ is called the *primitive* of the analytic function $f(z)$ in the domain $\mathfrak{G}$ if in this domain the relation $\Phi'(z) = f(z)$ holds. It is obvious that the function $f(z)$ has an assemblage of different primitives, but, as may be readily proved, all the primitives of this function differ solely in constant terms.*

*The collection of all primitives of the function $f(z)$ is called the indefinite integral of the function $f(z)$.* Whence it follows that just as in the case of a function of a real variable, we have the formula

$$\int_{z_1}^{z_2} f(\zeta)\, d\zeta = F(z_2) - F(z_1)$$

where $F(z)$ is any primitive of $f(z)$. Indeed, the integral on the left is independent of the integration path. It can therefore be represented as

$$\int_{z_1}^{z_2} f(\zeta)\, d\zeta = \int_{z_0}^{z_2} f(\zeta)\, d\zeta - \int_{z_0}^{z_1} f(\zeta)\, d\zeta$$

where $z_0$ is an arbitrary point of the domain $\mathfrak{G}$. According to (1-50), each of the integrals on the right of this formula is a value of the definite primitive at the appropriate points, and since all primitives differ only by a constant, it is immaterial what primitive is substituted into the given formula.

As an example of what will be of essential interest later on, consider the function

$$f(z) = \int_{1}^{z} \frac{d\zeta}{\zeta} \qquad (1\text{-}52)$$

Since the integrand function is analytic over the entire complex $z$-plane, except at the point $z = 0$, the expression (1-52) is meaningful provided that the integration curve does not pass through $z = 0$. Here, in any singly connected domain $\mathfrak{G}$ of the complex plane not containing the point $z = 0$, the function $f(z)$ is a single-valued analytic function of $z$ that does not depend on the choice of integration path in the formula (1-52). We consider as such a domain the extended

* Indeed, since $\Phi'(z) = \Phi_1'(z) - \Phi_2'(z) \equiv 0$, where $\Phi_1(z)$ and $\Phi_2(z)$ are different primitives of the function $f(z)$, it follows from (1-21) that all partial derivatives of the real part and imaginary part of the function $\Phi(z)$ are identically zero, whence, by a familiar theorem of analysis, we get $\Phi(z) \equiv$ constant.

complex $z$-plane cut along the negative real axis, that is, the domain $-\pi < \arg z < \pi$. We will assume that the integration path in (1-52) lies entirely in the domain $-\pi < \arg z < \pi$; i.e. it does not intersect the cut and does not pass through the point $z = 0$. Then, choosing for the integration path in the formula (1-52) an appropriate segment of the real axis, for the real positive values $z = x$ we get

$$f(x) = \int_1^x \frac{dx}{x} = \ln x \tag{1-53}$$

That is, for the positive values of its argument the function $f(z)$ coincides with the logarithmic function of a real variable. Therefore, for the function (1-52) in the domain under investigation ($-\pi < \arg z < \pi$) we retain the earlier notation, putting

$$\ln z = \int_1^z \frac{dz}{z} \tag{1-54}$$

This equality (in which the integration path is chosen in the manner described above) may be regarded as the definition of a logarithmic function for all complex values of its argument, except values lying on the negative real axis $z = x \leqslant 0$. Later on (Chapter 3) we will study the properties of this function in detail; for the present we need only observe that by virtue of (1-51) we have the relation

$$(\ln z)' = \frac{1}{z} \tag{1-55}$$

In the domain $-\pi < \arg z < \pi$ the derivative of the logarithmic function has the same expression as for real positive values of the argument. It will be established later on that the function (1-54) is the inverse function of $w = e^z$ introduced in Section 1.4.

## 1.6. Cauchy's Integral

### *a. Deriving Cauchy's formula*

In Section 1.5 we proved Cauchy's theorem, which implies a number of important corollaries; in particular, it permits establishing a definite relation between the values of an analytic function in the interior points of the domain of its analyticity and the boundary values of the function. Our job now is to establish this relation.

Let the function $f(z)$ be analytic in a singly connected domain $\mathfrak{G}$ bounded by the contour $C$. Take an arbitrary interior point $z_0$ and construct a closed contour $\Gamma$ which lies entirely in $\mathfrak{G}$ and contains the point $z_0$. Consider the auxiliary function

$$\varphi(z) = \frac{f(z)}{z - z_0} \tag{1-56}$$

The function $\varphi(z)$ is obviously analytic everywhere in the domain $\mathfrak{G}$ except at the point $z_0$. Therefore, if in $\mathfrak{G}$ we take a closed contour $\gamma$ lying inside $\Gamma$ so that the point $z_0$ lies inside a domain bounded

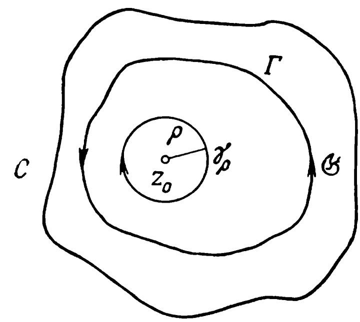


**Fig. 1.9**

by the contour $\gamma$, then the function $\varphi(z)$ will be analytic in the doubly connected domain $\mathfrak{G}^*$, which lies between the contours $\Gamma$ and $\gamma$. According to Cauchy's theorem, the integral of the function $\varphi(z)$ along the curve $\Gamma + \gamma$ is zero:

$$\int_{\Gamma^+} \frac{f(\zeta)}{\zeta - z_0} d\zeta + \int_{\gamma^-} \frac{f(\zeta)}{\zeta - z_0} d\zeta = 0$$

Reversing the sense of integration in the second integral, we can rewrite this equation as

$$\int_{\Gamma^+} \frac{f(\zeta)}{\zeta - z_0} d\zeta = \int_{\gamma^+} \frac{f(\zeta)}{\zeta - z_0} d\zeta \tag{1-57}$$

Since the integral on the left is independent of the choice of contour $\gamma$, the integral on the right has this property as well. For what follows it will be convenient to take as the integration contour $\gamma$ a circle $\gamma_\rho$ of a certain radius $\rho$ centred in the point $z_0$ (Fig. 1.9). Putting $\zeta = z_0 + \rho e^{i\varphi}$, we have

$$\int_{\Gamma^+} \frac{f(\zeta)}{\zeta - z_0} d\zeta = i \int_0^{2\pi} f(\zeta)\, d\varphi$$

We transform the latter integral as follows:

$$\int_0^{2\pi} f(\zeta)\,d\varphi = \int_0^{2\pi} [f(\zeta)-f(z_0)]\,d\varphi + \int_0^{2\pi} f(z_0)\,d\varphi$$
$$= \int_0^{2\pi} [f(\zeta)-f(z_0)]\,d\varphi + 2\pi f(z_0) \quad (1\text{-}58)$$

We now let $\rho$ approach zero. Since $f(z)$ is analytic and, consequently, continuous in the domain $\mathfrak{G}$, it follows that for any positive number $\varepsilon$ there is a value of $\rho$ such that $|f(\zeta) - f(z_0)| < \varepsilon$ for $|\zeta - z_0| < \rho$. Whence it follows that as $\rho \to 0$ there exists a limit

$$\lim_{\rho\to 0} \int_0^{2\pi} [f(\zeta)-f(z_0)]\,d\varphi = 0$$

Since in formula (1-58) the last term is independent of $\rho$, it follows that $\int_0^{2\pi} f(\zeta)\,d\varphi = 2\pi f(z_0)$ and consequently $\int_{\gamma^+} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta = 2\pi i f(z_0)$ and according to (1-57)

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(\zeta)}{\zeta - z_0}\,d\zeta \quad (1\text{-}59)$$

*The integral on the right-hand side of* (1-59) *expresses the value of the analytic function* $f(z)$ *at a certain point* $z_0$ *in terms of its value on any contour* $\Gamma$ *lying in the domain of analyticity of the function* $f(z)$ *and containing the point* $z_0$. This is *Cauchy's integral.* The formula (1-59) is often called *Cauchy's formula.*

*Note 1.* In (1-59) the integration is performed around the closed contour $\Gamma$ lying entirely within the domain of analyticity of the function $f(z)$ and containing the point $z_0$. Given the supplementary condition of continuity of $f(z)$ in the closed domain $\overline{\mathfrak{G}}$, a similar formula holds true (by virtue of Theorem 1.6) also when integrating along the boundary $C$ of the domain $\mathfrak{G}$.

*Note 2.* The considerations remain valid in the case of a multiply connected domain $\mathfrak{G}$ as well. Here, when deriving the basic formula (1-59), one should consider a closed contour $\Gamma$ such that can shrink to the point $z_0$ all the while remaining in $\mathfrak{G}$. Then it is easy to show that, provided that the function $f(z)$ is continuous in the closed domain $\overline{\mathfrak{G}}$ with a piecewise smooth boundary, formula (1-59) holds true for integration in the positive sense around the complete boundary $C$ of the given multiply connected domain.

### *b. Corollaries to Cauchy's formula*

There are several remarks to be made regarding formula (1-59).

1. An integral of the form $\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z_0}\,d\zeta$ around a closed contour $\Gamma$ lying entirely in the domain $\mathfrak{G}$ of analyticity of the function $f(z)$ is meaningful for any position of the point $z_0$ in the complex plane, provided that this point does not lie on the contour $\Gamma$. Then, if $z_0$ lies inside $\Gamma$, the value of the integral is equal to $f(z_0)$; if $z_0$ lies outside $\Gamma$, the value of the integral is zero, since in this case the integrand function is analytic everywhere inside $\Gamma$. And so

$$\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z_0}\,d\zeta=\begin{cases} f(z_0), & z_0 \text{ inside } \Gamma \\ 0, & z_0 \text{ outside } \Gamma \end{cases} \tag{1-60}$$

For $z_0 \in \Gamma$ the integral $I(z_0)=\frac{1}{2\pi i}\int_{\Gamma}\frac{f(\zeta)}{\zeta-z_0}\,d\zeta$ does not exist in the ordinary sense; however, given supplementary requirements on the behaviour of the function $f(\zeta)$ on the contour $\Gamma$, this integral can be imbued with definite meaning. Thus, if on the contour $\Gamma$ the function $f(\zeta)$ satisfies the Hölder condition

$$|f(\zeta_1)-f(\zeta_2)|\leqslant K\,|\zeta_1-\zeta_2|^{\gamma},\quad 0<\gamma<1$$

then a *Cauchy principal value* of the integral $I(z_0)$ exists:

$$PV\,I(z_0)=\lim_{\varepsilon\to 0}\frac{1}{2\pi i}\int_{\Gamma_\varepsilon}\frac{f(\zeta)}{\zeta-z_0}\,d\zeta$$

where $\Gamma_\varepsilon$ is a portion of $\Gamma$ exterior to the circle $|z-z_0|<\varepsilon$. Here,

$$PVI(z_0)=\frac{1}{2}f(z_0)$$

2. Let $f(z)$ be an analytic function in a singly connected domain $\mathfrak{G}$ and $z_0$ some interior point of the domain. Describe about this point as centre a circle of radius $R_0$ lying entirely in $\mathfrak{G}$. Then by Cauchy's formula we obtain

$$f(z_0)=\frac{1}{2\pi i}\int_{C_{R_0}}\frac{f(\zeta)}{\zeta-z_0}\,d\zeta$$

But on the circle $C_{R_0}$, $\zeta=z_0+R_0e^{i\varphi}$ and so

$$f(z_0)=\frac{1}{2\pi}\int_{0}^{2\pi}f(z_0+R_0e^{i\varphi})\,d\varphi \tag{1-61}$$

or

$$f(z_0)=\frac{1}{2\pi R_0}\int\limits_{C_{R_0}} f(\zeta)\,ds \tag{1-62}$$

This is termed the *mean-value formula* and expresses the value of an analytic function in the centre of a circle as the mean of its boundary values.

### c. The maximum-modulus principle of an analytic function

*Let a function $f(z)$ be analytic in a domain $\mathfrak{G}$ and continuous in a closed domain $\overline{\mathfrak{G}}$. Then either $|f(z)|\equiv$ constant or $|f(z)|$ attains maximum values only on the boundary of the domain.*

It is given that a real function of two real variables

$$|f(z)|=\sqrt{u^2(x,y)+v^2(x,y)}$$

is continuous in a closed domain. It therefore has a maximum value $M$ at some point $(x_0, y_0)$ of that domain. That is

$$M=|f(z_0)|\geqslant|f(z)|,\quad \begin{matrix} z_0=x_0+iy_0 \\ z\in\overline{\mathfrak{G}} \end{matrix} \tag{1-63}$$

Suppose that the point $z_0$ is an interior point of $\mathfrak{G}$. Construct in $\mathfrak{G}$ a circle $K_0$ of some radius $R$ centred in the point $z_0$, and write down the mean-value formula for $z_0$ and $R$. Taking into account (1-63), we get

$$2\pi M=\left|\int_0^{2\pi} f(\zeta)\,d\varphi\right|\leqslant\int_0^{2\pi}|f(\zeta)|\,d\varphi\leqslant 2\pi M$$

Consequently

$$\int_0^{2\pi}|f(\zeta)|\,d\varphi=2\pi M \tag{1-64}$$

From this relation, by virtue of the continuity of the function $f(\zeta)$ on the integration contour and due to the inequality (1-63), it follows that

$$|f(\zeta)|=M \quad \text{for} \quad \zeta=z_0+Re^{i\varphi} \tag{1-65}$$

Indeed, by (1-63) the absolute value $|f(\zeta)|$ cannot be greater than $M$ at any one point of integration. If we assume that at some point $\zeta_0$ of the integration path the absolute value $|f(\zeta_0)|$ is strictly less than $M$, then from the continuity of $|f(\zeta)|$ it follows that $|f(\zeta)|$ is strictly less than $M$ in some neighbourhood of the point $\zeta_0$ as well; i.e. there is an interval $[\varphi_1, \varphi_2]$ of integration on which

$$|f(\zeta)| \leqslant M - \varepsilon, \quad \varepsilon > 0$$

Then

$$\int_0^{2\pi} |f(\zeta)|\, d\varphi = \int_{\varphi_1}^{\varphi_2} |f(\zeta)|\, d\varphi + \int_0^{\varphi_1} |f(\zeta)|\, d\varphi + \int_{\varphi_2}^{2\pi} |f(\zeta)|\, d\varphi$$

$$\leqslant (M-\varepsilon)(\varphi_2-\varphi_1) + M[2\pi - (\varphi_2-\varphi_1)] < 2\pi M$$

but this contradicts (1-64). And so the relation (1-65) indeed holds true. This means that on a circle of radius $R$ centred in the point $z_0$ the function $|f(z)|$ has a constant value equal to its maximum value in the domain $\mathfrak{G}$. The same will occur on any circle of lesser

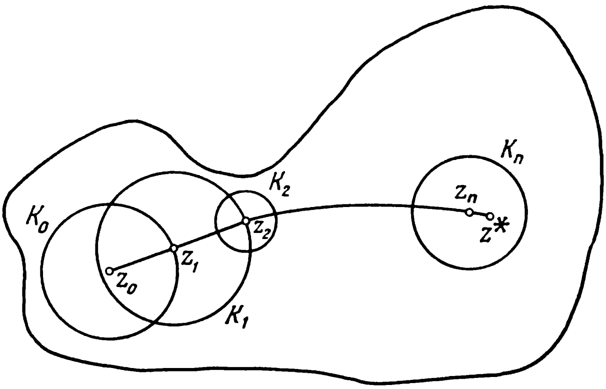


**Fig. 1.10**

radius with centre in the point $z_0$, and, consequently, in the whole of $K_0$. It is now easy to demonstrate that the function $|f(z)|$ has this same value also at any other interior point $z^*$ of the domain $\mathfrak{G}$. To do this, connect the points $z_0$ and $z^*$ by a curve $C$ lying entirely in $\mathfrak{G}$ and distant from its boundary by at least a certain positive number $d$. Take a point $z_1$, which is the last point of intersection of the curve $C$ with the circumference $K_0$ (Fig. 1.10). Since $|f(z_1)| = M$, by repeating the above reasoning we can show that inside $K_1 \subset \mathfrak{G}$ of radius $R_1 \leqslant d$, centred in $z_1$, the absolute value of the function $f(z)$ takes on a constant value equal to the maximum value $M$. Taking on curve $C$ a point $z_2$, which is the last point of intersection of the curve $C$ and the circumference $K_1$, and continuing the given process, we finally (after a finite number of steps) find that inside the circle $K_n$, to which the point $z^*$ belongs, the equality $|f(z)| = M$ holds true, thus proving the assertion.

We have thus demonstrated that if $|f(z)|$ takes on a maximum value $M$ in some interior point of a domain, then $|f(z)| \equiv M$ throughout the domain.*

Thus, if a function $|f(z)|$ is not a constant quantity in a domain $\mathfrak{G}$, then it cannot attain a maximum value in the interior points of $\mathfrak{G}$. But since a function that is continuous in a closed domain attains its maximum value in some point of the domain, in the latter case the function $|f(z)|$ must attain its maximum value at boundary points.

One final remark. Note that *if a function $f(z)$ that is analytic in a domain $\mathfrak{G}$ is not zero at any point of the domain and is continuous in $\overline{\mathfrak{G}}$, then the minimum-modulus principle holds true.* To prove this assertion, it suffices to consider the function $\varphi(z) = \frac{1}{f(z)}$ and take advantage of the principle of the maximum modulus of this function.

## 1.7. Integrals Dependent on a Parameter

### *a. Analytic dependence on a parameter*

When considering the Cauchy integral, we see that the integrand function depends on two complex variables: the variable of integration $\zeta$ and a fixed value of the variable $z_0$. Thus, Cauchy's integral is an integral which is dependent on a parameter $z_0$. It is natural to pose the question of the general properties of integrals with respect to a complex variable which depend on a parameter.

Let there be given a function of two complex variables** $\varphi(z, \zeta)$ uniquely defined for the values of the complex variable $z = x + iy$ from the domain $\mathfrak{G}$ and for the values of the complex variable $\zeta = \xi + i\eta$ which belong to a certain piecewise smooth curve $C$. The mutual positions of the domain $\mathfrak{G}$ and the curve $C$ may be quite arbitrary. Let the function of two complex variables $\varphi(z, \zeta)$ satisfy the following conditions:

(a) *The function $\varphi(z, \zeta)$ for any value of $\zeta \in C$ is an analytic function of $z$ in the domain $\mathfrak{G}$.*

---

* As follows from the relations (1-20), in this case the argument of an analytic function $f(z)$ also retains a constant value in the domain $\mathfrak{G}$, whence it follows that if the absolute value (modulus) of an analytic function is constant in some domain, then the function is identically equal to the constant in that domain.

** A function of two complex variables $z$, $\zeta$ is defined by a law which associates some complex number $w$ with each pair of values $z$, $\zeta$ from the domain of their definition. A short review of the theory of functions of many complex variables is given in Appendix 3.

for its derivative:

$$f''(z) = \frac{2}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^3} \, d\zeta \tag{1-71}$$

Since for any interior point $z$ of the domain $\mathfrak{G}$ a corresponding closed subdomain $\mathfrak{G}'$ may be constructed, the formulas (1-70) and (1-71) hold true at any point $z$. A more general theorem is also valid.

***Theorem 1.9.*** *Let a function $f(z)$ be analytic in the domain $\mathfrak{G}$ and continuous in the closed domain $\overline{\mathfrak{G}}$. Then there exists a derivative of any order of the function $f(z)$ in the interior points of $\mathfrak{G}$; for this derivative we have the formula*

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_{\Gamma} \frac{f(\zeta)}{(\zeta - z)^{n+1}} \, d\zeta \tag{1-72}$$

The proof of this theorem is obtained by repeating the foregoing reasoning an appropriate number of times.

And so if a function $f(z)$ is analytic in the domain $\mathfrak{G}$, then $f(z)$ has continuous derivatives of all orders in that domain. This property of an analytic function of a complex variable distinguishes it in a very essential way from the function of a real variable having a continuous first derivative in some domain. Generally speaking, in the latter case, the existence of higher derivatives does not follow from the existence of a first derivative.

Let us consider a number of important consequences of the property just established concerning an analytic function of a complex variable.

***Theorem 1.10 (Morera's theorem).*** *Let a function $f(z)$ be continuous in a singly connected domain $\mathfrak{G}$ and let the integral of $f(z)$ around any closed contour lying wholly in $\mathfrak{G}$ be zero. Then $f(z)$ is an analytic function in $\mathfrak{G}$.*

*Proof.* It has been proved earlier that under the hypothesis of the theorem, a function

$$F(z) = \int_{z_0}^{z} f(\zeta) \, d\zeta$$

where $z_0$ and $z$ are arbitrary points of the domain $\mathfrak{G}$ and the integral is taken along any path connecting these points in $\mathfrak{G}$, is analytic in that domain, and $F'(z) = f(z)$. But, as has already been established, the derivative of an analytic function is also an analytic function; that is, there exists a continuous derivative of the function $F'(z)$, namely the function $F''(z) = f'(z)$. The theorem is proved.

We note that Theorem 1.10 is, in a certain sense, the converse of Cauchy's theorem. It can readily be generalized to multiply connected domains.

***Theorem 1.11 (Liouville's theorem).*** *Let a function $f(z)$ be analytic throughout the complex plane, and let its modulus be uniformly bounded. Then $f(z)$ is identically equal to a constant.*

*Proof.* Write the value of the derivative $f'(z)$ at an arbitrary point $z$ using the formula (1-70):

$$f'(z)=\frac{1}{2\pi i}\int\limits_{C_R}\frac{f(\zeta)}{(\zeta-z)^2}\,d\zeta$$

The integration will be carried out around the circumference of a circle of radius $R$ with centre at the point $z$; i.e. $|\zeta - z| = R$. By hypothesis, there is a constant $M$ such that $|f(\zeta)| \leqslant M$ irrespective of $R$. Therefore

$$|f'(z)|\leqslant\frac{1}{2\pi}\int\limits_{C_R}\frac{|f(\zeta)|}{R^2}\,ds\leqslant\frac{M}{R}$$

Since the radius $R$ may be chosen arbitrarily large, and $f'(z)$ is independent of $R$, it follows that $|f'(z)| = 0$. Since the choice of the point $z$ is arbitrary, we conclude that $|f'(z)| \equiv 0$ over the entire complex plane. Whence it follows that $f(z) \equiv$ constant.

In Section 1.4 we introduced the trigonometric functions of a complex variable and demonstrated that they are analytic functions throughout the complex plane. By virtue of the theorem just proved, these functions cannot be uniformly bounded throughout the complex plane. Whence it follows, in particular, that there are values of the complex variable $z$ for which

$$|\sin z| > 1 \tag{1-73}$$

It is in this respect that the trigonometric functions of a complex variable differ essentially from the corresponding functions of a real variable.

(b) *The function* $\varphi(z, \zeta)$ *and its derivative* $\frac{\partial\varphi}{\partial z}(z, \zeta)$ *are continuous functions of the variables* $z$, $\zeta$ *taken together for an arbitrary variation of* $z$ *in the domain* $\mathfrak{G}$ *and of* $\zeta$ *on the curve* $C$.

The condition (b) signifies that the real and imaginary parts of the function $\frac{\partial\varphi}{\partial z}(z, \zeta)$ are continuous with respect to the variables $x$, $y$, $\xi$, $\eta$ taken together.

It is obvious that under the assumptions just made, the integral of the function $\varphi(z, \zeta)$ along the curve $C$ exists for any $z \in \mathfrak{G}$ and is a function of the complex variable $z$:

$$F(z) = \int_C \varphi(z, \zeta)\, d\zeta = U(x, y) + iV(x, y) \tag{1-66}$$

It is natural to pose the question of the properties of the function $F(z)$. It appears that under the given assumptions relative to the function $\varphi(z, \zeta)$, *the function* $F(z)$ *is an analytic function of the complex variable* $z$ *in the domain* $\mathfrak{G}$, *and the derivative of the function* $F(z)$ *may be computed by differentiating under the sign of the integral.*

To prove this assertion, let us consider the line integral

$$U(x, y) = \int_C u(x, y, \xi, \eta)\, d\xi - v(x, y, \xi, \eta)\, d\eta$$

Since it is assumed that the functions $u$ and $v$ possess partial derivatives with respect to $x$ and $y$, which are continuous with respect to both variables together, the partial derivatives of the function $U(x, y)$ with respect to the variables $x$, $y$ exist and may be computed by differentiating under the integral sign:

$$U_x(x, y) = \int_C u_x\, d\xi - v_x\, d\eta$$

$$U_y(x, y) = \int_C u_y\, d\xi - v_y\, d\eta$$

The functions themselves $U_x$ and $U_y$ are continuous functions of the variables $x$, $y$ in the domain $\mathfrak{G}$. On the basis of similar properties of the function $V(x, y)$ and taking advantage of the Cauchy-Riemann conditions for the function $\varphi(z, \zeta)$, we obtain

$$\begin{aligned} V_y(x, y) &= \int_C v_y\, d\xi + u_y\, d\eta = \int_C u_x\, d\xi - v_x\, d\eta = U_x \\ V_x(x, y) &= \int_C v_x\, d\xi + u_x\, d\eta = -\int_C u_y\, d\xi - v_y\, d\eta = -U_y \end{aligned} \tag{1-67}$$

Thus, for $F(z)$ the Cauchy-Riemann conditions are fulfilled: the partial derivatives of the functions $U(x, y)$ and $V(x, y)$ are continuous and connected by the relations (1-67), which proves the analyticity of $F(z)$ in the domain $\mathfrak{G}$.

Note that

$$F'(z) = U_x(x, y) + iV_x(x, y)$$
$$= \int_C u_x\,d\xi - v_x\,d\eta + i\int_C v_x\,d\xi + u_x\,d\eta = \int_C \frac{\partial\varphi}{\partial z}(z, \zeta)\,d\zeta \qquad (1\text{-}68)$$

Whence it follows that it is possible to compute the derivative of an integral by differentiating the integrand function with respect to a parameter. And if $\frac{\partial\varphi}{\partial z}$ satisfies the same conditions (a) and (b) as $\varphi(z, \zeta)$, then $F'(z)$ is also an analytic function in the domain $\mathfrak{G}$.

### *b. An analytic function and the existence of derivatives of all orders*

The above-considered property of integrals depending on a parameter permits establishing important characteristics of analytic functions. As we have seen, the value of a function $f(z)$ that is analytic in some domain $\mathfrak{G}$ bounded by a contour $\Gamma$ and continuous in a closed domain $\bar{\mathfrak{G}}$ can be expressed in the interior points of this domain in terms of the boundary values by means of the Cauchy integral:

$$f(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{\zeta - z}\,d\zeta \qquad (1\text{-}69)$$

Let us consider, in a domain $\mathfrak{G}$, a certain closed subdomain $\bar{\mathfrak{G}}'$, all of whose points are farther from the boundary $\Gamma$ of the domain $\mathfrak{G}$ than some positive number $d$ ($|z - \zeta| \geqslant d > 0$). The function $\varphi(z, \zeta) = \frac{f(\zeta)}{\zeta - z}$ is an analytic function of $z$ in the domain $\mathfrak{G}'$, and its partial derivative $\frac{\partial\varphi}{\partial z} = \frac{f(\zeta)}{(\zeta - z)^2}$ in this domain is a continuous function of its arguments. Thus, by virtue of the general properties of integrals dependent on a parameter, the derivative $f'(z)$ may be represented at the interior points of the domain $\mathfrak{G}'$ in the form

$$f'(z) = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{(\zeta - z)^2}\,d\zeta \qquad (1\text{-}70)$$

The integral (1-70) is again an integral that depends on a parameter, and its integrand function has the same properties as the integrand function of the integral (1-69). Consequently, $f'(z)$ is an analytic function of $z$ in the domain $\mathfrak{G}'$; the following formula holds true

# CHAPTER 2

# SERIES OF ANALYTIC FUNCTIONS

In this chapter we will examine the principal properties of functional series whose terms are functions of a complex variable. A special role in the theory of functions of a complex variable is played by series of analytic functions and, in particular, by power series of the form $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, where $c_n$ are specified complex constants and $z_0$ is a fixed point in the complex plane. A study of these series is very essential both for elucidating a number of general properties of functions of a complex variable and for solving a variety of problems that involve applications of the methods of the theory of functions of a complex variable.

## 2.1. Uniformly Convergent Series of Functions of a Complex Variable

### *a. Number series*

Let us first examine certain general properties of number series involving complex terms, that is, expressions of the form

$$\sum_{k=1}^{\infty} a_k \tag{2-1}$$

where $\{a_k\}$ is a given sequence of numbers with complex terms.

*The series* (2-1) *is convergent if the sequence* $\{S_n\}$ *of its partial sums* $S_n = \sum_{k=1}^{n} a_k$ *is convergent.* Here, the limit $S$ of the sequence $\{S_n\}$ is called the *sum of the series* (2-1). The series $\sum_{k=n+1}^{\infty} a_k$ is called the *remainder of the series* (2-1) after the $n$th term. In the case of a convergent series, the sum of its remainder after the $n$th term is denoted by $r_n$ and is ordinarily also called the remainder of the series (2-1). For a convergent series $S = S_n + r_n$ and for any $\varepsilon > 0$ there is a number $N$ such that $|r_n| < \varepsilon$ for $n \geqslant N$. From the definition of a convergent series it follows that a necessary and sufficient condition for convergence of the series is Cauchy's test.* Namely, the series (2-1) converges if and only if for any $\varepsilon > 0$ there is a number $N$ such that $\left|\sum_{k=n}^{n+p} a_k\right| < \varepsilon$ for $n \geqslant N$ and for any natural $p$.

*A necessary condition for convergence of the series* (2-1) is the requirement that $\lim_{n\to\infty} a_n = 0$. Indeed, from the convergence of this series, by virtue of Cauchy's test, it follows that for any $\varepsilon > 0$ it is possible to indicate an $N$ such that $|a_{n+1}| = |S_{n+1} - S_n| < \varepsilon$ for $n \geqslant N$.

If the series

$$\sum_{k=1}^{\infty} |a_k| \tag{2-2}$$

with real positive terms converges, then it is obvious that the series (2-1) will converge too. In this case it is termed an *absolutely convergent series*. One of the most frequently used methods of investigating the convergence of a series involving complex terms is the consideration of a series containing real terms which are the moduli of the terms of the original series. It will be recalled that *d'Alembert's test* and *Cauchy's test* for convergence are sufficient conditions for convergence of a series with real positive terms.

According to d'Alembert's test, the series (2-2) converges if, beginning with some number $N$, the ratio $\left|\frac{a_{n+1}}{a_n}\right| \leqslant l < 1$ for all $n \geqslant N$.

Note that if from some number $N$ onwards, the ratio $\left|\frac{a_{n+1}}{a_n}\right| \geqslant 1$, then the series (2-1) with complex terms diverges. Indeed, in this case all terms of the series (2-1), from $a_N$ onwards, satisfy the relation $|a_n| \geqslant |a_N| \neq 0$, i.e. the necessary condition for convergence of a series is not fulfilled.

In accordance with the Cauchy test, the series (2-2) converges if $\sqrt[n]{|a_n|} \leqslant q < 1$ for all $n \geqslant N$. If, from some $N$ onwards, for all $n \geqslant N$, we have the relation $\sqrt[n]{|a_n|} \geqslant 1$, then the series (2-1) diverges.

### b. Functional series. Uniform convergence

We now take up the study of functional series whose terms are functions of a complex variable. Let there be defined in a domain $\mathfrak{G}$ an infinite sequence $\{u_n(z)\}$ of single-valued functions of a complex

* This is a direct consequence of Cauchy's test for convergence of a numerical sequence $\{S_n\}$; see page 19.

variable. An expression of the form

$$\sum_{n=1}^{\infty} u_n(z) \tag{2-3}$$

will be called a *functional series*. For a fixed value of $z_0 \in \mathfrak{G}$, the series (2-3) is transformed into a number series of the type (2-1).

*The functional series* (2-3) *is convergent in the domain* $\mathfrak{G}$ *if for any* $z \in \mathfrak{G}$ *the corresponding number series converges.* If the series (2-3) converges in the domain $\mathfrak{G}$, then in this domain we can define a single-valued function $f(z)$ whose value at each point of $\mathfrak{G}$ is equal to the sum of the corresponding number series. This function is called the *sum of the series* (2-3) in the domain $\mathfrak{G}$. In this case, by virtue of the foregoing definitions, for any fixed point $z \in \mathfrak{G}$ and for any specified positive number $\varepsilon$ it is possible to indicate a number $N$ such that

$$\left| f(z) - \sum_{k=1}^{n} u_k(z) \right| < \varepsilon \quad \text{for} \quad n \geqslant N(\varepsilon, z)$$

Note that in the general case $N$ depends both upon $\varepsilon$ and $z$.

The concept of uniform convergence plays a special role in the theory of series of functions of a complex variable, just as it does in the case of a real variable. For example, as the reader will recall from the course of analysis, a convergent series of continuous functions does not by any means always converge to a continuous function. At the same time the sum of a uniformly convergent series of continuous functions is always a continuous function. Uniformly convergent series of functions of a complex variable, as in the case of a real variable, have a number of very important properties, which we will now examine. First a definition.

*If for any positive number* $\varepsilon$ *it is possible to indicate a number* $N(\varepsilon)$ *such that for* $n \geqslant N(\varepsilon)$ *the inequality*

$$\left| f(z) - \sum_{k=1}^{n} u_k(z) \right| < \varepsilon$$

*is fulfilled at once for all points* $z$ *of a domain* $\mathfrak{G}$, *then the series* (2-3) *is called uniformly convergent in the domain* $\mathfrak{G}$.

Denoting $r_n(z) = \sum_{k=n+1}^{\infty} u_k(z)$, the condition for the uniform convergence of a series (2-3) may be written in the form $|r_n(z)| < \varepsilon$ for $n \geqslant N(\varepsilon)$. A number of properties of uniformly convergent series will be given below.

We give a sufficient test for uniform convergence that is important in applications.

***Weierstrass' test.*** *If the terms of a functional series* (2-3) *can everywhere in a domain* $\mathfrak{G}$ *be bounded by terms of an absolutely convergent number series, then the series* (2-3) *converges uniformly in the domain* $\mathfrak{G}$.

*Proof.* By hypothesis we have the uniform evaluation

$$|u_n(z)| \leqslant |a_n|, \quad z \in \mathfrak{G} \tag{2-4}$$

Since the series $\sum_{n=1}^{\infty} |a_n|$ converges, then for any $\varepsilon > 0$ there exists an $N$ such that $\sum_{k=n+1}^{\infty} |a_k| < \varepsilon$ for $n \geqslant N$. But by (2-4), in the domain $\mathfrak{G}$ we have the inequality

$$\left|\sum_{k=n+1}^{\infty} u_k(z)\right| \leqslant \sum_{k=n+1}^{\infty} |u_k(z)| \leqslant \sum_{k=n+1}^{\infty} |a_k| < \varepsilon$$

for $n \geqslant N$. This proves the uniform convergence of the series (2-3) in the domain $\mathfrak{G}$.

One should bear in mind that the Weierstrass test is only a sufficient condition for uniform convergence. The following necessary and sufficient condition for uniform convergence holds.

***Cauchy's test.*** *A necessary and sufficient condition for uniform convergence of the series* (2-3) *in a domain* $\mathfrak{G}$ *is the existence for any* $\varepsilon > 0$ *of an* $N(\varepsilon)$ *such that the following relation holds simultaneously at all points of* $\mathfrak{G}$

$$|S_{n+m}(z) - S_n(z)| < \varepsilon \tag{2-5}$$

*for* $n \geqslant N$ *and for any natural number* $m$.

*Proof.* (1) *Necessity.* From the uniform convergence of the series (2-3) it follows that for any $\varepsilon > 0$ there exists an $N(\varepsilon)$ such that at all points $z$ of the domain $\mathfrak{G}$ we have the inequalities

$$|f(z) - S_n(z)| < \frac{\varepsilon}{2}, \quad |f(z) - S_{n+m}(z)| < \frac{\varepsilon}{2}$$

for $n \geqslant N$ and for any natural number $m$; whence (2-5) follows.

(2) *Sufficiency.* From the relation (2-5), by virtue of the Cauchy test for a number sequence with complex terms,* it follows that for any fixed $z \in \mathfrak{G}$ the sequence $\{S_n(z)\}$ is convergent. Hence, when (2-5) is fulfilled, the series (2-3) converges in the domain $\mathfrak{G}$ to a certain function $f(z) = \lim_{n\to\infty} S_n(z)$. But, because of (2-5),

$$\lim_{m\to\infty} |S_{n+m}(z) - S_n(z)| = |f(z) - S_n(z)| < \varepsilon \text{ for } n \geqslant N(\varepsilon)$$

at all points of the domain $\mathfrak{G}$ simultaneously. This proves the uniform convergence of the series (2-3) in the domain $\mathfrak{G}$.

---

* See Chapter 1, page 19.

### *c. Properties of uniformly convergent series. Weierstrass' theorems*

Let us now examine certain general properties of uniformly convergent series.

***Theorem 2.1.*** *If the functions $u_n(z)$ are continuous in a domain $\mathfrak{G}$, and the series $\sum_{n=1}^{\infty} u_n(z)$ converges in this domain uniformly to the function $f(z)$, then $f(z)$ is also continuous in the domain $\mathfrak{G}$.*

*Proof.* We consider the expression $|f(z+\Delta z) - f(z)|$, where the points $z$ and $z+\Delta z$ belong to the domain $\mathfrak{G}$. By virtue of the uniform convergence of the series $\sum_{n=1}^{\infty} u_n(z)$, for any $\varepsilon > 0$ there exists an $N$ such that the following inequalities are simultaneously valid

$$\left| f(z+\Delta z) - \sum_{k=1}^{N} u_k(z+\Delta z) \right| < \frac{\varepsilon}{3}, \quad \left| f(z) - \sum_{k=1}^{N} u_k(z) \right| < \frac{\varepsilon}{3} \qquad (2\text{-}6)$$

for any points $z$ and $z+\Delta z$ belonging to the domain $\mathfrak{G}$. By virtue of the continuity of the functions $u_k(z)$, at any point $z \in \mathfrak{G}$ for a given $\varepsilon$ and a chosen $N$ there exists a $\delta > 0$ such that

$$\left| \sum_{k=1}^{N} u_k(z+\Delta z) - \sum_{k=1}^{N} u_k(z) \right| \leqslant \sum_{k=1}^{N} |u_k(z+\Delta z) - u_k(z)| < \frac{\varepsilon}{3} \qquad (2\text{-}7)$$

for $|\Delta z| < \delta$. From (2-6), (2-7) and the fact that the absolute value of a sum does not exceed the sum of the absolute values, it follows that for any $\varepsilon > 0$ there exists a $\delta$ such that $|f(z+\Delta z) - f(z)| < \varepsilon$ for $|\Delta z| < \delta$. This proves the continuity of the function $f(z)$ in the domain $\mathfrak{G}$.

***Theorem 2.2.*** *If the series* (2-3) *of continuous functions $u_n(z)$ converges uniformly in a domain $\mathfrak{G}$ to the function $f(z)$, then the integral of this function along any piecewise smooth curve $C$ lying entirely in $\mathfrak{G}$ may be computed by a termwise integration of the series* (2-3), *that is,*

$$\int_C f(\zeta)\, d\zeta = \sum_{n=1}^{\infty} \int_C u_n(\zeta)\, d\zeta$$

*Proof.* Since the series (2-3) converges uniformly, then for any specified $\varepsilon > 0$ there exists a number $N$ such that for all points $\zeta \in \mathfrak{G}$

$$|r_n(\zeta)| < \frac{\varepsilon}{L} \quad \text{for} \quad n \geqslant N(\varepsilon)$$

where $L$ is the arc length of the curve $C$. Then

$$\left|\int\limits_C f(\zeta)\,d\zeta-\sum_{k=1}^{n}\int\limits_C u_n(\zeta)\,d\zeta\right|=\left|\int\limits_C r_n(\zeta)\,d\zeta\right|\leqslant\int\limits_C |r_n(\zeta)\,d\zeta|<\varepsilon$$

which proves the theorem.

Note that the properties of uniformly convergent series involving complex terms formulated in Theorem 2.1 and Theorem 2.2 are absolutely analogous to the corresponding properties of functional series involving real terms, and the proofs actually repeat those of the appropriate theorems of analysis.

We now consider a supremely important property of uniformly convergent series that characterizes the behaviour of a series whose terms are analytic functions.

***Theorem 2.3 (Weierstrass' first theorem).*** *Let the functions $u_n(z)$ be analytic in a domain $\mathfrak{G}$, and let the series $\sum_{n=1}^{\infty} u_n(z)$ converge uniformly to the function $f(z)$ in any closed subdomain $\overline{\mathfrak{G}}'$ of $\mathfrak{G}$. Then*

(1) *$f(z)$ is an analytic function in the domain $\mathfrak{G}$.*

(2) $f^{(k)}(z)=\sum_{n=1}^{\infty} u_n^{(k)}(z)$.

(3) *The series $\sum_{n=1}^{\infty} u_n^{(k)}(z)$ converges uniformly in any closed subdomain $\overline{\mathfrak{G}}'$ of the domain $\mathfrak{G}$.*

*Proof.* We will prove each one of the foregoing assertions.

(1) Consider an arbitrary interior point $z_0 \in \mathfrak{G}$ and construct a singly connected subdomain $\mathfrak{G}'$ of the domain $\mathfrak{G}$ containing the interior point $z_0$.

By Theorem 2.1, $f(z)$ is a continuous function in $\mathfrak{G}$. Consider the integral of $f(z)$ around an arbitrary closed contour $C$ lying entirely in the domain $\mathfrak{G}'$. By Theorem 2.2, this integral may be computed by termwise integration of the series (2-3). Then, since the functions $u_n(z)$ are analytic, we get

$$\int\limits_C f(\zeta)\,d\zeta=\sum_{n=1}^{\infty}\int\limits_C u_n(\zeta)\,d\zeta=0$$

Thus, all the conditions of Morera's theorem are fulfilled. Hence, $f(z)$ is an analytic function in the neighbourhood $\mathfrak{G}'$ of the point $z_0$. Since the choice of the point $z_0$ is arbitrary, $f(z)$ is analytic in the domain $\mathfrak{G}$. Note that for any natural number $n$ the function $r_n(z)=\sum_{j=n+1}^{\infty} u_j(z)=f(z)-\sum_{j=1}^{n} u_j(z)$, which is the sum of a finite number of analytic functions, is also analytic in $\mathfrak{G}$.

(2) Fix an arbitrary point $z_0 \in \mathfrak{G}$ and choose an arbitrary closed contour $C$ lying entirely in $\mathfrak{G}'$ and containing the point $z_0$ interior

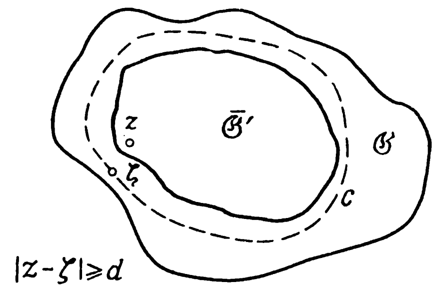


**Fig. 2.1**

to it. Denote by $d$ the minimal distance from the point $z_0$ to the contour $C$. Consider the series

$$\frac{f(z)}{(z-z_0)^{k+1}} = \sum_{n=1}^{\infty} \frac{u_n(z)}{(z-z_0)^{k+1}}$$

Since $\min\limits_{z \in C} |z-z_0| = d > 0$, this series, by virtue of the hypotheses of the theorem, converges uniformly on $C$. Therefore, by integrating it termwise along the contour $C$ and by expressing the derivative of an analytic function in terms of the Cauchy integral, we get $f^{(k)}(z_0) = \sum\limits_{n=1}^{\infty} u_n^{(k)}(z_0)$. Since $z_0$ is an arbitrary point of the domain $\mathfrak{G}$, assertion (2) is proved.

(3) Consider an arbitrary subdomain $\overline{\mathfrak{G}}'$ of $\mathfrak{G}$ and construct, in $\mathfrak{G}$, a closed contour $C$ containing $\overline{\mathfrak{G}}'$ so that the distance from an arbitrary point $z \in \overline{\mathfrak{G}}'$ to any point $\zeta \in C$ is not less than some positive number $d$, $|z-\zeta| \geqslant d > 0$ (Fig. 2.1). (It is obvious that an appropriate contour $C$ and number $d$ can be found for any subdomain $\overline{\mathfrak{G}}'$ of the domain $\mathfrak{G}$.) Since $r_n(z)$ is an analytic function in $\mathfrak{G}$, it follows that for any point $z \in \overline{\mathfrak{G}}'$ we have the relation $\frac{k!}{2\pi i} \int\limits_C \frac{r_n(\zeta)}{(\zeta-z)^{k+1}}\, d\zeta = r_n^{(k)}(z)$. And, in accordance with the assertion that has just been proved, $r_n^{(k)}(z)$ is the remainder of the series $\sum\limits_{n=1}^{\infty} u_n^{(k)}(z)$. By virtue of the uniform convergence of the

original series $\sum_{n=1}^{\infty} u_n(z)$, for every $\varepsilon > 0$ there is an $N$ such that on the contour $C$ for $n \geqslant N$ there is a uniform evaluation $|r_n(\zeta)| < \varepsilon \cdot \frac{2\pi d^{k+1}}{k! L}$, where $L$ is the length of the contour $C$. Then

$$|r_n^{(k)}(z)| \leqslant \frac{k!}{2\pi} \int_C \frac{|r_n(\zeta)|}{|\zeta - z|^{k+1}} ds < \varepsilon$$

for all $z \in \overline{\mathfrak{G}}'$ simultaneously, and this completes the proof of assertion (3). The foregoing proof refers to the case of a singly connected domain $\mathfrak{G}$. The case of a multiply connected domain is considered in analogous fashion. Thus, the theorem is proved.

Observe that this method permits proving uniform convergence of a series of derivatives only in any *closed subdomain* $\overline{\mathfrak{G}}'$ of domain $\mathfrak{G}$, even if the original series (2-3) converges uniformly in the closed domain as well. As elementary examples show, it does not follow from the uniform convergence of the series (2-3) in a closed domain $\overline{\mathfrak{G}}$ that in this domain a series composed of derivatives also converges uniformly. For example, the series $\sum_{n=1}^{\infty} \frac{z^n}{n^2}$ converges uniformly in the circle $|z| \leqslant 1$, and the series $\sum_{n=1}^{\infty} \frac{z^{n-1}}{n}$ composed of derivatives of terms of the original series cannot converge uniformly in the circle $|z| \leqslant 1$, since it diverges at $z = 1$. Thus, the assertion of (3) of the theorem on the uniform convergence of a series, composed of derivatives, only in a closed subdomain of the original domain cannot, generally speaking, be extended.

Let us make one more remark. When proving Theorem 2.3, we assumed uniform convergence of the series in any closed subdomain $\overline{\mathfrak{G}}'$ of domain $\mathfrak{G}$. It is clear that the theorem will all the more hold true in the case of uniform convergence of the series (2-3) in the closed domain $\overline{\mathfrak{G}}$. As the following theorem shows, the latter condition may be replaced by the condition of uniform convergence of the series (2-3) on the boundary $\Gamma$ of domain $\mathfrak{G}$.

***Theorem 2.4 (Weierstrass' second theorem).*** *Let the functions $u_n(z)$ be analytic in a domain $\mathfrak{G}$ and continuous in $\overline{\mathfrak{G}}$, and let the series $\sum_{n=1}^{\infty} u_n(z)$ converge uniformly on the boundary $\Gamma$ of this domain. Then the series $\sum_{n=1}^{\infty} u_n(z)$ converges uniformly in $\overline{\mathfrak{G}}$ as well.*

*Proof*. The difference of partial sums of the given series, i.e. the function $S_{n+p}(z) - S_n(z)$, as a finite sum of analytic functions, is analytic in $\mathfrak{G}$ and continuous in $\overline{\mathfrak{G}}$. From the uniform convergence on $\Gamma$ it follows that

$$|S_{n+p}(\zeta) - S_n(\zeta)| = |u_{n+p}(\zeta) + \ldots + u_{n+1}(\zeta)| < \varepsilon$$

for $n \geqslant N$ for any natural number $p$ and all points $\zeta \in \Gamma$ simultaneously. Consequently, by the maximum-modulus theorem of an analytic function, $|S_{n+p}(z) - S_n(z)| < \varepsilon$ for $n \geqslant N$ for any natural number $p$ and for all $z \in \overline{\mathfrak{G}}$. Thus, for the given series, Cauchy's condition is fulfilled, which proves the theorem.

*Note*. All the above-proved properties of functional series are clearly true for the functional sequences.

### *d. Improper integrals dependent on a parameter*

In Chapter 1 we considered the properties of integrals dependent on a parameter and confined ourselves to the case of proper integrals over a curve $C$ of finite length. The Weierstrass theorem permits generalizing the results to the case of improper integrals. We consider an improper integral of the first kind dependent on a parameter, $F(z) = \int_C f(z, \zeta)\, d\zeta$, along an unbounded curve $C$. Let the function of two complex variables $f(z, \zeta)$ defined for $z \in \mathfrak{G}$ and $\zeta \in C$ satisfy the same conditions as $\varphi(z, \zeta)$ in Sec. 1.7, namely:

(a) The function $f(z, \zeta)$ for any value $\zeta \in C$ is an analytic function of $z$ in the domain $\mathfrak{G}$.

(b) The function $f(z, \zeta)$ and its derivative $\frac{\partial f}{\partial z}(z, \zeta)$ are continuous functions with respect to the two variables $z$, $\zeta$ for $z \in \mathfrak{G}$ and $\zeta \in C$.

Let the improper integral of the first kind $\int_C f(z, \zeta)\, d\zeta$ converge uniformly with respect to the parameter $z$ in any closed subdomain $\overline{\mathfrak{G}}'$ of the domain $\mathfrak{G}$. This means that for any choice of a sequence of finite curves $C_n$ constituting a part of $C$, as $C_n \to C$, the sequence of functions $u_n(z) = \int_{C_n} f(z, \zeta)\, d\zeta$ converges uniformly in $\overline{\mathfrak{G}}'$ to the function $F(z)$.

It can readily be shown that if all the foregoing conditions are fulfilled, the function $F(z)$ is analytic in $\mathfrak{G}$ and

$$F'(z) = \int_C \frac{\partial f}{\partial z}(z, \zeta)\, d\zeta$$

Indeed, as was demonstrated in Sec. 1.7, the proper integrals, the functions $u_n(z) = \int\limits_{C_n} f(z, \zeta)\, d\zeta$, are analytic functions in $\mathfrak{G}$ and $u_n'(z) = \int\limits_{C_n} \frac{\partial f}{\partial z}(z, \zeta)\, d\zeta$. The sequence $\{u_n(z)\}$ converges to $F(z)$ uniformly in any $\overline{\mathfrak{G}}'$. Hence, by the Weierstrass theorem, the function $F(z)$ is analytic in $\mathfrak{G}$ and $F'(z) = \int\limits_{C} \frac{\partial f}{\partial z}(z, \zeta)\, d\zeta$.

## 2.2. Power Series. Taylor's Series

### *a. Abel's theorem*

In the preceding section we considered general functional series (2-3); the form of the functions $u_n(z)$, however, was not specified. Very important are the so-called power series for which $u_n(z) = c_n(z - z_0)^n$, where $c_n$ are some complex numbers and $z_0$ is a fixed point of the complex plane. The terms of the series $\sum\limits_{n=0}^{\infty} c_n(z - z_0)^n$ are analytic functions throughout the entire complex plane, and for this reason the general theorems of the preceding section may be applied in studying the properties of the series. As was established, many important properties are consequences of uniform convergence. Thus, in investigating the power series $\sum\limits_{n=0}^{\infty} c_n(z - z_0)^n$ it is important to establish the domain of its uniform convergence. We immediately observe that the domain of convergence of a power series is determined by the form of the coefficients $c_n$. For example, the series $\sum\limits_{n=0}^{\infty} n!\,(z - z_0)^n$ converges only at one point $z = z_0$. Indeed, the ratio of the absolute values of two successive terms of the series $\left|\frac{u_{n+1}}{u_n}\right| = (n + 1)\,|z - z_0| > 1$ for any fixed value $z \neq z_0$, beginning with some $N(z)$; according to the considerations on page 59, this indicates that the given series is divergent. On the other hand, by means of d'Alembert's test it is easy to establish the absolute convergence of the series $\sum\limits_{n=0}^{\infty} \frac{(z - z_0)^n}{n!}$ for any $z$.

The following theorem is essential for determining the domain of convergence of a power series.

***Theorem 2.5 (Abel's theorem).*** *If a power series* $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ *converges at some point* $z_1 \neq z_0$, *then it converges absolutely at any point* $z$ *which satisfies the condition* $|z-z_0| < |z_1-z_0|$; *in a circle* $|z-z_0| \leqslant \rho$ *of radius* $\rho$ *less than* $|z_1-z_0|$, *the series converges uniformly.*

*Proof*. Take an arbitrary point $z$ which satisfies the condition $|z-z_0| < |z_1-z_0|$ and consider the series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$. Denote $|z-z_0| = q\,|z_1-z_0|$, $q<1$. By virtue of the necessary condition for convergence of the series $\sum_{n=0}^{\infty} c_n (z_1-z_0)^n$, its terms tend to zero as $n \to \infty$. Consequently, there exists a constant $M$ such that $|c_n| \cdot |z_1-z_0|^n \leqslant M$. Whence, for the coefficients $c_n$ of the given power series, we get the evaluation $|c_n| \leqslant \frac{M}{|z_1-z_0|^n}$. Then

$$\left|\sum_{n=0}^{\infty} c_n (z-z_0)^n\right| \leqslant \sum_{n=0}^{\infty} |c_n| \cdot |z-z_0|^n \leqslant M \sum_{n=0}^{\infty} \left|\frac{z-z_0}{z_1-z_0}\right|^n \tag{2-8}$$

By hypothesis, $q = \left|\frac{z-z_0}{z_1-z_0}\right| < 1$. The series $\sum_{n=0}^{\infty} q^n$, which is the sum of an infinite geometric progression with common ratio less than unity, converges. Then from (2-8) it follows that the series under consideration converges too. In order to prove the uniform convergence of the series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ in the circle $|z-z_0| \leqslant \rho < |z_1-z_0|$, it suffices, by virtue of the Weierstrass test, to construct a convergent number series that dominates the given functional series in the domain. It is obvious that such a series is $M \sum_{n=0}^{\infty} \frac{\rho^n}{|z_1-z_0|^n}$, which is also the sum of an infinite geometric progression with common ratio less than unity. The theorem is completely proved.

From Abel's theorem we can derive a number of important corollaries.

***Corollary 1.*** If a power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ diverges at some point $z_1$, then it diverges at all points $z$ which satisfy the inequality $|z-z_0| > |z_1-z_0|$.

Assuming the contrary, we find that by Abel's theorem the series should converge in any circle of radius $\rho < |z - z_0|$ in particular at the point $z_1$, which contradicts the hypothesis.

Consider the least upper bound $R$ of the distances $|z - z_0|$ from the point $z_0$ to the points $z$ at which the series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges. If $R \neq \infty$, then at all points $z'$ satisfying the condition $|z' - z_0| > R$, the given power series diverges. Let $R$ be strictly greater than zero, then the circle $|z - z_0| < R$ is the greatest domain of convergence of the given series. The series diverges everywhere outside this circle; at the boundary points $|z - z_0| = R$ it may either converge or diverge. *The domain* $|z - z_0| < R$ $(R > 0)$ *is called the circle of convergence of the power series, and the number R is its radius of convergence.*

Thus, we have established

***Corollary 2.*** For any power series there exists a number $R$ such that inside the circle $|z - z_0| < R$ the given power series converges and outside the circle it diverges.

In the circle $|z - z_0| \leqslant \rho < R$ of any radius $\rho$ less than the radius of convergence $R$, the power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges uniformly. Observe that, depending on the form of the coefficients, the radius of convergence of a power series may have any value between 0 and $\infty$. The first limiting case will correspond to a series convergent only at the point $z_0$, the second, to a series convergent throughout the entire complex plane. Examples of such series have already been given. The radius of convergence of a power series may be determined in terms of its coefficients $c_n$.

***Corollary 3.*** Inside the circle of convergence, a power series converges to an analytic function. Indeed, the terms of the power series $u_n(z) = c_n (z - z_0)^n$ are functions which are analytic throughout the complex plane; the series converges uniformly in any closed subdomain of the circle of convergence. Hence, by Weierstrass' first theorem, the sum of the series is an analytic function.

***Corollary 4.*** A power series inside the circle of convergence may be integrated and differentiated term by term any number of times, and the radius of convergence of the series obtained is equal to the radius of convergence of the original series. This property is also a direct consequence of the theorems of Abel and Weierstrass.

***Corollary 5.*** The coefficients of the power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ are expressible in terms of the values of the sum of the series $f(z)$ and its derivatives in the centre of the circle of convergence by the formulas

$$c_n = \frac{1}{n!} f^{(n)}(z_0) \tag{2-9}$$

Setting $z = z_0$ in the expression of the sum of the power series $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$, we get $f(z_0) = c_0$; differentiating the series termwise and setting $z = z_0$ in the expression for the derivative $f'(z) = \sum_{n=1}^{\infty} c_n n (z - z_0)^{n-1}$, we get $f'(z_0) = c_1$; analogously, setting $z = z_0$ in the expression for the $k$th derivative,

$$f^{(k)}(z) = \sum_{n=k}^{\infty} c_n n (n-1) \ldots (n-k+1)(z-z_0)^{n-k}$$

we get $f^{(k)}(z_0) = c_k \cdot k!$.

***Corollary 6.*** The radius of convergence $R$ of the power series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ is determined by the formula* $R = \frac{1}{l}$, where $l = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}$ is the upper bound** of the sequence $\{\sqrt[n]{|c_n|}\}$. First suppose that $0 < l < \infty$. We have to show that at any point $z_1$ which satisfies the condition $|z_1 - z_0| < \frac{1}{l}$ the series converges, and at any point $z_2$ which satisfies the condition $|z_2 - z_0| > \frac{1}{l}$ it diverges. Since $l$ is the upper bound of the sequence $\{\sqrt[n]{|c_n|}\}$, for any $\varepsilon > 0$ there is a number $N$ beginning with which $\sqrt[n]{|c_n|} < l + \varepsilon$. On the other hand, for the same $\varepsilon$ it is possible to find infinitely many terms of the sequence $\{\sqrt[n]{|c_n|}\}$ that are greater than $l - \varepsilon$. Take an arbitrary point $z_1$ that satisfies the inequality $l|z_1 - z_0| < 1$ and for $\varepsilon$ take the number $\frac{1 - l|z_1 - z_0|}{2|z_1 - z_0|} > 0$. Then

$$\sqrt[n]{|c_n|}\,|z_1 - z_0| < (l + \varepsilon)|z_1 - z_0| = \frac{1 + l|z_1 - z_0|}{2} = q < 1$$

Whence it follows that the series $\sum_{n=0}^{\infty} c_n (z_1 - z_0)^n$ is dominated by the geometric progression $\sum_{n=0}^{\infty} q^n$ with common ratio less than unity. This proves its convergence. Now taking some point

* This formula is frequently called the Cauchy-Hadamard formula.

** The upper bound $\overline{x}$ of a number sequence $\{x_n\}$ is the greatest limit point of the sequence.

$z_2$ that satisfies the inequality $l \mid z_2 - z_0 \mid > 1$ and choosing for $\varepsilon$ the number $\frac{l|z_2 - z_0| - 1}{|z_2 - z_0|} > 0$, we get

$$\sqrt[n]{|c_n|}\,|z_2 - z_0| > (l - \varepsilon)\,|z_2 - z_0| = 1$$

for an infinity of values of $n$. Whence $|c_n (z_2 - z_0)^n| > 1$, and this, on the basis of the necessary condition for convergence, indicates that the series $\sum_{n=0}^{\infty} c_n (z_2 - z_0)^n$ diverges.

*Note*. We carried out the proof for the case $0 < l < \infty$. Let us now consider the limiting cases.

For $l = 0$ the series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ converges at any point $z$, that is, $R = \infty$. Indeed, in this case, for any $\varepsilon > 0$ there is a number $N$ such that $\sqrt[n]{|c_n|} < \varepsilon$ from $N$ onwards. Taking for $\varepsilon$ the number $\frac{q}{|z - z_0|}$, where $z$ is an arbitrary point of the complex plane and $0 < q < 1$, we get $|c_n (z - z_0)^n| < q^n$. This proves the convergence of the series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$.

For $l = \infty$ the series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$ diverges at any point $z \neq z_0$, i.e. $R = 0$. Indeed, in this case, for any number $M$ there are infinitely many coefficients $c_n$ such that $\sqrt[n]{|c_n|} > M$. Let us consider an arbitrary point $z \neq z_0$ and choose $M$ so that $M\,|z - z_0| = q > 1$. Then an infinity of terms of the series $\sum_{n=0}^{\infty} C_n (z - z_0)^n$ satisfies the condition $|c_n (z - z_0)^n| > 1$; this proves its divergence.

Thus, the Cauchy-Hadamard formula $R = \frac{1}{l}$, where $l = \overline{\lim_{n\to\infty}} \sqrt[n]{|c_n|}$, holds true for any value of $l$.

We illustrate with an example that will be important later on. Let us consider the power series $\sum_{n=0}^{\infty} (z - z_0)^n$, all coefficients $c_n$ of which are equal to 1. By d'Alembert's test we find that the series converges in a circle $|z - z_0| < 1$ to some analytic function. To find this function, we apply the direct definition of the sum of a series as the limit of the partial sums:

$$f(z) = \lim_{n\to\infty} S_n(z) = \lim_{n\to\infty} \frac{1 - (z - z_0)^n}{1 - (z - z_0)} = \frac{1}{1 - (z - z_0)} \qquad (2\text{-}10)$$

Here we have obviously taken advantage of the formula (which holds true in the domain of complex numbers as well) of the sum of a geometric progression with a finite number of terms and the possibility of a passage to the limit in the numerator of the fraction, the denominator of which is nonzero. The equation (2-10) signifies that the formula for the sum of an infinitely decreasing geometric progression holds true in the complex domain as well.

### b. Taylor's series

Thus, a power series inside the circle of convergence defines a certain analytic function. The following question naturally arises: can we associate, with a function that is analytic inside a certain circle, a power series convergent to the given function in this circle? The following theorem gives the answer.

***Theorem 2.6 (Taylor's theorem).*** *A function $f(z)$ that is analytic inside a circle $|z-z_0|<R$ may be represented in the circle as a convergent power series $f(z)=\sum_{n=0}^{\infty} c_n (z-z_0)^n$, the series being defined uniquely.*

*Proof*. Choose an arbitrary point $z$ inside the circle $|z-z_0|< R$ and construct a circle $C_\rho$ of radius $\rho<R$ centred at the point $z_0$ and containing the interior point $z$ (Fig. 2.2). Such a construction

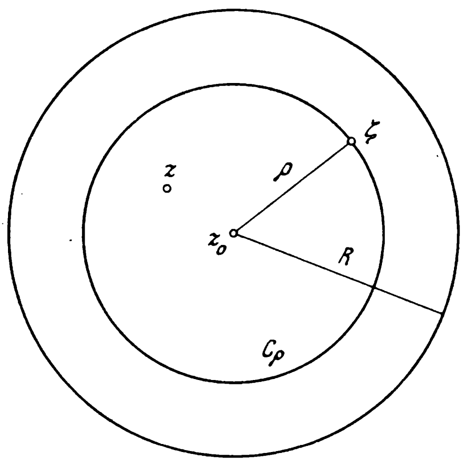


**Fig. 2.2**

is obviously possible for any point $z$ of the given domain. Since $z$ is an interior point of the domain $|z-z_0|<\rho$ in which the function $f(z)$ is analytic, it follows by Cauchy's formula that

$$f(z)=\frac{1}{2\pi i}\int_{C_\rho}\frac{f(\zeta)}{\zeta-z}\,d\zeta \tag{2-11}$$

Transform the integrand of (2-11)

$$\frac{1}{\zeta-z}=\frac{1}{\zeta-z_0}\cdot\frac{1}{1-\frac{z-z_0}{\zeta-z_0}}=\frac{1}{\zeta-z_0}\sum_{n=0}^{\infty}\frac{(z-z_0)^n}{(\zeta-z_0)^n} \qquad (2\text{-}12)$$

Here we took advantage of formula (2-10) and the obvious relation $\left|\frac{z-z_0}{\zeta-z_0}\right|<1$. For $\zeta\in C_\rho$ the series (2-12) converges uniformly in $\zeta$, since it is dominated by the convergent number series $\sum_{n=0}^{\infty}\frac{|z-z_0|^n}{\rho^{n+1}}$ ($|z-z_0|<\rho$). Putting (2-12) into (2-11) and integrating termwise, we obtain

$$f(z)=\sum_{n=0}^{\infty}\frac{1}{2\pi i}\int_{C_\rho}\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}(z-z_0)^n \qquad (2\text{-}13)$$

Introducing the notation

$$c_n=\frac{1}{2\pi i}\int_{C_\rho}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta \qquad (2\text{-}14)$$

rewrite (2-13) in the form of a power series convergent at the chosen point $z$:

$$f(z)=\sum_{n=0}^{\infty}c_n(z-z_0)^n \qquad (2\text{-}15)$$

In formula (2-14) the circle $C_\rho$ may be replaced, by virtue of Cauchy's theorem, by any closed contour $C$ lying in the domain $|z-z_0|<R$ and containing the interior point $z_0$. Since $z$ is an arbitrary point of the given domain, it follows that the series (2-15) converges to $f(z)$ everywhere inside the circle $|z-z_0|<R$, and in the circle $|z-z_0|\leqslant\rho<R$ the series converges uniformly. So the function $f(z)$ that is analytic inside the circle $|z-z_0|<R$ can, in this circle, be expanded in a convergent power series. On the basis of formula (1-72) for derivatives of an analytic function, the coefficients (2-14) of the expansion are of the form

$$c_n=\frac{1}{2\pi i}\int_{C}\frac{f(\zeta)\,d\zeta}{(\zeta-z_0)^{n+1}}=\frac{f^{(n)}(z_0)}{n!} \qquad (2\text{-}16)$$

It remains to prove the uniqueness of the expansion (2-15). Suppose we have the following expansion:

$$f(z)=\sum_{n=0}^{\infty}c_n'(z-z_0)^n \qquad (2\text{-}15')$$

where at least one coefficient $c'_n \neq c_n$. The power series (2-15′) converges in the circle $|z - z_0| < R$, and so on the basis of formula (2-9), $c'_n = \frac{f^{(n)}(z_0)}{n!}$, which coincides with the expression (2-16) for the coefficients $c_n$. This proves the uniqueness of definition of the coefficients.

The expansion of a function, which is analytic in the circle $|z - z_0| < R$, into the convergent power series (2-15) is often called *Taylor's expansion*, and the series (2-15) is termed *Taylor's series*.

The theorem that has just been proved establishes a reciprocal one-to-one correspondence between a function analytic in the neighbourhood of some point $z_0$ and a power series centred at this point. This means that the concept of an analytic function as an infinitely differentiable function is equivalent to a function that can be represented in the form of the sum of a power series.* This is not only very important in constructing the theory of analytic functions, but finds extensive application in the solution of practical problems.

Note also that if a function $f(z)$ is analytic in a domain $\mathfrak{G}$ and $z_0$ is an interior point of this domain, then the radius of convergence of Taylor's series $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!}(z - z_0)^n$ of this function is not less than the distance from the point $z_0$ to the boundary of the domain $\mathfrak{G}$.

### c. Examples

1. To take a simple example, we consider the Taylor expansion of the function $f(z) = \frac{1}{1+z^2}$. This function is analytic throughout the complex plane with the exception of the points $z_{1,2} = \pm i$ at which the denominator of the fraction vanishes. And so by virtue of Theorem 2.6 this function can be expanded into a Taylor series in any circle of the complex plane that does not contain the points

* Note that no similar equivalence occurs for functions of a real variable. Indeed, from the existence, on an interval $[a, b]$, of all derivatives of a function $f(x)$ there does not follow the possibility of expanding this function in a power series of the form $f(x) = \sum_{n=0}^{\infty} c_n (x - x_0)^n$, where $x_0 \in [a, b]$, that is convergent over the whole interval $[a, b]$. For example, the function $f(x) = \frac{1}{1+x^2}$ has derivatives of all orders for any real $x$; however, for $x_0 = 0$ the power series $\sum_{n=0}^{\infty} (-1)^n x^{2n}$ converges to the given function only in the interval $-1 < x < 1$ and not over the whole real $x$-axis.

$z_{1,2} = \pm i$. We begin with the circle $|z| < 1$. Under the condition $|z| < 1$, the expression $\frac{1}{1+z^2}$ may be considered to be the sum of an infinitely decreasing geometric progression. And so, by (2-10),

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n z^{2n} \tag{2-17}$$

which yields the desired expansion. Note that the radius of convergence of the series (2-17) is equal to unity, that is, it is determined by the distance from the centre of the expansion to the boundary of the region of analyticity of the function $f(z) = \frac{1}{1+z^2}$.

Now let us find the Taylor expansion of the function $f(z) = \frac{1}{1+z^2}$ in the circle $|z-1| < \sqrt{2}$. In this case, determining the coefficients $c_n$ of the series $\sum_{n=0}^{\infty} c_n (z-1)^n$ via the formula (2-16) involves rather unwieldy computations. And so we represent the function as $\frac{1}{1+z^2} = \frac{1}{2i}\left\{\frac{1}{z-i} - \frac{1}{z+i}\right\}$ and take advantage of (2-10), which in this case holds true provided $|z-1| < \sqrt{2}$, to get

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n \frac{1}{2i}\left[\frac{1}{(1-i)^{n+1}} - \frac{1}{(1+i)^{n+1}}\right](z-1)^n$$

Using the exponential form of writing the complex numbers, $1-i = \sqrt{2}\, e^{-i\frac{\pi}{4}}$, $1+i = \sqrt{2}\, e^{i\frac{\pi}{4}}$, it is now easy to obtain

$$\frac{1}{1+z^2} = \sum_{n=0}^{\infty} (-1)^n \frac{\sin (n+1)\frac{\pi}{4}}{2^{\frac{n+1}{2}}} (z-1)^n \tag{2-18}$$

As follows from the Cauchy-Hadamard formula, the radius of convergence of the series (2-18) is equal to $\sqrt{2}$, which is to say it is again determined by the distance from the centre of the expansion to the boundary of the region of analyticity of the function at hand.

2. By way of illustration, let us consider the Taylor-series expansion of the function $f(z) = \ln z = \int_1^z \frac{d\zeta}{\zeta}$ introduced in Chapter 1 (page 46). It was earlier established that this is an analytic function over the entire complex plane cut along the negative part of the real axis, and consequently inside the circle $|z-1| < 1$ as well. Assuming $z_0 = 1$ and computing the coefficients $c_n$ by formula (2-16),

we get

$$c_0 = \ln 1 = 0; \quad c_1 = \frac{1}{z}\Big|_{z=1} = 1$$

$$c_n = \frac{1}{n!}(-1)^{n-1}\frac{(n-1)!}{z^n}\Big|_{z=1} = (-1)^{n-1}\frac{1}{n}, \quad n = 2, 3, \ldots$$

Whence

$$\ln z = \sum_{n=1}^{\infty} (-1)^{n-1}\frac{(z-1)^n}{n} \tag{2-19}$$

As will readily be seen if we apply d'Alembert's test, the circle $|z-1| < 1$ is the circle of convergence of the series (2-19).

## 2.3. Uniqueness of Definition of an Analytic Function

The properties of functions of a complex variable that we have already studied permit concluding that in order to define a function analytic in a given domain, one need not specify the values of the function throughout the domain. For instance, by specifying the values of an analytic function on the boundary of the domain we can, with the aid of Cauchy's integral, define its values at all interior points of the domain. Thus, a function that is analytic in a given domain is defined by specifying incomplete information about its values in that domain. The natural question arises: what is the minimum information needed to completely define a function analytic in a given domain?

### *a. Zeros of an analytic function*

Let us first introduce the concept of a zero of an analytic function. Let $f(z)$ be an analytic function in a domain $\mathfrak{G}$. The point $z_0 \in \mathfrak{G}$ is called a *zero* of $f(z)$ if $f(z_0) = 0$. From the power-series expansion of $f(z)$ in the neighbourhood of the point $z_0$, $f(z) = \sum_{n=0}^{\infty} c_n (z-z_0)^n$, it follows that in the given case the coefficient $c_0 = 0$. If not only the coefficient $c_0$, but also the coefficients $c_1$, $c_2, \ldots, c_{k-1}$ are equal to zero, and the coefficient $c_k$ is different from zero, then the point $z_0$ is called a *zero of order k* of the function $f(z)$. According to formula (2-9), in a zero of order $k$, not only the function itself but also its first $k-1$ derivatives are equal to zero, and the $k$th derivative is nonzero. In the neighbourhood of a zero of order $k$, the power-series expansion of the function $f(z)$ is of the form

$$f(z)=\sum_{n=k}^{\infty} c_n (z-z_0)^n$$

$$=(z-z_0)^k \sum_{n=0}^{\infty} c_{n+k} (z-z_0)^n=(z-z_0)^k \varphi(z) \quad (2\text{-}20)$$

where $\varphi(z)$ is an analytic function in the neighbourhood of the point $z_0$, the power-series expansion of which function is of the form $\varphi(z) = \sum_{n=0}^{\infty} c_{n+k} (z - z_0)^n$ and $\varphi(z_0) \neq 0$. Note that the last series converges in the same circle as the original one.

### b. Uniqueness theorem

Let us now state the basic proposition of this section.

**Theorem 2.7.** *Let a function $f(z)$ be an analytic function in a domain $\mathfrak{G}$ and let it vanish at various points $z_n \in \mathfrak{G}$, $n = 1, 2, \ldots$. If a sequence $\{z_n\}$ converges to a limit $a$ belonging to that domain, then the function $f(z)$ is identically zero in the domain $\mathfrak{G}$.*

*Proof.* Since $a \in \mathfrak{G}$, the function $f(z)$ may be expanded in a power series in the neighbourhood of the given point, $f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$, and the radius $R_0$ of convergence of the given series is not less than the distance of point $a$ from the boundary of the domain. From the definition of continuity of the function $f(z)$ it follows that $f(a) = 0$. Whence $c_0 = 0$, and the expansion of $f(z)$ in the neighbourhood of $z = a$ is of the form

$$f(z)=(z-a) f_1(z), \quad \text{where } f_1(z)=\sum_{n=0}^{\infty} c_{n+1} (z-a)^n$$

We assume that all points of the sequence $\{z_n\}$ are different from $a$. This does not diminish the generality of our reasoning, since only one of these points could be equal to $a$. By the latter condition $f_1(z_n) = 0$, and by the definition of a continuous function, $f_1(a) = 0$. Whence $c_1 = 0$, and the expansion of $f_1(z)$ in the neighbourhood of $a$ takes on the form $f_1(z) = (z - a) f_2(z)$, where $f_2(z) = \sum_{n=0}^{\infty} c_{n+2}(z-a)^n$. As before, we also find that $f_2(a) = 0$, i.e. $c_2 = 0$. Continuing this process indefinitely, we find that all the coefficients $c_n$ in the expansion of $f(z)$ in the power series $f(z) = \sum_{n=0}^{\infty} c_n (z - a)^n$ in the neighbourhood of the point $a$ are equal to zero. This means that $f(z) \equiv 0$

within the circle $|z - a| < R_0$. Let us now take up the proof* that the function $f(z)$ is identically equal to zero throughout the domain $\mathfrak{G}$. It will suffice to show that $f(z_1) = 0$, where $z_1$ is an arbitrary point of the domain $\mathfrak{G}$ lying outside the circle $|z - a| < R_0$. To do this, connect the points $a$ and $z_1$ by a rectifiable curve $L$ lying entirely in $\mathfrak{G}$ and distant from its boundary by $d > 0$. Since any point of the circle $|z - a| < R_0$ lying within the domain $\mathfrak{G}$ may be regarded as the limit of a sequence of zeros of the function $f(z)$, it is possible, by choosing as the new centre of expansion the last point $z = a_1$ of intersection of the curve $L$ with the circle $|z - a| = R_0$, to find that $f(z) \equiv 0$ inside the circle $|z - a_1| < R_1$, where $R_1 \geqslant d$. Continuing in similar fashion, we cover the entire curve $L$ with a finite number of circles with radii not less than $d$, within which $f(z) \equiv 0$. In the process, the point $z = z_1$ is inside the last circle, thus, $f(z_1) = 0$. Since $z_1$ is an arbitrary point of the domain $\mathfrak{G}$, it follows that $f(z) \equiv 0$ in $\mathfrak{G}$.

This theorem has a number of important corollaries.

***Corollary 1.*** The function $f(z) \not\equiv 0$ and is analytic in the domain $\mathfrak{G}$; in any closed bounded subdomain $\overline{\mathfrak{G}}'$ of $\mathfrak{G}$ it has only a finite number of zeros.

If the totality of zeros of the function $f(z)$ in the domain $\overline{\mathfrak{G}}'$ is infinite, then by Theorem 1.2 we can extract from it a convergent sequence $\{z_n\} \to a$, the limit $a$ of this sequence belonging to $\overline{\mathfrak{G}}'$ Whence $f(z) \equiv 0$ in $\mathfrak{G}$, which contradicts the hypothesis.

***Corollary 2.*** If the point $z_0 \in \mathfrak{G}$ is a zero of infinite order** of the function $f(z)$ (i.e., all coefficients $c_n \equiv 0$ in the expansion of $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ about the point $z_0$), then $f(z) \equiv 0$ in the domain $\mathfrak{G}$.

***Corollary 3.*** An analytic function can have an infinite number of zeros only in an open or unbounded domain.

The function of a complex variable that is analytic throughout the complex plane ($z \neq \infty$) is called an *entire* (or *integral*) *function*. From what has been considered it follows that an entire function in any bounded part of the complex plane has only a finite number of zeros. Consequently, all the zeros of an entire function can be arranged in some kind of order, for example in the order of increasing absolute values. In the extended plane, an entire function can only have a countable set of zeros, and the limit point of this set is the point at infinity of the complex plane. Entire functions play an important role both in the theory of functions of a complex variable and its applications.

---

* This proof is analogous to that of the theorem on pages 51-53.

** It is obvious that in this case both the function $f(z)$ and all its derivatives at the point $z_0$ are equal to zero.

***Theorem 2.8.*** *Let the functions $f(z)$ and $\varphi(z)$ be analytic in a domain $\mathfrak{G}$. If in $\mathfrak{G}$ there is a sequence, that converges to some point $a \in \mathfrak{G}$, of different points $\{z_n\}$ at which the values of the functions $f(z)$ and $\varphi(z)$ coincide, then $f(z) \equiv \varphi(z)$ in $\mathfrak{G}$.*

To prove this theorem, it is sufficient to establish, with the aid of Theorem 2.7, that the function $\psi(z) = f(z) - \varphi(z) \equiv 0$ in $\mathfrak{G}$.

Theorem 2.8 is exceedingly important, since it signifies that in a given domain $\mathfrak{G}$ only a single analytic function can exist that assumes specified values in the sequence of points $\{z_n\}$ convergent to the point $a \in \mathfrak{G}$. This theorem is called the *uniqueness theorem of definition of an analytic function.*

The following corollaries to the uniqueness theorem find frequent application.

***Corollary 1.*** If the functions $f_1(z)$ and $f_2(z)$, analytic in a domain $\mathfrak{G}$, coincide on some curve $L$ belonging to the given domain, then they are identically equal in the domain $\mathfrak{G}$.

***Corollary 2.*** If the functions $f_1(z)$ and $f_2(z)$, analytic respectively in the domains $\mathfrak{G}_1$ and $\mathfrak{G}_2$ which have a common subdomain $\mathfrak{G}$, coincide in $\mathfrak{G}$, then there exists a unique analytic function $F(z)$ such that

$$F(z) \equiv \begin{cases} f_1(z), & z \in \mathfrak{G}_1 \\ f_2(z), & z \in \mathfrak{G}_2 \end{cases}$$

The uniqueness theorem and its corollaries can also be given the following forms.

(1) Let there be chosen, in a domain $\mathfrak{G}$, a sequence of different points $z_n \in \mathfrak{G}$ convergent to the point $a \in \mathfrak{G}$. Then in this domain there can exist only one analytic function $f(z)$ that assumes specified values at the points $z_n$.

(2) Let a certain curve $L$ be given in a domain $\mathfrak{G}$. Then in $\mathfrak{G}$ there can exist only one analytic function $f(z)$ that assumes specified values on $L$.

(3) Let there be given, in a domain $\mathfrak{G}$, a certain subdomain $\mathfrak{G}'$. Then in $\mathfrak{G}$ there will be a unique analytic function $f(z)$ that takes on specified values in the subdomain $\mathfrak{G}'$.

If there exists a function $f(z)$ defined in the domain $\mathfrak{G}$ [as mentioned in (1), (2), (3)], then it can be called an analytic continuation into $\mathfrak{G}$ from the set $\{z_n\}$, from the line $L$ or the subdomain $\mathfrak{G}'$.

Note that specification of the values of an analytic function on an appropriate set of points cannot be performed in arbitrary fashion. However, we will not discuss the requirements that these values must satisfy so that they may be continued analytically in the domain $\mathfrak{G}$.

# CHAPTER 3

# ANALYTIC CONTINUATION. ELEMENTARY FUNCTIONS OF A COMPLEX VARIABLE

In this chapter we will examine a number of fundamental consequences of the theorem on uniqueness of definition of an analytic function. It has been established that an analytic function is uniquely defined by specification of its values on a certain set of points in the domain of its definition. This circumstance permits constructing an analytic continuation of elementary functions of a real variable into the complex domain and to elucidate their properties in this domain. We will also briefly consider the general principles of analytic continuation.

## 3.1. Elementary Functions of a Complex Variable. Continuation from the Real Axis

### *a. Continuation from the real axis*

The theorem on the uniqueness of definition of an analytic function permits extending elementary functions of a real variable automatically to the complex domain. First observe the validity of the following assertion: let there be given, on an interval $[a, b]$ of the real $x$-axis, a continuous function $f(x)$ of a real variable; then in some domain $\mathfrak{G}$ of the complex plane that contains the interval $[a, b]$ of the real axis there can exist only one analytic function $f(z)$ of the complex variable $z$ that takes on the given values of $f(x)$ on the interval $[a, b]$. We call the function $f(z)$ *an analytic continuation of the function $f(x)$ of the real variable $x$ into the complex domain* $\mathfrak{G}$.

We now consider some examples of the construction of analytic continuations of elementary functions of a real variable. Among the elementary functions of a real variable, of particular importance are the exponential function $e^x$ and trigonometric functions $\sin x$ and $\cos x$. It will be recalled that these functions can be specified by their Taylor-series expansions:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \tag{3-1}$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} \tag{3-2}$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \tag{3-3}$$

Note that these series converge for any value of $x$.

Consider the following power series in the complex plane:

$$\sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{3-4}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{3-5}$$

$$\sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \tag{3-6}$$

For real $z = x$, the expressions (3-4), (3-5), (3-6), and (3-1), (3-2), (3-3) coincide respectively.

As follows from Abel's theorem, the domain of convergence of the series (3-4) to (3-6) is the entire plane of the complex variable, i.e. these series are entire functions of the complex variable $z$ which are analytic continuations onto the entire complex plane of the elementary functions $e^x$, $\sin x$ and $\cos x$ of a real variable. It is natural to preserve the earlier notation for these functions. Let us put

$$e^z = \sum_{n=0}^{\infty} \frac{z^n}{n!} \tag{3-7}$$

$$\sin z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n+1}}{(2n+1)!} \tag{3-8}$$

$$\cos z = \sum_{n=0}^{\infty} (-1)^n \frac{z^{2n}}{(2n)!} \tag{3-9}$$

With the aid of the function $e^z$ construct hyperbolic functions of the complex variable

$$\cosh z = \frac{e^z + e^{-z}}{2} \tag{3-10}$$

and

$$\sinh z = \frac{e^z - e^{-z}}{2} \tag{3-11}$$

These functions are also entire functions by virtue of the general properties of analytic functions.

Similarly, we can construct the remaining trigonometric functions $\tan z = \frac{\sin z}{\cos z}$, $\operatorname{cosec} z = \frac{1}{\sin z}$, etc., with the aid of the basic trigonometric functions $\sin z$ and $\cos z$, by means of a formal transfer of the appropriate definitions to the complex domain. These functions are not entire, since their analyticity breaks down at those points of the $z$-plane where the denominators of the expressions defining them vanish.

As will be shown below, many of the basic properties of the corresponding elementary functions of a real variable are preserved for all the constructed functions of a complex variable. This will be established on the basis of certain general propositions; for the present we will construct the continuation of two more elementary functions into the complex domain. Consider the following power series:

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(x-1)^n}{n} \tag{3-12}$$

and

$$x + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \ldots (2n-1)\, x^{2n+1}}{2^n \cdot n!\,(2n+1)} \tag{3-13}$$

The first series is known to converge in the interval $0 < x < 2$ and the second in the interval $-1 < x < 1$ to the functions of the real variable $\ln x$ and $\arcsin x$, respectively. It is easy to establish that the power series

$$\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n} \tag{3-12$'$}$$

and

$$z + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \ldots (2n-1)\, z^{2n+1}}{2^n \cdot n!\,(2n+1)} \tag{3-13$'$}$$

converge: the first converges inside the circle $|z-1| < 1$, the second, within the circle $|z| < 1$; and on appropriate intervals of the real axis they coincide with the series (3-12) and (3-13). Therefore, the analytic functions of the complex variable $z$ defined by means of the series (3-12′) and (3-13′) inside their circles of convergence are analytic continuations of the elementary functions $\ln x$ and $\arcsin x$ of the real variable $x$ onto the appropriate complex domain. We again retain the earlier notation for these functions, putting

$$\ln z = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{(z-1)^n}{n} \tag{3-14}$$

and

$$\arcsin z = z + \sum_{n=1}^{\infty} \frac{1 \cdot 3 \ldots (2n-1)\, z^{2n+1}}{2^n \cdot n!\,(2n+1)} \tag{3-15}$$

Note that the functions (3-14) and (3-15), unlike the earlier introduced functions (3-7) to (3-9), are not entire functions, since the series defining them do not converge on the entire complex plane but only inside circles of unit radius. The properties of these functions will also be considered somewhat later. However, it may be observed that the function (3-14) in the circle $|z-1| < 1$ coincides with the function $\ln z = \int_1^z \frac{d\zeta}{\zeta}$ (which was introduced by a different method in Chapter 1, page 46), since both these analytic functions are defined in the indicated domain and coincide on the common interval of the real axis $0 < x < 2$ with one and the same function $\ln x$. We will therefore use the same notation for both functions. Thus, the function $f(z) = \int_1^z \frac{d\zeta}{\zeta}$ defined on the complex plane $z$ cut along the negative real axis is also an analytic continuation of the function $\ln x$ onto the appropriate domain.

In conclusion we observe that if the function $f(x)$ of the real variable $x$ is specified by its power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x-x_0)^n \tag{3-16}$$

convergent on the interval $[a, b]$, then there exists an analytic function $f(z)$ of the complex variable $z$, which is an analytic continuation of $f(x)$ into the complex domain $\mathfrak{G}$ that contains the interval $[a, b]$ of the real axis. This circumstance permits calling the function of the real variable $f(x)$, which can be represented by the series (3-16), an analytic function. Recall that the function of a real variable that can be represented on an interval $[a, b]$ by the power series (3-16) has derivatives of all orders on this interval. Obviously, the derivative $f^{(n)}(z)$ is the analytic continuation of the derivative $f^{(n)}(x)$ into the domain $\mathfrak{G}$.

*b. Continuation of relations*

We now consider further consequences of the theorem on uniqueness of definition of an analytic function. This theorem permits not only constructing analytic continuations of elementary functions of a real variable but also *analytically continuing into the complex domain relations* which occur between appropriate functions of a real variable. As typical instances, consider first relations of the form

$$\sin^2 x + \cos^2 x = 1 \tag{3-17}$$

$$e^{\ln x} = x \tag{3-18}$$

and, secondly, relations of the form

$$e^{x_1} \cdot e^{x_2} = e^{x_1 + x_2} \tag{3-19}$$

$$\cos(x_1 + x_2) = \cos x_1 \cdot \cos x_2 - \sin x_1 \cdot \sin x_2 \tag{3-20}$$

The relations (3-17) and (3-18) establish a connection between different functions of one real variable; relations (3-19) and (3-20) involve functions of several variables. These are some of the basic relations for elementary functions of real variables. It is natural to ask whether they will hold true for analytic continuations of elementary functions into the complex domain.

We will establish the fact that the identity (3-17) remains valid for the complex domain as well. To do this, consider the function

$$F(z) = \sin^2 z + \cos^2 z - 1$$

of the complex variable $z$. According to the general properties of analytic functions (see Chapter 1, page 33), $F(z)$ is an entire function of $z$, and for real values of $z = x$ [by virtue of (3-17)] $F(x) \equiv 0$. Whence, by the uniqueness theorem, we find that throughout the complex $z$-plane the relation

$$\sin^2 z + \cos^2 z \equiv 1 \tag{3-21}$$

is fulfilled. Similar reasoning will suffice to prove the validity, in the complex domain, of the expression (3-18) and of other relations between different analytic functions of one complex variable. However, there is no need to carry out a special investigation each time. We can formulate a general theorem.

Let there be given a function $F[w_1, \ldots, w_n]$ of the complex variables $w_1, \ldots, w_n$, which is analytic with respect to each variable* $w_i \in D_i$ and such that it itself and its partial derivatives $\frac{\partial F}{\partial w_i}$ are

---

* We shall call a function of many complex variables $F(z_1, \ldots, z_n)$ defined for the values $z_i \in D_i$, an analytic function of each of its variables $z_i$

continuous in all the variables $w_1, \ldots, w_n$. A function $F[w_1, \ldots, w_n]$ having these properties will be called an analytic function of many complex variables. Let there be given $n$ functions $f_1(z), \ldots, f_n(z)$ of the complex variable $z$ which are defined in the domain $\mathfrak{G}$ of the complex $z$-plane, and let $f_i(z) \in D_i$.

We will say that the functions $f_i(z)$ satisfy the *relation* $F[f_1(z), \ldots, f_n(z)] = 0$ on the set $M$ if this relation is satisfied at all points $z \in M$. In the sequel we consider relations specified solely by analytic functions of many complex variables. Then we have

***Theorem 3.1.*** *If the functions $f_i(z)$ are analytic functions of $z$ in the domain $\mathfrak{G}$ containing an interval $[a, b]$ of the real x-axis, then from the relation $F[f_1(x), \ldots, f_n(x)] = 0$ for $a \leqslant x \leqslant b$, there follows the relation $F[f_1(z), \ldots, f_n(z)] = 0$ for $z \in \mathfrak{G}$.*

*Proof.* To prove the theorem it is sufficient to show that under the formulated conditions the function $\Phi(z) = F[f_1(z), \ldots, f_n(z)]$ is an analytic function of the complex variable $z$ in the domain $\mathfrak{G}$. We carry out the proof for the case of two variables $w_i$, that is, when $\Phi(z) = F[f_1(z), f_2(z)]$. In the domain $\mathfrak{G}$ fix an arbitrary point $z_0 \in \mathfrak{G}$ and denote $f_1(z_0) = w_1^0$ and $f_2(z_0) = w_2^0$. Form the expression

$$\Phi(z_0 + \Delta z) - \Phi(z_0) = F[w_1^0 + \Delta w_1, w_2^0 + \Delta w_2] - F[w_1^0, w_2^0] \quad (3\text{-}22)$$

where $\Delta w_1$, $\Delta w_2$ are increments of the functions $f_1(z)$ and $f_2(z)$, which correspond to the increment $\Delta z$ of the independent variable $z$. Since, by assumption, there exist partial derivatives of the function $F$ which are continuous in all the variables, it follows that (3-22) may be transformed to

$$\Phi(z_0 + \Delta z) - \Phi(z_0) = \frac{\partial F}{\partial w_1}(w_1^0, w_2^0 + \Delta w_2)\,\Delta w_1 + \eta_1 \Delta w_1 + \frac{\partial F}{\partial w_2}(w_1^0, w_2^0)\,\Delta w_2 + \eta_2 \cdot \Delta w_2 \quad (3\text{-}23)$$

where the functions $\eta_1$ and $\eta_2$ are infinitesimal as $\Delta w_1$ and $\Delta w_2$ approach zero, and, thus, as $\Delta z \to 0$. We now form the difference quotient $\frac{\Delta\Phi}{\Delta z}$ and, passing to the limit as $\Delta z \to 0$, since the partial

---

$(i = 1, 2, \ldots, m;\ m \leqslant n)$ if for any $i = 1, 2, \ldots, m$ the corresponding function $\Phi_i(z_i) = F(z_1^0, \ldots, z_{i-1}^0, z_i, z_{i+1}^0, \ldots, z_n^0)$ of one complex variable $z_i$ obtained for arbitrary fixed values of the remaining variables $z_j = z_j^0$ $(j \neq i)$ is an analytic function of the given variable. We will call the derivatives of the function $\Phi_i(z_i)$ with respect to the appropriate variables, partial derivatives of the function $F(z_1, \ldots, z_n)$ of many complex variables $\Phi_i'(z_i) = \frac{\partial F(z_1, \ldots, z_n)}{\partial z_i}$. A brief survey of the theory of functions of many complex variables is given in Appendix III.

derivatives of the functions $F$ are continuous in all the variables, we get

$$\lim_{\Delta z\to 0}\frac{\Phi(z_0+\Delta z)-\Phi(z_0)}{\Delta z}=\frac{\partial F}{\partial w_1}(w_1^0, w_2^0)\, f_1'(z_0)+\frac{\partial F}{\partial w_2}(w_1^0, w_2^0)\, f_2'(z_0)$$

This proves the existence of the derivative $\Phi'(z_0)$ at the point $z_0$. On the basis of the assumptions that have been made, the function $\Phi'(z)$ is continuous at the point $z_0$, and since $z_0$ is an arbitrary point of the domain $\mathfrak{G}$ it follows that the function $\Phi(z)$ is analytic in the domain $\mathfrak{G}$. The proof is quite analogous for a larger number of variables $w_i$.

Theorem 3.1 permits analytically continuing into the complex domain relations of the form (3-17), (3-18) between elementary functions of one real variable. This is essential in the study of various properties of elementary functions of a complex variable. Appropriate examples will be given later on, for the present we confine ourselves to the following remarks regarding Theorem 3.1.

***Corollary.*** *If the conditions of Theorem 3.1 are fulfilled and the functions $f_1(z)$ are respectively equal: $f_1(z) = f(z)$, $f_2(z) = f'(z)$, ... ..., $f_{n+1}(z) = f^{(n)}(z)$, then from the relation*

$$F[f(x), \ldots, f^{(n)}(x)] = 0 \text{ for } a < x < b \tag{3-24}$$

*there follows*

$$F[f(z), \ldots, f^{(n)}(z)] = 0, \quad z \in \mathfrak{G} \tag{3-25}$$

This signifies that if the function $f(x)$ of a real variable is a solution of the differential equation (3-24), then its analytic continuation $f(z)$ into the domain $\mathfrak{G}$ satisfies, in that domain, the differential equation (3-25), which is an analytic continuation of the relation (3-24) into $\mathfrak{G}$.

Let us now substantiate the analytic continuation of relations of the form (3-19) and (3-20). We will not consider each one separately but will state a general theorem.

***Theorem 3.2.*** *Let the functions $w_1 = f_1(z_1), \ldots, w_n = f_n(z_n)$ be analytic functions of the complex variables $z_1, \ldots, z_n$ in the domains $\mathfrak{G}_1, \ldots, \mathfrak{G}_n$ containing the intervals $[a_i, b_i]$ $(i = 1, \ldots, n)$ of the real x-axis. Let the function $F[w_1, \ldots, w_n]$ be analytic with respect to each of the variables $w_1, \ldots, w_n$ in their range. Then from the relation $F[f_1(x_1), \ldots, f_n(x_n)] = 0$ for $a_i \leqslant x \leqslant b_i$ there follows the relation $F[f_1(z_1), \ldots, f_n(z_n)] = 0$ for $z_i \in \mathfrak{G}_i$.*

*Proof.* Fix the values of the variables $x_2 = x_2^0, \ldots, x_n = x_n^0$ and consider the function $\Phi_1(z_1) = F[f_1(z_1), f_2(x_2^0), \ldots, f_n(x_n^0)]$. This function, being a composite function of the complex variable $z_1$ by virtue of the assertion on page 33 of Chapter 1, is an analytic function of the complex variable $z_1 \in \mathfrak{G}_1$. Therefore, by the theorem on uniqueness of definition of an analytic function it follows from the relation $F[f_1(x_1), f_2(x_2^0), \ldots, f_n(x_n^0)] = 0$ for $a_1 \leqslant x_1 \leqslant b_1$, that $F[f_1(z_1), f_2(x_2^0), \ldots, f_n(x_n^0)] = 0$ for $z_1 \in \mathfrak{G}_1$. Note that by virtue of the arbitrariness of $x_2^0, \ldots, x_n^0$ there follows from this that $F[f_1(z_1), f_2(x_2), \ldots, f_n(x_n)] = 0$. Now fix an arbitrary value of the complex variable $z_1^0 \in \mathfrak{G}_1$ and consider the function $\Phi_2(z_2) = F[f_1(z_1^0), f_2(z_2), f_3(x_3^0), \ldots, f_n(x_n^0)]$ of the complex variable $z_2 \in \mathfrak{G}_2$. The function $\Phi_2(z_2)$, just like $\Phi_1(z_1)$, is an analytic function of the variable $z_2 \in \mathfrak{G}_2$. Therefore, from the relation $F[f_1(z_1^0), f_2(x_2), f_3(x_3^0), \ldots, f_n(x_n^0)] = 0$ for $a_2 \leqslant x_2 \leqslant b_2$ there follows $F[f_1(z_1^0), f_2(z_2), f_3(x_3^0), \ldots, f_n(x_n^0)] = 0$, for $z_2 \in \mathfrak{G}_2$. Since the choice of $z_1^0$ is arbitrary, we find that the relation $F[f_1(x_1), f_2(x_2), f_3(x_3^0), \ldots, f_n(x_n^0)] = 0$ for $a_1 \leqslant x_1 \leqslant b_1$, $a_2 \leqslant x_2 \leqslant b_2$ implies the relation $F[f_1(z_1), f_2(z_2), f_3(x_3^0), \ldots, f_n(x_n^0)] = 0$ for $z_1 \in \mathfrak{G}_1$, $z_2 \in \mathfrak{G}_2$.

Continuing in analogous fashion, we prove the theorem. It will be noted that the proof of the theorem does not depend on the mutual arrangement of the domains $\mathfrak{G}_i$.

Theorem 3.2 permits constructing analytic continuations of relations of the form (3-19) and (3-20). For example, consider (3-19) and introduce the functions $w_1, w_2, w_3$ of the complex variables $z_1, z_2$ and $z_3 = z_1 + z_2$

$$w_1 = e^{z_1}, \quad w_2 = e^{z_2}, \quad w_3 = e^{z_3} = e^{z_1+z_2} \tag{3-26}$$

Consider the function of three complex variables

$$F[w_1, w_2, w_3] = w_3 - w_1 \cdot w_2 \tag{3-27}$$

Since the functions (3-26), (3-27) are entire functions of their variables, and $F = 0$ for $z_1 = x_1$, $z_2 = x_2$, $z_3 = x_3$ $(-\infty < x_i < \infty)$, all the conditions of Theorem 3.2 are fulfilled and this proves the validity of the relation (3-19) for any values of the complex variables $z_1$ and $z_2$.

### c. Properties of elementary functions

Now let us study in more detail the basic properties of the earlier introduced elementary functions of a complex variable. By virtue of Theorem 3.1 and Theorem 3.2 for all values of the complex variable $z_1$ we have the relations

$$\sin^2 z + \cos^2 z = 1 \tag{3-28}$$

$$\cosh^2 z - \sinh^2 z = 1 \tag{3-29}$$

and other familiar identities for the various trigonometric and hyperbolic functions of one complex variable. We also have the relations

$$e^{z_1+z_2} = e^{z_1} \cdot e^{z_2} \tag{3-30}$$

$$\sin (z_1 + z_2) = \sin z_1 \cos z_2 + \cos z_1 \sin z_2 \tag{3-31}$$

$$\cos (z_1 + z_2) = \cos z_1 \cos z_2 - \sin z_1 \sin z_2 \tag{3-32}$$

and other trigonometric formulas which are analytic continuations into the complex domain of familiar relations for elementary functions of a real variable.

We will establish a relationship between the exponential function and the trigonometric functions of a complex variable. To do this, we return to the expression (3-7) for the function $e^z$ and make the substitution $z = i\zeta$. Then

$$e^{i\zeta} = \sum_{n=0}^{\infty} i^n \frac{\zeta^n}{n!}$$

Breaking up this absolutely convergent series into a sum of two series, we get

$$e^{i\zeta} = \sum_{n=0}^{\infty} (-1)^n \frac{\zeta^{2n}}{(2n)!} + i \sum_{n=0}^{\infty} (-1)^n \frac{\zeta^{2n+1}}{(2n+1)!}$$

that is,

$$e^{i\zeta} = \cos \zeta + i \sin \zeta \tag{3-33}$$

It is obvious that this identity holds for all values of the complex variable $\zeta$.

The relation (3-33) which establishes a relationship between the exponential function and the trigonometric functions of a complex variable is called *Euler's formula*, which yields the following very important formulas*

$$\cos z = \frac{1}{2} (e^{iz} + e^{-iz}) \tag{3-34}$$

and

$$\sin z = \frac{1}{2i} (e^{iz} - e^{-iz}) \tag{3-35}$$

Using these formulas and formulas (3-10) and (3-11), it is easy to set up relations connecting the trigonometric and hyperbolic functions of a complex variable:

$$\sin z = -i \sinh iz, \quad \cos z = \cosh iz \tag{3-36}$$

In particular,

$$\sin iy = i \sinh y, \quad \cos iy = \cosh y \tag{3-37}$$

---

* Recall that in Chapter 1, using these formulas, we defined the functions $\cos z$ and $\sin z$, and also formally introduced the Euler relation.

We will establish certain other important properties of these functions. First, however, note that by virtue of formula (3-30) we have the relation

$$w = e^z = e^{x+iy} = e^x \cdot e^{iy} \tag{3-38}$$

Whence it follows that $|w| = e^x$ and $\arg w = y$.

Now let us consider the function $w = \ln z = \int_0^z \frac{d\zeta}{\zeta}$, which is an analytic continuation of $\ln x$ into the complex plane cut along the negative real axis. Since for real positive $x$, the function $\ln x$ is inverse of the exponential function, then by Theorem 3.1, in the interval $-\pi < \arg z < \pi$, the relation

$$e^{\ln z} = z \tag{3-39}$$

holds, which is an analytic continuation of the relation $e^{\ln x} = x$ $(x > 0)$ into the complex plane. Thus, *the function* $\ln z$ *is the inverse of the function* $e^w$.

Note an important corollary to formula (3-39). By virtue of this formula and of (3-38) it follows from $w = u + iv = \ln z$ that

$$z = e^w = e^{u+iv} = e^u \cdot e^{iv} \tag{3-40}$$

Whence $|z| = e^u$, $\arg z = v$, and since $u$ and $|z|$ are real variables, we finally get

$$u = \ln |z|, \quad v = \arg z \tag{3-41}$$

where the symbol $\ln |z|$ denotes the real logarithmic function of a real positive argument. Thus, for the function $\ln z$ of a complex variable we get an algebraic notation in the form

$$\ln z = \ln |z| + i \arg z \tag{3-42}$$

From (3-42) we obtain the values: $\ln i = i\frac{\pi}{2}$, $\ln (1) = 0$, $\ln (-i) = -i\frac{\pi}{2}$, $\ln (1+i) = \ln \sqrt{2} + i\frac{\pi}{4}$ and so forth.

In similar fashion, on the basis of Theorem 3.1, it is easy to show that the function $\arcsin z$ defined by formula (3-15), is also the inverse of the function $\sin z$, i.e.

$$\sin (\arcsin z) = z \tag{3-43}$$

Above we established a relationship between an exponential function and the trigonometric functions. It is quite obvious that the inverse functions of the given ones, say $\ln z$ and $\arcsin z$, are also connected by definite relations.

By virtue of (3-43), from the expression $w = \arcsin z$ there follows $z = \sin w$, which, according to (3-35), may be rewritten as

$$z = \frac{1}{2i}(e^{iw} - e^{-iw}) \tag{3-44}$$

or

$$e^{2iw} - 2ize^{iw} - 1 = 0 \tag{3-45}$$

Solving the quadratic equation (3-45) for $e^{iw}$, we have

$$e^{iw} = iz + \sqrt{1 - z^2} \tag{3-46}$$

We do not write the sign $\pm$ in front of the radical because the function $\sqrt{1 - z^2}$ of the complex variable $z$ is itself a multiple-valued function (see page 28, Chapter 1). Here the choice of branch of the multiple-valued function $\sqrt{1 - z^2}$ is made so that the function at hand $w = \arcsin z$ is an analytic continuation of the corresponding function of a real variable. From this last condition, it follows that the value of the root to be taken must be positive for positive real values of the radicand. From (3-39) and (3-46) it follows that

$$iw = \ln [iz + \sqrt{1 - z^2}]$$

whence we finally obtain

$$w = \arcsin z = -i \ln [iz + \sqrt{1 - z^2}] \tag{3-47}$$

At first glance, this expression is rather complicated, and one is inclined to have doubts as to whether it indeed yields real values of $\arcsin x$ for real values $z = x$ satisfying the condition $|x| \leqslant 1$. It is easy to dispel any doubts, however. Denote $\zeta = iz + \sqrt{1 - z^2}$. For real values $z = x$ satisfying the condition $|x| \leqslant 1$, we obtain $|\zeta| = \sqrt{x^2 + 1 - x^2} = 1$ and $\arg \zeta = \arctan \frac{x}{\sqrt{1 - x^2}} = \arcsin x$. Whence, by virtue of formula (3-42), we have $-i \ln \zeta = -i [\ln 1 + i \arg \zeta] = \arg \zeta = \arcsin x$.

Since the function (3-42) is defined for all values of its argument in the complex plane with a cut along the negative real axis, it follows that formula (3-47) yields an analytic continuation of the function $\arcsin z$ into a certain domain of the $z$-plane. Then the points $z = \pm 1$ turn out to be singular in a certain sense. Indeed, as a result of circling any one of these points in the $z$-plane around a closed curve belonging to a sufficiently small $\varepsilon$-neighbourhood of the point, upon continuous variation of the function (3-47) it will change its value, since in a single circuit about the point $z = 1$ or $z = -1$ the function $\sqrt{1 - z^2}$ changes its value.* For this rea-

* See pages 29-30.

son, for the domain of single-valued definition of the function (3-47) one can choose, say, the extended $z$-plane with cuts along intervals of the real axis $[-\infty, -1]$, $[1, \infty]$.

### d. Mappings of elementary functions

To conclude this section, which is devoted to the elementary functions of a complex variable, we consider certain geometric properties of mappings performed by these functions. We start with the simplest examples.

*Example 1.* In Chapter 1 we considered the elementary power function $w = z^2$. We will now examine a mapping by the function

$$w = z^n \tag{3-48}$$

where $n$ is an arbitrary integer. This function is obviously an entire function. In the study of the geometric properties of its mapping, it is convenient to use the exponential notation of complex numbers: $z = \rho e^{i\varphi}$, $w = re^{i\psi} = \rho^n e^{in\varphi}$, from which it follows that any sector* with central angle $\alpha = \frac{2\pi}{n}$ of the $z$-plane is mapped by the given function onto the extended complex $w$-plane. Different interior points of this sector are mapped onto different points of the $w$-plane. In the process, the boundaries of the sector are mapped into one and the same ray $\psi = \psi_0$ in the $w$-plane. In order to establish a reciprocal one-to-one correspondence between the univalence domain of the function $z^n$ and the $w$-plane, we will take it that there is a cut in the $w$-plane along the ray $\psi = \psi_0$, and with the boundaries of the given sector of the $z$-plane are associated different lips of the cuts. For example, the sector $0 \leqslant \varphi \leqslant \frac{2\pi}{n}$ of the $z$-plane is mapped by the function (3-48) onto the extended $w$-plane, and both boundaries of the sector, rays $I$ and $II$ in Fig. 3.1, go into the positive real $u$-axis of the $w$-plane. The sector $\frac{2\pi}{n} \leqslant \varphi \leqslant \frac{4\pi}{n}$ is also mapped onto the extended $w$-plane, etc. Therefore, the geometric image of the function $w = z^n$ is the $w$-plane repeated $n$ times. Thus, the mapping of the extended $z$-plane onto the extended $w$-plane by this function is not one-to-one. However, if for the geometric image of the $w$ function we consider a more complicated manifold than the ordinary complex plane, the one-to-one nature of the mapping can be preserved. Suppose we have $n$ sheets of the $w$-plane cut along the positive real axis, on each of which $\arg w$ varies over the interval $2\pi (k - 1) \leqslant \arg w \leqslant 2\pi k$, where $k = 1, 2, \ldots, n$. Then, the

* By a sector we mean a closed domain together with its boundaries.

function (3-48) associates with the sector $\frac{2\pi}{n}(k-1) \leqslant \varphi \leqslant \frac{2\pi}{n} k$ of the plane $z$ the $k$th sheet of the $w$-plane; the ray $\varphi = \frac{2\pi}{n}(k-1)$ goes to the upper lip of the cut of the $k$th sheet, and the ray $\varphi = \frac{2\pi k}{n}$ to the lower lip of the cut of that sheet. Let us construct a continuous geometric manifold out of these sheets so that to a continuous

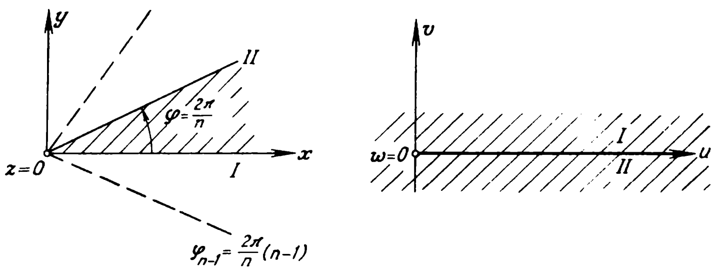


**Fig. 3.1**

motion of a point in the $z$-plane there corresponds a continuous motion of the point $w$ on the given manifold. Note that the lower lip of the cut of the $k$th sheet and the upper lip of the cut of the $(k+1)$th sheet have the same argument $\psi_k = 2\pi \cdot k$. When the point $z$, moving continuously in the $z$-plane, passes from one sector to another, its corresponding point $w$ passes from one sheet of the $w$-plane to an adjacent sheet. Clearly, in order to retain the continuity of mapping, we have to join adjacent sheets, the lower lip of the cut of the $k$th sheet with the upper lip of the cut of the $(k+1)$th sheet. Then the upper lip of the cut of the first sheet and the lower lip of the cut of the $n$th sheet remain free. Let the point $z$ make a complete circuit about the point $z = 0$ in the $z$-plane, successively passing through all $n$ sectors of the plane, beginning with the first sector and returning to its original position. Then the point $w$ corresponding to it will pass through $n$ sheets, and we have to join the free lips of the cuts of the 1st and $n$th sheets so that it can return to the first sheet. Thus, the function $w = z^n$ associates with the extended $z$-plane $n$ sheets of the $w$-plane which are joined as indicated above. Such a geometric manifold is a special case of a so-called *Riemann surface*. The function $w = z^n$ is an $n$-valent function.

*Example 2*. Consider the mapping by the function $w = e^z$. From (3-38) it follows that this function associates with every complex number $z = x + iy$ a complex number $w$, the modulus of which is $e^x$ and the argument is $y$. This implies that the exponential function

$w = e^z$ maps the straight line $y = y_0$ of the $z$-plane onto the ray $\arg w = y_0$ of the $w$-plane. As is readily seen, a strip of the $z$-plane bounded by the straight lines $y = 0$ and $y = 2\pi$ will go into the extended $w$-plane, and the straight boundary lines $y = 0$ and $y = 2\pi$ will be mapped onto one and the same ray of the $w$-plane—the positive real $u$-axis (Fig. 3.2). There is thus established a one-to-one mapping of the open domain $0 < y < 2\pi$ onto the $w$-plane with

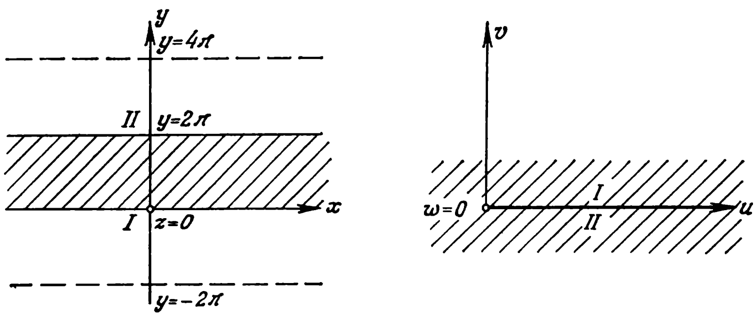


**Fig. 3.2**

removed positive real $u$-axis. In order to establish a one-to-one mapping of the corresponding closed domains, we will assume that a cut has been made along the positive real $u$-axis and a one-to-one correspondence established between the upper lip of the cut and the straight line $y = 0$, and also between the lower lip of the cut and the straight line $y = 2\pi$ of the $z$-plane. Thus, *the exponential function $e^z$ performs a one-to-one mapping of the strip $0 \leqslant y \leqslant 2\pi$ of the z-plane onto the extended w-plane cut along the positive real axis.** In similar fashion it is established that the exponential function performs a one-to-one mapping of any strip $2\pi \cdot n \leqslant y \leqslant 2\pi (n + 1)$ $(n = 0, \pm 1, \ldots)$ of the $z$-plane onto the same extended $w$-plane cut along the positive real $u$-axis. In the process, the points $z_0 = x_0 + iy_0$ and $z_1 = x_0 + i(y_0 + 2\pi k)$ $(k = \pm 1, \pm 2, \ldots)$ pass into one and the same point of the $w$-plane. This means that *an exponential function is an infinitely many-sheeted periodic function of the complex variable z with imaginary period* $2\pi i$. The domain of its univalence is any strip $y_0 < y < y_0 + 2\pi$ that is mapped onto the extended $w$-plane cut along the ray $\arg w = y_0$. Note that the argument $w$ in planes corresponding to different strips $2\pi \cdot n \leqslant y \leqslant 2\pi (n + 1)$ $(n = 0, \pm 1, \ldots)$ varies respectively within different limits. We thus obtain an infinite set of different sheets of the

* Then the boundary $y = 0$ of the strip goes into the upper lip of the cut of the $w$-plane, and the boundary $y = 2\pi$ into the lower lip.

$w$-plane cut along the positive real $u$-axis. For a continuous motion of $w$ to correspond to the continuous motion of point $z$ in the $z$-plane, during which it passes from one strip to another, the appropriate sheets of the $w$-plane have to be joined together; here it is obvious that the upper lip of the cut of the $n$th sheet must be joined to the lower lip of the cut of the $(n - 1)$th sheet and the lower lip of the cut of the $n$th sheet must be joined to the upper lip of the cut of the $(n + 1)$th sheet. The resulting geometric manifold forms a *Riemann surface with infinitely many sheets.*

Similar reasoning also applies to trigonometric functions of a complex variable. Note straight off that by virtue of formulas (3-34) and (3-35) *the trigonometric functions are infinitely many-sheeted functions of a complex variable z that are periodic with real period* $2\pi$. As in the case of the function $e^z$, it is easy to consider the geometric properties of mappings accomplished by trigonometric functions. We confine ourselves to the function $\cos z$. With the aid of the above-established properties of trigonometric functions we get

$$\cos z = \cos (x + iy) = u(x, y) + iv(x, y)$$
$$= \cos x \cdot \cosh y - i \sin x \cdot \sinh y$$

which implies that the function $\cos z$ maps the straight line $x = x_0$ of the $z$-plane into a branch of the hyperbola

$$\frac{u^2}{\cos^2 x_0} - \frac{v^2}{\sin^2 x_0} = 1 \tag{3-49}$$

in the $w$-plane. For $0 < x_0 < \frac{\pi}{2}$, the straight line $x = x_0$ goes into the right branch of the hyperbola, and the straight line $x = \pi - x_0$ goes into the left branch. As may readily be established, all hyperbolas (3-49) are confocal, their foci lying in the points $\pm 1$ of the real $u$-axis. The straight line $x_0 = \frac{\pi}{2}$ is mapped by the function $\cos z$ onto the imaginary $v$-axis of the $w$-plane, and the straight lines $x_0 = 0$ and $x_0 = \pi$, into the rays $[1, \infty]$ and $[-\infty, -1]$ of the real $u$-axis of the $w$-plane. Note that in the motion of the point $z$ along a given straight line (say the straight line $x_0 = 0$) the corresponding ray is traversed twice. Thus, *the function* $\cos z$ *executes a one-to-one mapping of the strip* $0 \leqslant x \leqslant \pi$ *of the z-plane onto the extended w-plane cut along the rays of the real axis* $[1, \infty]$ *and* $[-\infty, -1]$. In this case, the upper semi-strip $0 \leqslant x \leqslant \pi$, $y > 0$ goes into the lower half-plane $v < 0$, and the lower semi-strip $0 \leqslant x \leqslant \pi$, $y < 0$ goes into the upper half-plane $v > 0$ (this is indicated by the appropriate hatching in Fig. 3.3). It is easy to see that the next strip $\pi \leqslant x \leqslant 2\pi$ is mapped by the function $\cos z$ onto the same extended $w$-plane with cuts along rays of the real axis $[1, \infty]$ and $[-\infty, -1]$. Since $\cos (z + \pi) = -\cos z$, the upper semi-strip $\pi \leqslant x \leqslant 2\pi$, $y > 0$

goes into the upper half-plane $v > 0$, and the lower semi-strip $\pi \leqslant x \leqslant 2\pi$, $y < 0$ goes into the lower half-plane $v < 0$ (Fig. 3.3). The situation is obviously similar for any strip $n\pi \leqslant x \leqslant (n+1)\pi$. Whence it follows that the strip $n\pi < x < (n+1)\pi$ is the domain of univalence of the function $\cos z$. The function $\cos z$ is a function of infinitely many sheets, and its range is the Riemann surface of

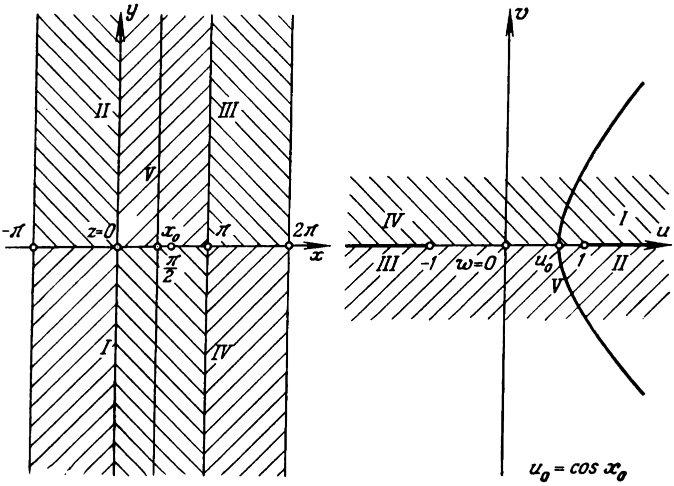


Fig. 3.3

infinitely many sheets resulting from cutting the $w$-planes along rays of the real axis $[-\infty, -1]$ and $[1, \infty]$ and joining them along the appropriate lips of the cuts.

To conclude this discussion of the basic properties of an exponential function and the trigonometric functions, we investigate the problem of the zeros of these functions. The exponential function $w = e^z$ does not vanish for any value of the complex variable $z$, as follows from formula (3-38). All the zeros of trigonometric functions lie on the real axis. Indeed, if $\sin z = 0$, then $e^{iz} - e^{-iz} = 0$, $e^{2iz} = 1$. But if the complex numbers are equal, then their arguments differ by a multiple of $2\pi$, whence $z = n\pi$, which proves the assertion.

## 3.2. Analytic Continuation. The Riemann Surface

### *a. Basic principles. The concept of a Riemann surface*

The principal task of analytic continuation is the extension of the values of a function $f(z)$ specified in a certain domain $\mathfrak{G}'$ to a larger domain $\mathfrak{G}$.

Let there be given in the complex plane two domains $\mathfrak{G}_1$ and $\mathfrak{G}_2$ having a common portion* $\mathfrak{G}_{12}$ (Fig. 3.4). Let the single-valued analytic functions $f_1(z)$ and $f_2(z)$ be given, respectively, in $\mathfrak{G}_1$ and $\mathfrak{G}_2$, and let them identically coincide in the intersection $\mathfrak{G}_{12}$. Then the function $F(z)$ defined by the relations

$$F(z) = \begin{cases} f_1(z), & z \in \mathfrak{G}_1 \\ f_2(z), & z \in \mathfrak{G}_2 \end{cases} \tag{3-50}$$

is analytic in the extended domain $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$ and coincides with $f_1(z)$ in $\mathfrak{G}_1$ and with $f_2(z)$ in $\mathfrak{G}_2$.

The function $F(z)$ is called the *analytic continuation of the function* $f_1(z)$ $(f_2(z))$ *into the domain* $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$. The function

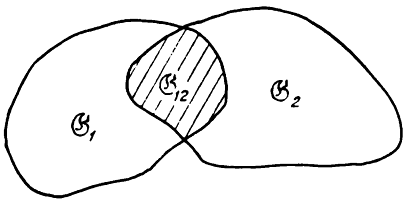


**Fig. 3.4**

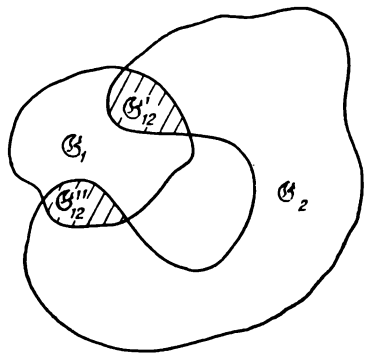


**Fig. 3.5**

$f_2(z)$ $(f_1(z))$ is also called the analytic continuation of the function $f_1(z)$ $(f_2(z))$ into the domain $\mathfrak{G}_2$ $(\mathfrak{G}_1)$.

It is readily seen that the analytic continuation of $F(z)$ of the function $f_1(z)$ into the domain $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$ is defined uniquely. Indeed, an assumption that there are two different functions in the domain $\mathfrak{G}$ identically coinciding with $f_1(z)$ in the domain $\mathfrak{G}_1$ leads to a contradiction with the theorem of the uniqueness of definition of an analytic function that was proved in the preceding chapter.

The foregoing method of analytic continuation of a function $f_1(z)$ from a domain $\mathfrak{G}_1$ into a broader domain $\mathfrak{G}$ is the simplest form of the *principle of analytic continuation.*

---

* Various cases are possible here. For example: (a) the domain $\mathfrak{G}_1$ lies within the domain $\mathfrak{G}_2$, then $\mathfrak{G}_{12}$ obviously coincides with $\mathfrak{G}_1$; (b) the intersection $\mathfrak{G}_{12}$ is a singly connected or multiply connected domain; (c) the intersection $\mathfrak{G}_{12}$ consists of several (perhaps, infinitely many) separate connected domains.

Now let us examine the case when the functions $f_1(z)$ and $f_2(z)$ coincide identically only on the part $\mathfrak{G}'_{12}$ of the overlap, $\mathfrak{G}_{12}$, of the domains $\mathfrak{G}_1$ and $\mathfrak{G}_2$ (Fig. 3.5). Consider the domain $\hat{\mathfrak{G}} = \mathfrak{G}_1 + \mathfrak{G}_2 - \mathfrak{G}''_{12}$, where $\mathfrak{G}''_{12} = \mathfrak{G}_{12} - \mathfrak{G}'_{12}$ is that part of the overlap $\mathfrak{G}_{12}$ in which the functions $f_1(z)$ and $f_2(z)$ are different. According to earlier considerations, a unique analytic function $\hat{F}(z)$ is defined in $\hat{\mathfrak{G}}$, which function is an analytic continuation of $f_1(z)$ specified in the domain $\mathfrak{G}_1 - \mathfrak{G}''_{12}$ into the domain $\hat{\mathfrak{G}}$. This function coincides indentically with the function $f_1(z)$ in the domain $\mathfrak{G}_1 - \mathfrak{G}''_{12}$ and with $f_2(z)$ in the domain $\mathfrak{G}_2 - \mathfrak{G}''_{12}$. The function $\hat{F}(z)$ may be analytically continued to the set $\mathfrak{G}''_{12}$ in two ways:

$$F_1(z) = \begin{cases} \hat{F}(z), & z \in \hat{\mathfrak{G}} \\ f_1(z), & z \in \mathfrak{G}''_{12} \end{cases} \tag{3-51}$$

or

$$F_2(z) = \begin{cases} \hat{F}(z), & z \in \hat{\mathfrak{G}} \\ f_2(z), & z \in \mathfrak{G}''_{12} \end{cases} \tag{3-52}$$

This naturally makes it necessary for us to consider the *multiple-valued analytic function* $F(z)$ which is defined in the domain $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$ and takes on different values at the same points of the $\mathfrak{G}''_{12}$ portion of the domain $\mathfrak{G}$. In particular, in the given case we obtain a double-valued analytic function $F(z)$ which at one and the same point $z_0 \in \mathfrak{G}''_{12}$ assumes two different values that coincide with the values of the functions $f_1(z)$ or $f_2(z)$ at that point.

When dealing with the multiple-valued function $F(z)$ having different values at one and the same point of the complex plane, one encounters difficulties in choosing its values at a given point. To make the choice of these values more convenient, one frequently makes use of the concept of a *branch* of an analytic function,* which is single-valued and continuous in an appropriate part of the domain of definition of the function $F(z)$. However, there is a still more convenient approach which permits regarding a given function as single-valued but defined on a more complicated manifold than the ordinary plane of a complex variable that has been used up to now. Returning to the earlier example of the double-valued function $F(z)$, we will consider that the domains $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are joined along the overlapping portion $\mathfrak{G}'_{12}$ in which the functions $f_1(z)$ and $f_2(z)$ coincide, and the two sheets $\mathfrak{G}''_{12}$ that belong to the domains $\mathfrak{G}_1$ and $\mathfrak{G}_2$ are left free.

Then, on the geometric manifold thus obtained, which is a union of the domains $\mathfrak{G}_1$ and $\mathfrak{G}_2$ joined along $\mathfrak{G}'_{12}$ (so that the points

* That was our approach in Chapter 1 when studying the function $z = \sqrt{w}$.

belonging to $\mathfrak{G}''_{12}$ overlap twice), the function $F(z)$ is a single-valued analytic function.

A manifold constructed in this manner is called a *Riemann surface of the analytic function* $F(z)$, which is an analytic continuation of the function $f_1(z)$ ($f_2(z)$), and the separate sheets of the repeating domains are different *sheets* of the Riemann surface.

Thus, instead of considering a multiple-valued function in the complex $z$-plane, we can consider a single-valued function on a Riemann surface. As in the simplest case considered at the beginning of this subsection this method of analytic continuation of a function $f_1(z)$ from a domain $\mathfrak{G}_1$ into a broader domain (which then represents a Riemann surface) is a particular form of the general *principle of analytic continuation*. Clearly it is possible, in a similar manner, to construct analytic continuations of single-valued analytic functions specified on a Riemann surface. We will then, naturally, arrive at Riemann surfaces of many sheets; these would form a geometric manifold which one and the same domain of the complex plane enters not as two sheets but as many sheets. Appropriate examples will be considered in Subsection 3.2.c. We now consider another mode of analytic continuation.

### b. *Analytic continuation across a boundary*

In a number of cases, the following method is used for analytic continuation of a function $f_1(z)$ originally specified in a domain $\mathfrak{G}_1$. Let the domains $\mathfrak{G}_1$ and $\mathfrak{G}_2$ have the piecewise smooth curve $\Gamma_{12}$

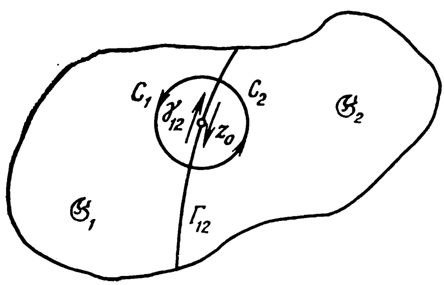


**Fig. 3.6**

(Fig. 3.6) as their common boundary and let there be given the analytic functions $f_1(z)$ and $f_2(z)$ which are respectively continuous in $\mathfrak{G}_1 + \Gamma_{12}$ and $\mathfrak{G}_2 + \Gamma_{12}$ and coincide on $\Gamma_{12}$. Consider the set of points $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2 + \Gamma_{12}$. Since the points $z \in \Gamma_{12}$ are interior points of this set, the set $\mathfrak{G}$ is a domain. We will show that the function $F(z)$ defined by means of the relations

$$F(z)=\begin{cases} f_1(z), & z\in\mathfrak{G}_1+\Gamma_{12} \\ f_2(z), & z\in\mathfrak{G}_2+\Gamma_{12}\end{cases} \tag{3-53}$$

is analytic in the domain $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2+\Gamma_{12}$. It obviously suffices to prove that for each point $z_0$ of $\mathfrak{G}$ lying on the curve $\Gamma_{12}$ it is possible to indicate a neighbourhood such that in it the function $F(z)$ is analytic. Take an arbitrary point $z_0\in\Gamma_{12}$ and construct a circle $C_0$ centred in this point and lying entirely in $\mathfrak{G}$. Consider an integral of the Cauchy type:

$$\Phi(z)=\frac{1}{2\pi i}\int_{C_0}\frac{F(\zeta)}{\zeta-z}d\zeta \tag{3-54}$$

By virtue of the earlier established properties of integrals dependent on a parameter (Chapter 1, page 53), the function $\Phi(z)$ is an analytic function of $z$ for any position of the point $z$ not lying on the curve $C_0$. We will show that when the point $z$ lies inside the circle $C_0$, $\Phi(z)\equiv F(z)$. Indeed, represent the integral (3-54) in the form

$$\frac{1}{2\pi i}\int_{C_0}\frac{F(\zeta)}{\zeta-z}\,d\zeta=\frac{1}{2\pi i}\int_{C_1+\gamma_{12}}\frac{f_1(\zeta)}{\zeta-z}\,d\zeta+\frac{1}{2\pi i}\int_{\gamma_{12}+C_2}\frac{f_2(\zeta)}{\zeta-z}\,d\zeta \tag{3-55}$$

where $C_1$ and $C_2$ are parts of the circle $C_0$ lying in $\mathfrak{G}_1$ and $\mathfrak{G}_2$ ($C_0=C_1+C_2$) and $\gamma_{12}$ is a portion of the curve $\Gamma_{12}$ lying inside the circle $C_0$. If the point $z$ belongs to the domain $\mathfrak{G}_1$, then by Cauchy's theorem,* we have

$$\frac{1}{2\pi i}\int_{C_1+\gamma_{12}}\frac{f_1(\zeta)}{\zeta-z}\,d\zeta=f_1(z),\quad \frac{1}{2\pi i}\int_{\gamma_{12}+C_2}\frac{f_2(\zeta)}{\zeta-z}\,d\zeta=0 \tag{3-56}$$

whence $\Phi(z)=f_1(z)=F(z)$ for $z\in\mathfrak{G}_1$. Similarly, $\Phi(z)= =f_2(z)=F(z)$ for $z\in\mathfrak{G}_2$. In the point $z_0$, which belongs to $\gamma_{12}$, by virtue of continuity of the functions $\Phi(z)$, $f_1(z)$, $f_2(z)$ inside the circle $C_0$, we will also have $\Phi(z_0)=f_1(z_0)=f_2(z_0)=F(z_0)$, whence it follows that $F(z)$ is an analytic function in the domain $\mathfrak{G}$.

As in the preceding case, we will say here that the function $f_1(z)$ ($f_2(z)$) specified in the domain $\mathfrak{G}_1$ ($\mathfrak{G}_2$) *is analytically continued into the domain* $\mathfrak{G}_2$ ($\mathfrak{G}_1$). The above-constructed function $F(z)$ is *an analytic continuation of the function* $f_1(z)$ *into the domain* $\mathfrak{G}= =\mathfrak{G}_1+\mathfrak{G}_2+\Gamma_{12}$. This construction is a special form of *the general principle of analytic continuation—analytic continuation across the boundary of a domain.* And also, as in the previous cases, in continuing across a boundary, we may find it necessary to consider a

* The applicability of Cauchy's theorem to the integrals on the right of (3-55) is obvious due to the assumption that the curve $\Gamma_{12}$ is piecewise smooth and due to the choice of the curve $C_0$.

single-valued analytic function on a Riemann surface in cases when the domains $\mathfrak{G}_1$ and $\mathfrak{G}_2$ have aside from the common portion of the boundary $\Gamma_{12}$, a nonempty intersection $\mathfrak{G}_{12}$, in which the functions $f_1(z)$ and $f_2(z)$ are not identically equal to each other.

Let us now consider a number of examples in applying the general principles of analytic continuation that lead both to multiple-valued and single-valued functions.

### *c. Examples in constructing analytic continuations. Continuation across a boundary*

Consider some examples of the construction of an analytic continuation of a function $f_1(z)$ originally specified in a domain $\mathfrak{G}_1$ of the complex $z$-plane. As was noted above, in a number of cases we find it necessary to examine functions that are multiple-valued in the complex plane.

In Chapter 1 we already had an elementary example of a multiple-valued function of a complex variable, the function $w = \sqrt{z}$*, which is the inverse of the power function $z = w^2$. We now consider this and a few other functions from the general viewpoint of analytic continuation.

*Example 1.* The function $w = \sqrt[n]{z}$. According to the rule of extracting the $n$th root of a complex number, to one value of $z$ there correspond $n$ distinct complex numbers $w$ computed from the formula

$$w = re^{i\psi} = \sqrt[n]{\rho}\, e^{i\frac{\varphi+2\pi k}{n}} \quad (k = 0, 1, \ldots, n-1) \tag{3-57}$$

where $z = \rho e^{i\varphi}$ and $\varphi$ is one of the values of $\operatorname{Arg} z$. The function $w = \sqrt[n]{z}$ is a multiple-valued function having $n$ distinct branches. We will assume that $\varphi$ varies within the interval $0 \leqslant \varphi \leqslant 2\pi$ and we will choose that branch of the function $w = \sqrt[n]{z}$ which is an analytic continuation of the real function $u = \sqrt[n]{x}$ of the real positive variable $x > 0$. Clearly, it will be

$$w_1 = \sqrt[n]{\rho}\, e^{i\frac{\varphi}{n}} \quad (0 \leqslant \varphi \leqslant 2\pi) \tag{3-58}$$

The domain $\mathfrak{G}_1$ of definition of the function $w_1$ is the $z$-plane cut along the positive real $x$-axis. The upper lip of the cut corresponds to the value $\arg z = 0$, the lower lip, to the value $\arg z = 2\pi$. Obviously, the function $w_1$, which is the inverse of $z = w^n$, maps the closed domain $\overline{\mathfrak{G}}_1$ of the $z$-plane one-to-one onto the sector

* Here we have changed the designations of dependent variable and independent variable.

$0 \leqslant \arg w \leqslant \frac{2\pi}{n}$ of the $w$-plane. By virtue of the general properties of analytic functions (see Chapter 1, page 33), the function $w_1$ in the domain $\mathfrak{G}_1$ is a single-valued analytic function whose derivative is computed from the formula

$$w_1'(z) = \frac{1}{n} z^{\frac{1}{n}-1} = \frac{1}{n} \rho^{\frac{1}{n}-1} \cdot e^{i\varphi \frac{1-n}{n}}$$

Now consider the closed domain $\overline{\mathfrak{G}}_2$—the same $z$-plane with a cut along the positive real $x$-axis, but such a plane in which the argument $z$ varies within the interval $2\pi \leqslant \arg z \leqslant 4\pi$. The upper lip of the cut corresponds to the value $\arg z = 2\pi$, the lower lip, to the value $\arg z = 4\pi$. In this domain consider the function

$$w_2(z) = \sqrt[n]{\rho}\, e^{i\frac{(\varphi+2\pi)}{n}} \quad (0 \leqslant \varphi \leqslant 2\pi) \tag{3-59}$$

This function maps one-to-one the closed domain $\overline{\mathfrak{G}}_2$ onto the sector $\frac{2\pi}{n} \leqslant \arg w \leqslant \frac{4\pi}{n}$ of the $w$-plane and is a single-valued analytic function of $z$ in the domain $\mathfrak{G}_2$. The closed domains $\overline{\mathfrak{G}}_1$ and $\overline{\mathfrak{G}}_2$ have a common portion of the boundary $\Gamma_{12}$—the ray $\arg z = 2\pi$—on which the functions $w_1$ and $w_2$, continuous, respectively, in $\mathfrak{G}_1 + \Gamma_{12}$ and $\mathfrak{G}_2 + \Gamma_{12}$, coincide. Therefore, by virtue of the principle of analytic continuation across a boundary, the function $w_2(z)$ is an analytic continuation of the function $w_1(z)$ into the domain $\mathfrak{G}_2$. On the other hand, $\overline{\mathfrak{G}}_1$ and $\overline{\mathfrak{G}}_2$ actually overlap in the $z$-plane, since the points of a complex plane with equal moduli and with arguments differing by $2\pi$ coincide. Since the functions (3-58) and (3-59) have different values at one and the same point $z$, it follows that by earlier considerations in order that the function

$$F_1(z) = \begin{cases} w_1(z), & z \in \mathfrak{G}_1 + \Gamma_{12} \\ w_2(z), & z \in \mathfrak{G}_2 + \Gamma_{12} \end{cases} \tag{3-60}$$

should be single-valued in the domain of its definition $R_1 = \mathfrak{G}_1 + \mathfrak{G}_2 + \Gamma_{12}$, we have to assume that the manifold $R_1$ is a Riemann surface made up of the sheets $\overline{\mathfrak{G}}_1$ and $\overline{\mathfrak{G}}_2$ joined together. Clearly, the sheets should be joined along the common portion of the boundary $\Gamma_{12}$, the ray $\arg z = 2\pi$, by connecting the lower lip of the cut of domain $\overline{\mathfrak{G}}_1$ with the upper lip of the cut of the domain $\overline{\mathfrak{G}}_2$. Repeating our reasoning, we will find that the function

$$w_{k+1}(z) = \sqrt[n]{\rho}\, e^{i\frac{\varphi+2\pi k}{n}} \quad (0 \leqslant \varphi \leqslant 2\pi) \tag{3-61}$$

defined in the closed domain $\overline{\mathfrak{G}}_{k+1}$ $(2\pi k \leqslant \arg z \leqslant 2\pi(k+1))$ is an analytic continuation of the function $w_k(z)$ defined in $\overline{\mathfrak{G}}_k$.

Note that the function $w_{n+1}(z)$ identically coincides with the function $w_1(z)$. And so it is natural to consider the single-valued analytic function

$$F(z)=\begin{cases} w_1(z), & z\in \mathfrak{G}_1+\Gamma_{12} \\ w_2(z), & z\in \mathfrak{G}_2+\Gamma_{12}+\Gamma_{23} \\ \dots\dots\dots\dots & \\ w_n(z), & z\in \mathfrak{G}_n+\Gamma_{n-1\,n} \end{cases} \qquad (3\text{-}62)$$

defined on the Riemann surface $R=\mathfrak{G}_1+\mathfrak{G}_2+\dots+\mathfrak{G}_n+$ $+\Gamma_{12}+\dots+\Gamma_{n-1\,n}$ built up in the way mentioned above by joining the $n$ sheets that form the $z$-plane with a cut along the positive real $x$-axis. Then the upper lip of the cut ($\arg z=0$) on the

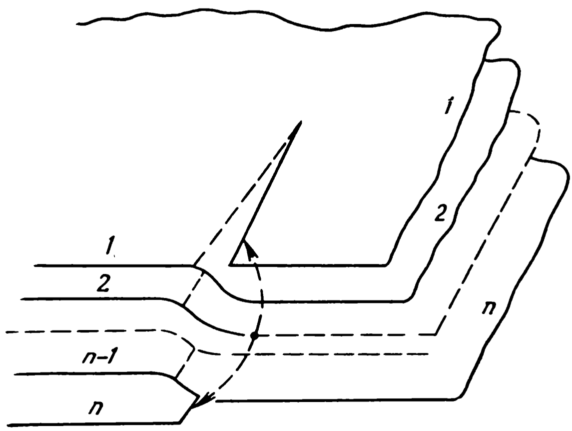


**Fig. 3.7**

first sheet $\mathfrak{G}_1$ and the lower lip of the cut ($\arg z=2\pi n$) on the $n$th sheet $\mathfrak{G}_n$ remain free. In order to retain the continuity of the function $F(z)$ throughout the domain of its definition, we will join these lips of the cuts (Fig. 3.7).* The function (3-62) is called the complete analytic function $w=\sqrt[n]{z}$, and the thus constructed closed manifold $R$ is termed the complete Riemann surface of this function. On each sheet of the Riemann surface is defined a separate branch of the given multiple-valued function.

We bring attention to the following circumstance. Fix on the $z$-plane a certain point $z_0$ and draw through it a closed curve $C$.

---

* Join cut sheets of paper to get a better pictorial view of what occurs. However, the last joint is physically impossible and can only be visualized mentally.

Then if arg $z$ varies continuously in motion round the curve $C$, and $C$ intersects the branch cut in the $z$-plane, then two cases are a priori possible in one complete circuit of the curve $C$ (Fig. 3.8). In the first case, the point $z = 0$ lies outside the curve $C$. Therefore, starting from the point $z = z_0$ $(\arg z_0 = \varphi_0)$ on the $k$th sheet,* we return, after a circuit of this curve, to the original point $z_0$ on the same $k$th $(\arg z_0 = \varphi_0)$ sheet, although we crossed onto other sheets

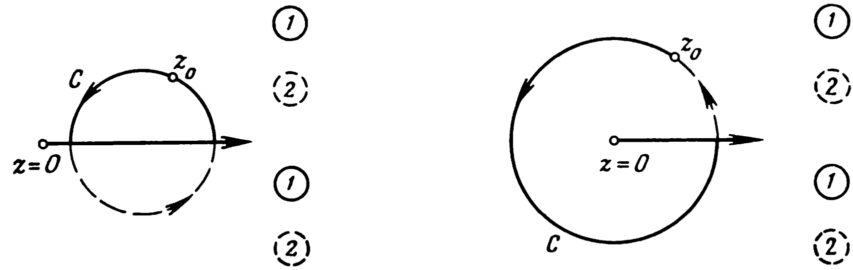


**Fig. 3.8**

as we intersected the branch cut. In the second case, the point $z = 0$ lies inside the curve $C$. And so, starting from the point $z = z_0$ $(\arg z_0 = \varphi_0)$ on the $k$th sheet, we return, after traversing the curve $C$, to the point $z = z_0$ not on the original $k$th sheet, but, say, on the $(k + 1)$th sheet $(\arg z_0 = \varphi_0 + 2\pi)$. The point $z_0$, which is encircled via any closed curve in a sufficiently small neighbourhood of the point and during the circuit of which we pass from one sheet of the Riemann surface of the analytic function $F(z)$ to another sheet, is called the *branch point* of the function $F(z)$. It is easy to see that this definition of a branch point is equivalent to the definition given on page 29 of Chapter 1. Obviously, in the case at hand of the function $w = \sqrt[n]{z}$, the branch points are $z = 0$ and $z = \infty$.

*Example 2.* The function $w = \operatorname{Ln} z$.

In the closed domain $\mathfrak{G}_0$, which is the $z$-plane cut along the negative real axis $-\pi \leqslant \arg z \leqslant \pi$, consider the function $\ln z$, which was discussed in the preceding section:

$$w_0 = \ln(z) = \ln|z| + i \arg z, \quad -\pi \leqslant \arg z \leqslant \pi \qquad (3\text{-}63)$$

We know that this single-valued analytic function is an analytic continuation of the real function $u = \ln x$ and is the inverse of the function $z = e^w$. Therefore the function (3-63) maps the domain $\mathfrak{G}_0$ of the $z$-plane onto the strip $-\pi < \mathrm{I}\quad w < \pi$ of the $w$-plane.

* Fig. 3.8 corresponds to the case $k = 1$.

In the closed domain $\overline{\mathfrak{G}}_1$ ($\pi \leqslant \arg z \leqslant 3\pi$) consider the function

$$w_1 = \ln_1 (z) = \ln |z| + i \arg z, \quad \pi \leqslant \arg z \leqslant 3\pi \qquad (3\text{-}64)$$

Clearly, the function $w_1(z)$ is an analytic continuation of $w_0(z)$ into the domain $\mathfrak{G}_1$. Analogously, the function

$$w_{-1}(z) = \ln_{-1}(z) = \ln |z| + i \arg z, \quad -3\pi \leqslant \arg z \leqslant -\pi \quad (3\text{-}64')$$

defined in the closed domain $\overline{\mathfrak{G}}_{-1}$ ($-3\pi \leqslant \arg z \leqslant -\pi$) is an analytic continuation of the function $w_0(z)$ in the domain $\mathfrak{G}_{-1}$. The same goes for the function $w_k(z)$:

$$w_k(z) = \ln_k(z) = \ln |z| + i \arg z, \quad \pi(2k-1) \leqslant \arg z \leqslant \leqslant \pi(2k+1) \qquad (3\text{-}65)$$

defined in the closed domain $\overline{\mathfrak{G}}_k$, $\pi(2k-1) \leqslant \arg z \leqslant \pi(2k+1)$, which is the analytic continuation of the function $w_{k-1}(z)$. The function $w_k(z)$, which uniquely maps the domain $\mathfrak{G}_k$ onto the strip $\pi(2k-1) < \operatorname{Im} w < \pi(2k+1)$, is also the inverse function of $z = e^w$. Unlike the preceding case, not one of the functions $w_k(z)$ ($k \neq 0$) is identically equal to the function $w_0(z)$. Therefore the given process of analytic continuation should be carried out indefinitely both for $k > 0$ and for $k < 0$. Thus the complex analytic function

$$F(z) = \operatorname{Ln} z = \ln |z| + i \operatorname{Arg} z = \begin{cases} \vdots \\ w_1(z), & z \in \mathfrak{G}_1 + \Gamma_{0,1} + \Gamma_{12} \\ w_0(z), & z \in \mathfrak{G}_0 + \Gamma_{0,1} + \Gamma_{0,-1} \\ w_{-1}(z), & z \in \mathfrak{G}_{-1} + \Gamma_{0,-1} + \Gamma_{-1,-2} \\ \vdots \end{cases} \qquad (3\text{-}66)$$

is an infinitely-valued function in the ordinary $z$-plane and single-valued on the infinitely-valent Riemann surface $R = \sum_{n=-\infty}^{\infty} \overline{\mathfrak{G}}_n$ composed of infinitely many sheets $\overline{\mathfrak{G}}_n$ by joining the upper lip of the cut of each $(k+1)$th sheet with the lower lip of the cut of the preceding $k$th sheet. As in the previous case, the points $z = 0$ and $z = \infty$ are branch points of the function $\operatorname{Ln} z$.

Note again that the function $w = \operatorname{Ln} z$ is the inverse of the function $z = e^w$. This permits defining the *power function* $z^\alpha$ for any complex value of $\alpha$ in the form

$$z^\alpha = (e^{\operatorname{Ln} z})^\alpha = e^{\alpha \operatorname{Ln} z} \qquad (3\text{-}67)$$

### d. Examples in constructing analytic continuations. Continuation by means of power series

In the cases we have examined, the various branches of an analytic function were specified explicitly in the entire complex plane and the analytic continuation was constructed by an appropriate joining of the domains of definition of the branches. We now consider yet another method of construction of the analytic continuation of an analytic function originally specified in some domain $\mathfrak{G}_1$ of the complex $z$-plane.

Let the function $f_1(z)$ be analytic in the domain $\mathfrak{G}_1$. Choose an arbitrary point $z_0 \in \mathfrak{G}_1$ and expand $f_1(z)$ in a power series in the neighbourhood of this point:

$$f_1(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n \tag{3-68}$$

Consider the series on the right of (3-68). A priori, there are two possible cases (Fig. 3.9). In the first case, the radius of convergence

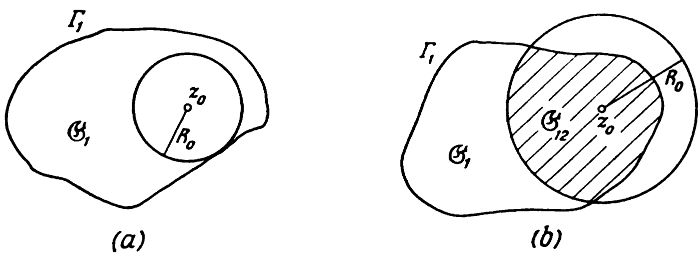


**Fig. 3.9**

$R_0$ of the series (3-68) does not exceed the distance from the point $z_0$ to the boundary $\Gamma_1$ of the domain $\mathfrak{G}_1$. In this case, the expansion (3-68) does not go beyond the boundary of the domain $\mathfrak{G}_1$ of the original definition of the analytic function $f_1(z)$. In the second case, the radius of convergence $R_0$ of the series (3-68) exceeds the distance from the point $z_0$ to the boundary $\Gamma_1$ of the domain $\mathfrak{G}_1$. In this case, the domain $\mathfrak{G}_2$ which is the circle $|z - z_0| < R_0$ is no longer a subdomain of $\mathfrak{G}_1$ but only has a common overlapping portion $\mathfrak{G}_{12}$. In $\mathfrak{G}_2$, the convergent power series (3-68) defines an analytic function $f_2(z)$ that coincides with $f_1(z)$ in $\mathfrak{G}_{12}$. This function $f_2(z)$ is the analytic continuation of $f_1(z)$ into the domain $\mathfrak{G}_2$. Consequently, there is defined in the domain $\mathfrak{G} = \mathfrak{G}_1 + \mathfrak{G}_2$ the analytic function

$$F(z) = \begin{cases} f_1(z), & z \in \mathfrak{G}_1 \\ f_2(z), & z \in \mathfrak{G}_2 \end{cases} \tag{3-69}$$

Thus, in the case at hand, the expansion (3-68) takes us beyond the boundary $\Gamma_1$ of the domain $\mathfrak{G}_1$ of the original definition of the analytic function $f_1(z)$. Reasoning similarly for some point $z_1$ of the constructed domain $\mathfrak{G}_2$, then for point $z_2$ of $\mathfrak{G}_3$ and so forth, we get the analytic continuation of the function $f_1(z)$ along a *chain of domains* $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_n, \ldots$. Here, there may be such overlappings of domains of the chain that make it necessary to consider the function $F(z)$ as a single-valued analytic function defined not in the ordinary complex $z$-plane, but on a Riemann surface.

Let us take an instance of this method of analytic continuation.

*Example 3.* Let the function $f_1(z)$ be originally specified by its power series

$$f_1(z) = \sum_{n=0}^{\infty} z^n \tag{3-70}$$

This series converges inside the circle $|z| < 1$ to the analytic function $f_1(z) = \frac{1}{1-z}$. Everywhere outside the circle $|z| < 1$, the series diverges; hence, $f_1(z)$ is not defined outside the circle $|z| < 1$. Choose some point $z_0$ inside the circle $|z| < 1$, and construct the power-series expansion of $f_1(z)$, $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, centred in this point. Computing the coefficients $c_n$ from formula (2-16), we get $c_n = \frac{1}{(1-z_0)^{n+1}}$. It is easy to show that the radius of convergence of the given series $\rho(z_0)$ is $|1 - z_0|$. As follows from elementary geometric reasoning, when the point $z_0$ does not lie on a segment of the real axis $[0, 1]$, the circle of convergence of the given series goes beyond the original circle of convergence $|z| < 1$. Hence, the function $f_2(z) = \sum_{n=0}^{\infty} \frac{(z-z_0)^n}{(1-z_0)^{n+1}}$ is an analytic continuation of the function $f_1(z)$ into the domain $|z - z_0| < |1 - z_0|$.

Note that the power series defining the function $f_2(z)$ is also readily summable, and $f_2(z) = \frac{1}{1-z}$. Therefore, taking as the new centre of expansion the point $z_1$ inside the circle $|z - z_0| < |1 - z_0|$, we get the series $\sum_{n=0}^{\infty} \frac{(z-z_1)^n}{(1-z_1)^{n+1}}$ which converges inside the circle $|z - z_1| < |1 - z_1|$ to the function $f_3(z) = \frac{1}{1-z}$, which coincides with $f_2(z)$ and $f_1(z)$ in the overlapping parts of the circle $|z - z_1| < |1 - z_1|$ and of the domains of definition of the appropriate functions. Thus, $f_3(z)$ is an analytic continuation of $f_1(z)$ into a new domain. Note that for any choice of the point $z_1$, the boundary of the appropriate circle of convergence will pass through the point $z = 1$ (Fig. 3.10). In similar fashion it is possible to construct the analytic continuation of the function $f_1(z)$ into the extended plane of a complex variable, except the point $z = 1$. Then, the function $F(z) = \frac{1}{1-z}$ defined everywhere and analytic everywhere, except at the point $z = 1$, is the analytic continuation of $f_1(z)$ obtained by means of power series.

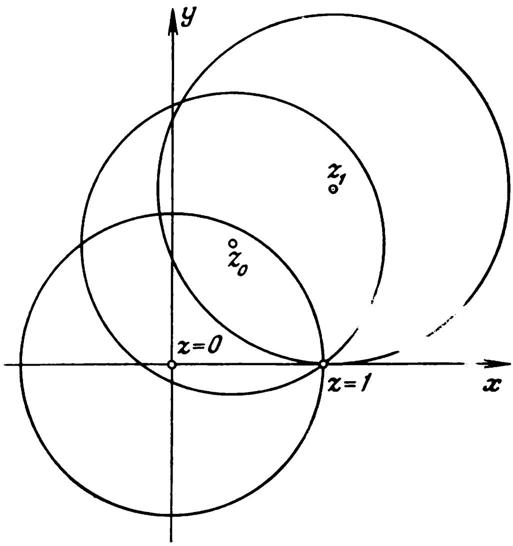


Fig. 3.10

We have thus been able to extend the domain of the original specification of the analytic function $F(z)$—the circle $|z| < 1$ in which the function $f_1(z)$ was specified—to a greater domain. Observe that although there are numerous cases of overlapping of the constructed chain of domains, the resulting analytic function $F(z) = \frac{1}{1-z}$ is single-valued throughout the domain of its definition, that is, in the extended $z$-plane with the point $z = 1$ removed. A further analytic continuation of the function $F(z)$ to a greater domain is now impossible. The point $z = 1$, which is the limit of the domain of analyticity of the function $F(z)$ is, in a definite sense, a singular point of this function. The behaviour of an analytic function in the neighbourhood of such points deserves a more detailed study. This will be done later on.

### *e. Regular and singular points of an analytic function*

Let a function $f(z)$ be given in a domain $\mathfrak{G}$ bounded by a contour $\Gamma$. *The point $z_0 \in \overline{\mathfrak{G}}$ is called a regular point of the function $f(z)$ if there is a convergent power series $\sum_{n=0}^{\infty} c_n (z - z_0)^n$, which in the overlapping part of the domain $\mathfrak{G}$ and of its circle of convergence $|z - z_0| < \rho(z_0)$ converges to the function $f(z)$.* Only one restriction is imposed on the value of the number $\rho(z_0)$: $\rho(z_0)$ is strictly greater than zero. The points $z \in \overline{\mathfrak{G}}$, which are not regular points of the function $f(z)$, are called *singular points*. Clearly, if $f(z)$ is analytic in the domain $\mathfrak{G}$, then all interior points of the domain are regular points of the function $f(z)$. The points of the boundary $\Gamma$ may be either regular or singular points of the analytic function $f(z)$. It is obvious that all the points of the boundary $\Gamma$ lying inside the circle $|z - z_0| < \rho(z_0)$ centred in some regular point $z_0 \in \overline{\mathfrak{G}}$ are also regular points of the function $f(z)$. Thus, in the above example, all the points of the boundary $|z| = 1$ of the domain of the original definition of the function $f_1(z) = \sum_{n=0}^{\infty} z^n$, with the exception of $z = 1$, are regular points. The sole singular point of this function can only be $z = 1$. It is also a singular point of the function $F(z) = \frac{1}{1-z}$, which is the analytic continuation of the function $f_1(z)$ into the extended domain. Similarly, the points $z = 0, \infty$ are singular points of the functions $\sqrt[n]{z}$ and $\operatorname{Ln} z$, considered in Subsection 3.2.c.

Let the analytic function $f_1(z)$ be originally given in the domain $\mathfrak{G}_1$ and let all points of the connected section $\Gamma'$ of the boundary $\Gamma$ of this domain be regular points of the function $f_1(z)$. Then from the foregoing reasoning it follows that $f_1(z)$ may be analytically continued across $\Gamma'$ into a greater domain. It may turn out that all the points of the boundary $\Gamma$ of the domain $\mathfrak{G}_1$ of the original specification of the analytic function $f(z)$ are regular. In this case, the function $f(z)$ will be called *analytic in the closed domain $\overline{\mathfrak{G}}_1$*. It follows from earlier reasoning that *a function which is analytic in a closed domain $\overline{\mathfrak{G}}_1$ may be analytically continued into the greater domain $\mathfrak{G}$ which contains $\overline{\mathfrak{G}}_1$.*

Analytic continuation across a portion of a boundary containing only singular points of the function $f_1(z)$ is obviously impossible.

We give an example of an analytic function (specified in a bounded domain) that cannot be continued to a greater domain.

*Example 4.* Consider the analytic function $f(z)$ specified by the power series

$$f(z)=\sum_{n=0}^{\infty} z^{2^n} \tag{3-71}$$

As is readily determined by means of elementary characteristics, the series (3-71) converges inside the circle $|z|<1$. For a real $x \to 1$, the sum $\sum_{n=0}^{\infty} x^{2^n}$ increases indefinitely; the point $z=1$ is thus a singular point of $f(z)$. We will show that the points $z_{k,m}= e^{i\frac{2\pi}{2^k}m}$, where $m=1, 2, 3, \ldots, 2^k$ ($k$ is any natural number) are also singular points of the function $f(z)$. To do this, consider the point $\tilde{z}_{k,m}=\rho\cdot e^{i\frac{2\pi}{2^k}m}$ $(0<\rho<1)$ and represent the value of $f(z)$ at this point in the form

$$f(\tilde{z}_{k,m})=\sum_{n=0}^{k-1} \tilde{z}_{k,m}^{2^n}+\sum_{n=k}^{\infty} \tilde{z}_{k,m}^{2^n} \tag{3-72}$$

The first term in (3-72), which is the sum of a finite number of terms, is bounded in absolute value, and the second, by virtue of choice of the point $\tilde{z}_{k,m}$, may be transformed to

$$\sum_{n=k}^{\infty} \tilde{z}_{k,m}^{2^n}=\sum_{n=k}^{\infty} \rho^{2^n} \tag{3-73}$$

As $\rho \to 1$, the sum of the expression on the right of (3-73) increases without bound. This proves that the points $z_{k,m}$ are singular points of the function $f(z)$. But as $k \to \infty$, these points are everywhere dense* on the circle $|z|=1$, thus implying that the function (3-71) indeed cannot be extended across any arc of the circle.

While constructing the analytic continuation of the function $F(z)=\frac{1}{1-z}$ by means of power series we saw that the boundary of the circle of convergence of every element $f_k(z)$ of it passes through the point $z=1$, which is a singular point of the function. Thus, on the boundary of the circle of convergence of any one of the constructed power series there lies a singular point of the analytic function to which the series converges. This property is a general consequence of the following theorem.

***Theorem 3.3.*** *On the boundary of the circle of convergence of a power series, there is at least one singular point of the analytic function $F(z)$ to which point the series converges.*

---

* What this means is that there will be points of the sequence $\{z_{k,m}\}$ in any $\varepsilon$-neighbourhood of every point of the circle $|z|=1$.

*Proof*. Suppose that all points of the circumference $C_0$ of the circle $K_0$ of convergence of the series $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$ are regular, i.e. for any point $\hat{z} \in C_0$ there exists a $\rho(\hat{z}) > 0$ such that in the common part of the circle $K_0$ and of its circle of convergence $|z - \hat{z}| < \rho(\hat{z})$ the corresponding series $\sum_{n=0}^{\infty} c_n(\hat{z})(z - \hat{z})^n$ converges to $f(z)$. Let the radius of the circle $K_0$ be $R_0$.

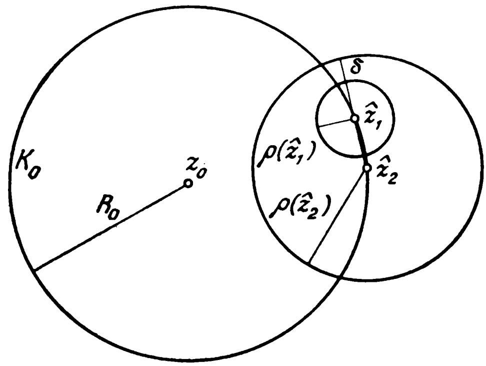


**Fig. 3.11**

Consider the function $\rho(\hat{z})$ defined on the circumference $C_0$. We will show that for any two points $\hat{z}_1$ and $\hat{z}_2$ on $C_0$ the condition

$$|\rho(\hat{z}_1) - \rho(\hat{z}_2)| \leqslant |\hat{z}_1 - \hat{z}_2| \tag{3-74}$$

is fulfilled. Indeed, suppose that it is not fulfilled; for example, $\rho(\hat{z}_2) - \rho(\hat{z}_1) = |\hat{z}_1 - \hat{z}_2| + \delta$, where $\delta > 0$. Then the circle $|z - \hat{z}_1| < \rho(\hat{z}_1)$ of convergence of the series $\sum_{n=0}^{\infty} c_n(\hat{z}_1)(z - \hat{z}_1)^n = f_1(z)$ lies inside the circle $|z - \hat{z}_2| < \rho(\hat{z}_2)$ of convergence of the series $\sum_{n=0}^{\infty} c_n(\hat{z}_2)(z - \hat{z}_2)^n = f_2(z)$ (Fig. 3.11). In the overlapping portion of these circles and of the circle $K_0$ both series converge to the same function $f(z)$. Hence, the function $f_2(z)$ is an analytic continuation of the function $f_1(z)$. This means that in the circle $|z - \hat{z}_1| < \rho(z_1) + \delta$ there is defined the analytic function $f_2(z)$, which coincides with $f_1(z)$ in the circle $|z - \hat{z}_1| < \rho(\hat{z}_1)$. By Taylor's theorem it then follows that the radius of convergence of the series $\sum_{n=0}^{\infty} c_n (\hat{z}_1)(z-\hat{z}_1)^n$ is at least $\rho(\hat{z}_1)+\delta$, which contradicts the starting data. The condition (3-74) is thus established.

This condition implies the uniform continuity of the function $\rho(\hat{z})$ on the curve $C_0$. Indeed, the relation $|\rho(\hat{z}_1)-\rho(\hat{z}_2)|<\varepsilon$ is fulfilled for any prescribed $\varepsilon>0$ provided the condition $|\hat{z}_1-\hat{z}_2|<\varepsilon$ is fulfilled. Since the function $\rho(\hat{z})>0$, it is bounded from below and by virtue of continuity attains its greatest lower bound $\rho(\hat{z})\geqslant\rho(\hat{z}_0)=\rho_0>0$ on $C_0$. This inequality holds true because for all $\hat{z}\in C_0$ the strict inequality $\rho(\hat{z})>0$ is fulfilled.

By the uniqueness of analytic continuation it may be asserted that in the circle $|z-z_0|<R_0+\rho_0$ is defined a single-valued analytic function $F(z)$ that coincides with the function $f(z)$ in the circle $|z-z_0|<R_0$. Hence, the radius of convergence of the original power series $\sum_{n=0}^{\infty} c_n(z-z_0)^n$ should be $R_0+\rho_0$, and not $R_0$. But this contradicts the hypothesis of the theorem. Thus, the supposition that all points of the boundary of the circle of convergence are regular leads to a contradiction. The theorem is proved.

From Theorem 3.3 it follows that *the radius of the circle of convergence of a power series is determined by the distance from the centre of convergence to the nearest singular point* of the analytic function to which the given series converges

### *f. The concept of a complete analytic function*

The foregoing considerations have made it possible to construct an analytic continuation of a function $f_1(z)$ given in a domain $\mathfrak{G}_1$ to a greater domain $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2$ or to a corresponding Riemann surface. As we have seen it is possible to regard an analytic continuation along a chain of domains $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_n$ having overlapping portions $\mathfrak{G}_{i,\,i+1}'$ in which the analytic functions $f_1(z)$, $f_2(z), \ldots, f_n(z)$ specified in the domains $\mathfrak{G}_1, \mathfrak{G}_2, \ldots, \mathfrak{G}_n$ coincide. We then obtain, in the domain $\mathfrak{G}=\mathfrak{G}_1+\mathfrak{G}_2+\ldots+\mathfrak{G}_n$ or on a corresponding Riemann surface $R$, a single-valued analytic function $F(z)$, which is an analytic continuation of the function $f_1(z)$.

If the analytic function $f_1(z)$ is originally specified in a domain $\mathfrak{G}_1$, then, by constructing different chains of domains that go beyond $\mathfrak{G}_1$, we can obtain an analytic continuation of the function $f_1(z)$ to various domains containing $\mathfrak{G}_1$. Here, the essential thing is the concept of a complete analytic function.

*The function $F(z)$ obtained by means of an analytic continuation along all possible chains of domains extending beyond the domain $\mathfrak{G}_1$*

*of the original specification of the analytic function $f_1(z)$ is called the complete analytic function. Its domain of definition $R$ is called the natural domain of existence of the complete analytic function.*

According to the reasoning just carried out, the natural domain of existence $R$ of a complete analytic function $F(z)$ may be a Riemann surface. Observe that an analytic continuation of the function $F(z)$ beyond the boundary $\Gamma$ of its natural domain of existence $R$ is then impossible. All points of this boundary are then singular points of the function $F(z)$. This can easily be proved. Assume that the point $z_0 \in \Gamma$ is a regular point of the function $F(z)$. In that case, by the definition of a regular point, there exists inside the circle $|z - z_0| < \rho(z_0)$ a certain analytic function $\Phi(z)$ that coincides with $F(z)$ in the common part of the given circle and of the domain $\mathfrak{G}$. But the circle $|z - z_0| < \rho(z_0)$ definitely goes beyond the domain $\mathfrak{G}$, and so $\Phi(z)$ is an analytic continuation of the complete analytic function across the boundary of its natural domain of existence, which is not possible.

In the examples examined in the earlier parts of this section, we constructed a series of complete analytic functions and their natural domains of existence. Thus, the natural domains of existence of the complete analytic functions $\sqrt[n]{z}$ and $\operatorname{Ln} z$ are the $n$-valent and infinitely-valent Riemann surfaces, respectively; the natural domain of existence of the complete analytic function $\frac{1}{1-z}$ is the extended complex plane with point $z = 1$ removed; the natural domain of existence of the function (3-71) considered in Example 4 is the unit circle $|z| < 1$.

Here, the domain $\mathfrak{G}_1$ of the original specification of the analytic function $f_1(z)$ is such that an analytic continuation of the function $f_1(z)$ across the boundary $\Gamma_1$ of the domain $\mathfrak{G}_1$ is impossible. This implies that $f_1(z)$ is a complete analytic function and $\mathfrak{G}_1$ is its natural domain of existence. However, if the domain $\mathfrak{G}_1$ is such that an analytic continuation of $f_1(z)$ into a greater domain is possible, then the function $f_1(z)$ is called an *element of the complete analytic function $F(z)$*. The analytic continuation $f_2(z)$ of the function $f_1(z)$, specified in the domain $\mathfrak{G}_1$, into the domain $\mathfrak{G}_2$ having with $\mathfrak{G}_1$ the overlapping portion $\mathfrak{G}_{12}$ will be called the *direct analytic continuation of the function $f_1(z)$*.

# CHAPTER 4

# THE LAURENT SERIES AND ISOLATED SINGULAR POINTS

In this chapter we will study the behaviour of a single-valued analytic function in the neighbourhood of its isolated singular points. A knowledge of this behaviour not only permits penetrating more deeply into the nature of analytic functions, but also finds direct practical utilization in numerous applications in the theory of functions of a complex variable.

In earlier chapters we saw the great role played by power series, in particular, the Taylor series in studying the properties of analytic functions in a domain where there are no singular points of the functions under study. An analogous role, in the study of the properties of analytic functions in the neighbourhood of their isolated singular points, is played by the Laurent series.

## 4.1. The Laurent Series

### *a. The domain of convergence of a Laurent series*

Consider a series of the form

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \tag{4-1}$$

where $z_0$ is a fixed point in the complex plane, $c_n$ are certain complex numbers, and the summation is over both positive and negative values of the index $n$. The series (4-1) is called the *Laurent series.* Let us determine its domain of convergence. To do this, represent (4-1) as

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n = \sum_{n=0}^{\infty} c_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n} \tag{4-2}$$

It is obvious that the domain of convergence of (4-1) is the common part of the domains of convergence of each of the terms of the right side of (4-2). The domain of convergence of the series $\sum_{n=0}^{\infty} c_n (z-z_0)^n$ is a circle of a certain radius $R_1$ centred at $z_0$ (as was established in Chapter 2, the value of $R_1$ may, as a particular case, be zero or infinite). Inside the circle of convergence this series converges to a certain analytic function of a complex variable:

$$f_1(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n, \quad |z - z_0| < R_1 \tag{4-3}$$

To determine the domain of convergence of the series $\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}$ make the change of variable $\zeta = \frac{1}{z - z_0}$. This series will then become $\sum_{n=1}^{\infty} c_{-n}\zeta^n$ which is an ordinary power series convergent, within its circle of convergence, to some analytic function $\varphi(\zeta)$ of the complex variable $\zeta$. Denote the radius of convergence of the resulting power series by $\frac{1}{R^2}$. Then

$$\varphi(\zeta) = \sum_{n=1}^{\infty} c_{-n}\zeta^n, \quad |\zeta| < \frac{1}{R_2} \tag{4-4}$$

Returning to the earlier variable and putting $\varphi(\zeta(z)) = f_2(z)$, we get

$$f_2(z) = \sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}, \quad |z - z_0| > R_2 \tag{4-5}$$

This implies that the domain of convergence of the series $\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}$ in negative powers of the difference $(z - z_0)$ is the domain exterior to the circumference $|z - z_0| = R_2$ (the value of $R_2$, like that of $R_1$, may, in a particular case, be zero or infinite).

So each of the power series of the right side of (4-2) converges in its domain of convergence to an appropriate analytic function. If $R_2 < R_1$, then there is a common domain of convergence of these series—the *annulus* $R_2 < |z - z_0| < R_1$ in which the series (4-1) converges to the analytic function

$$f(z) = f_1(z) + f_2(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n, \quad R_2 < |z - z_0| < R_1 \tag{4-6}$$

Since the series (4-3) and (4-4) are ordinary power series, it follows that in this domain the function $f(z)$ has all the properties of the sum of a power series. This means that *the Laurent series* (4-1) *converges inside its annulus of convergence to some function* $f(z)$, *which is analytic in that annulus.*

If $R_2 > R_1$, the series (4-3) and (4-5) do not have a common domain of convergence, so in this case the series (4-1) does not converge anywhere to any function.

### b. Expansion of an analytic function in a Laurent series

The natural question arises: Is it possible to associate a function, which is analytic in some annular region, with a Laurent series convergent to the function in the given annulus? The answer is found in the following theorem.

***Theorem 4.1.*** *The function $f(z)$, analytic in the annulus $R_2 < |z - z_0| < R_1$, is uniquely represented in the annulus by a convergent Laurent series.*

*Proof.* Fix an arbitrary point $z$ inside an annulus $R_2 < |z - z_0| < R_1$ and construct circles $C_{R_1'}$ and $C_{R_2'}$ centred at $z_0$ and of radii

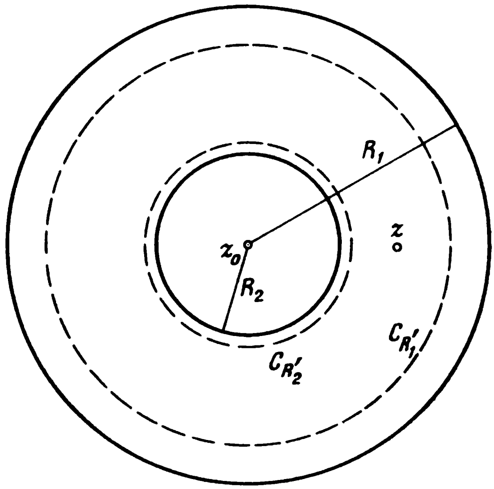


**Fig. 4.1**

which satisfy the conditions $R_2 < R_2' < R_1' < R_1$, $R_2' < |z-z_0| < R_1'$ (Fig. 4.1). According to Cauchy's formula for a multiply connected domain, we have the relation

$$f(z) = \frac{1}{2\pi i} \int\limits_{C_{R_1'}} \frac{f(\zeta)}{\zeta - z} d\zeta + \frac{1}{2\pi i} \int\limits_{C_{\overline{R}_2'}} \frac{f(\zeta)}{\zeta - z} d\zeta \qquad (4\text{-}7)$$

The inequality $\left|\frac{z-z_0}{\zeta - z_0}\right| \leqslant q < 1$ holds true on $C_{R_1'}$. And so, representing the fraction $\frac{1}{\zeta - z}$ in the form

$$\frac{1}{\zeta - z} = \frac{1}{(\zeta - z_0) - (z - z_0)} = \frac{1}{\zeta - z_0} \cdot \frac{1}{1 - \frac{z - z_0}{\zeta - z_0}} = \frac{1}{\zeta - z_0} \sum_{n=0}^{\infty} \left(\frac{z - z_0}{\zeta - z_0}\right)^n$$

and performing term-by-term integration, which is possible by virtue of the uniform convergence of the series in variable $\zeta$ (for details see Chapter 2), we get

$$f_1(z)=\frac{1}{2\pi i}\int\limits_{C_{R_1'}}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\sum_{n=0}^{\infty}c_n(z-z_0)^n \tag{4-8}$$

where

$$c_n=\frac{1}{2\pi i}\int\limits_{C_{R_1'}}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta,\quad n\geqslant 0 \tag{4-9}$$

Since the inequality $\left|\frac{\zeta-z_0}{z-z_0}\right|<1$ holds on $C_{R_2'}$, we again have

$$\frac{1}{\zeta-z}=-\frac{1}{z-z_0}\sum_{n=0}^{\infty}\left(\frac{\zeta-z_0}{z-z_0}\right)^n$$

Term-by-term integration of this series yields

$$f_2(z)=\frac{1}{2\pi i}\int\limits_{C_{\overline{R_2'}}}\frac{f(\zeta)}{\zeta-z}\,d\zeta=\sum_{n=1}^{\infty}\frac{c_{-n}}{(z-z_0)^n} \tag{4-10}$$

where

$$c_{-n}=-\frac{1}{2\pi i}\int\limits_{C_{\overline{R_2'}}}f(\zeta)(\zeta-z_0)^{n-1}\,d\zeta \tag{4-11}$$

Reversing the direction of integration in (4-11), rewrite the expression in the form

$$c_{-n}=\frac{1}{2\pi i}\int\limits_{C_{R_2'}}\frac{f(\zeta)}{(\zeta-z_0)^{-n+1}}\,d\zeta,\quad n>0 \tag{4-12}$$

Note that the integrand functions in (4-9) and (4-12) are analytic in the annulus $R_2<|z-z_0|<R_1$. And so by virtue of Cauchy's theorem, the values of the corresponding integrals will not change under an arbitrary deformation of the contours of integration in the domain of analyticity of the integrand functions. This permits us to combine formulas (4-9) and (4-12)

$$c_n=\frac{1}{2\pi i}\int\limits_{C}\frac{f(\zeta)}{(\zeta-z_0)^{n+1}}\,d\zeta,\quad n=0,\ \pm 1,\ \pm 2,\ \ldots \tag{4-13}$$

where $C$ is an arbitrary closed contour lying in the annulus $R_2<|z-z_0|<R_1$ and containing the point $z_0$ inside. Returning

to (4-7), we obtain

$$f(z)=\sum_{n=0}^{\infty} c_n (z-z_0)^n+\sum_{n=1}^{\infty} \frac{c_{-n}}{(z-z_0)^n}=\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n \quad (4\text{-}14)$$

where the coefficients $c_n$ for all values of the index $n$ are determined by the uniform formula (4-13). Since $z$ is an arbitrary point inside the annulus $R_2 < |z - z_0| < R_1$, it follows that the series (4-14) converges to the function $f(z)$ everywhere inside the annulus; and in the closed annulus $R_2 < \overline{R}'_2 \leqslant |z - z_0| \leqslant \overline{R}'_1 < R_1$ the series converges to the function $f(z)$ uniformly. It remains to prove the uniqueness of the expansion (4-14). Assume that we have another expansion:

$$f(z)=\sum_{n=-\infty}^{\infty} c'_n (z-z_0)^n$$

where at least one coefficient $c'_n \neq c_n$. Then everywhere inside the annulus $R_2 < |z - z_0| < R_1$ we have the equality

$$\sum_{n=-\infty}^{\infty} c_n (z-z_0)^n=\sum_{n=-\infty}^{\infty} c'_n (z-z_0)^n \quad (4\text{-}15)$$

Draw a circle $C_R$ of radius $R$, $R_2 < R < R_1$, centred at the point $z_0$. The series (4-15) converge on $C_R$ uniformly. Multiply them by $(z - z_0)^{-m-1}$, where $m$ is a fixed integer, and integrate termwise. Consider $\int\limits_{C_R} (z - z_0)^{n-m-1}\,dz$. Putting $z - z_0 = Re^{i\varphi}$, we have

$$\int\limits_{C_R} (z-z_0)^{n-m-1}\,dz=R^{n-m} i\int\limits_0^{2\pi} e^{i(n-m)\varphi}\,d\varphi=\begin{cases} 0, & n\neq m \\ 2\pi i, & n=m \end{cases} \quad (4\text{-}16)$$

Taking into account (4-16), we find that after the indicated integration of the expression (4-15), the infinite sums in the left and right members of this expression will have only one term each that is different from zero. And so we get $c_m = c'_m$. Since $m$ is an arbitrary number, this proves the uniqueness of the expansion (4-14). The theorem is completely proved.

From the results obtained, it follows that the annular region $R_2 < |z - z_0| < R_1$, on the boundaries of which there is at least one singular point (singularity) of the analytic function $f(z)$ to which the series (4-1) converges, is the exact domain of convergence of the Laurent series (4-1). This assertion is a corollary to Theorem 3.3.

## 4.2. A Classification of the Isolated Singular Points of a Single-Valued Analytic Function

*A point* $z_0$ *is an isolated singular point of a function* $f(z)$ *if* $f(z)$ *is single-valued and analytic in the annulus* $0 < |z - z_0| < R_1$ *and the point* $z_0$ *is a singular point of the function* $f(z)$. The function $f(z)$ may not be defined at the point $z_0$ itself. Let us study the behaviour of $f(z)$ in the neighbourhood of $z_0$. According to the preceding section, the function $f(z)$ in the neighbourhood of the point $z_0$ may be expanded in the Laurent series (4-14), which is convergent in the annulus $0 < |z - z_0| < R_1$. Three different cases are then possible:

(1) The resulting Laurent series does not contain terms involving negative powers of the difference $(z - z_0)$.

(2) The Laurent series contains a finite number of terms with negative powers of the difference $(z - z_0)$.

(3) The Laurent series contains an infinite number of terms containing negative powers of the difference $(z - z_0)$.

The foregoing serves as a basis for classifying isolated singular points. Let us examine each one of the above cases in detail.

(1) The Laurent series of the function $f(z)$ in the neighbourhood of its isolated singular point $z_0$ does not involve terms with negative powers of the difference $(z - z_0)$, i.e., $f(z) = \sum_{n=0}^{\infty} c_n (z - z_0)^n$. It is readily seen that as $z \to z_0$ there is a limiting value of the function $f(z)$, and it is equal to $c_0$. If $f(z)$ was not defined at $z_0$, then we redefine it putting $f(z_0) = c_0$. If the originally specified value of $f(z_0)$ does not coincide with $c_0$, we change the value of the function $f(z)$ at the point $z_0$, putting $f(z_0) = c_0$. The function $f(z)$ thus defined will be analytic everywhere inside the circle $|z - z_0| < R_1$. We have thus removed the discontinuity of the function $f(z)$ at the point $z_0$. Therefore, an isolated singularity $z_0$ of $f(z)$ for which an expansion of $f(z)$ in a Laurent series about $z_0$ does not contain terms with negative powers of the difference $(z - z_0)$ is called a *removable singularity*.

The foregoing proves the following theorem.

**Theorem 4.2.** *If a point* $z_0$ *is a removable singularity of an analytic function* $f(z)$, *then there exists a limiting value* $\lim_{z \to z_0} f(z) = c_0$, *where* $|c_0| < \infty$.

Note that in the neighbourhood of a removable singularity the function $f(z)$ is bounded and can be represented in the form

$$f(z) = (z - z_0)^m \varphi(z) \tag{4-17}$$

where $m \geqslant 0$ is an integer and $\varphi(z_0) \neq 0$. Here, if $\lim_{z \to z_0} f(z) = 0$, then in the representation (4-17) the number $m > 0$ determines the order of the zero of the function $f(z)$ at the point $z_0$.

The converse also holds true. We will prove it in a stronger formulation.

***Theorem 4.3.*** *If a function $f(z)$ which is analytic in the annulus $0 < |z - z_0| < R_1$ is bounded ($|f(z)| < M$ for $0 < |z - z_0| < R_1$), then the point $z_0$ is a removable singularity of $f(z)$.*

*Proof.* Expand the function $f(z)$ in the Laurent series (4-14) and consider the expression (4-13) for the coefficients of the series:

$$c_n = \frac{1}{2\pi i} \int\limits_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta$$

For the contour of integration take a circle of radius $\rho$ centred at the point $z_0$. Then, by hypothesis, we have the upper evaluation

$$|c_n| < M\rho^{-n} \tag{4-18}$$

We will consider coefficients with negative index $n < 0$. Since the value of the coefficients $c_n$ is not dependent on $\rho$, from (4-18) we get $c_n = 0$ for $n < 0$, which proves the theorem.

(2) The Laurent series of the function $f(z)$ about its isolated singularity $z_0$ contains a finite number $m$ of terms involving negative powers of $(z - z_0)$, that is $f(z) = \sum\limits_{n=-m}^{\infty} c_n (z - z_0)^n$. In this case, the point $z_0$ is called a *pole of order m of the function $f(z)$*. The behaviour of an analytic function in the neighbourhood of its pole is determined by the following theorem.

***Theorem 4.4.*** *If a point $z_0$ is the pole of an analytic function $f(z)$, then as $z \to z_0$ the absolute value of the function $f(z)$ increases without bound no matter how $z$ approaches $z_0$.*

*Proof.* Represent the function $f(z)$ about the point $z_0$ as

$$\begin{aligned} f(z) &= \frac{c_{-m}}{(z - z_0)^m} + \ldots + \frac{c_{-1}}{z - z_0} + \sum_{n=0}^{\infty} c_n (z - z_0)^n \\ &= (z - z_0)^{-m} \{c_{-m} + c_{-m+1}(z - z_0) + \ldots + c_{-1}(z - z_0)^{m-1}\} \\ &\quad + \sum_{n=0}^{\infty} c_n (z - z_0)^n = (z - z_0)^{-m} \varphi(z) + \sum_{n=0}^{\infty} c_n (z - z_0)^n. \end{aligned} \tag{4-19}$$

The function $\varphi(z)$ is obviously a bounded analytic function about the point $z_0$. From the representation (4-19) it follows that, as $z \to z_0$, the absolute value of the function $f(z)$ increases without bound irrespective of the approach path of $z$ to $z_0$, which completes the proof. Note that if we redefine the function $\varphi(z)$ at the point

$z_0$, putting $\varphi(z_0) = c_{-m} \neq 0$, then formula (4-19) may be rewritten as

$$f(z) = \frac{\psi(z)}{(z - z_0)^m} \tag{4-20}$$

where $\psi(z)$ is an analytic function and $\psi(z_0) \neq 0$; the number $m$ is called the order of the pole.

The converse of Theorem 4.4 is also valid.

***Theorem 4.5.*** *If a function $f(z)$ which is analytic about its isolated singularity $z_0$ increases indefinitely in absolute value for any approach path of $z$ to $z_0$, then the point $z_0$ is a pole of the function $f(z)$.*

*Proof.* It is obvious that, by hypothesis, for any number $A > 0$ there is an ε-neighbourhood of $z_0$ such that in it $|f(z)| > A$. Consider the function $g(z) = \frac{1}{f(z)}$. In the indicated ε-neighbourhood of $z_0$ this function is analytic and bounded and $\lim_{z \to z_0} g(z) = 0$. Therefore, on the basis of Theorem 4.3, the point $z_0$ is a removable singularity of the function $g(z)$, and $g(z)$, by virtue of formula (4-17), can be represented in the neighbourhood of $z_0$ as $g(z) = (z-z_0)^m \varphi(z)$, where $\varphi(z)$ is analytic; $\varphi(z_0) \neq 0$ and $m > 0$. Then for the original function $f(z)$ we have the representation $f(z) = \frac{1}{g(z)} = \frac{1}{(z-z_0)^m} \times \frac{1}{\varphi(z)}$ in the neighbourhood of $z_0$. Because $\varphi(z_0) \neq 0$, it can be rewritten in the form $f(z) = \frac{\psi(z)}{(z - z_0)^m}$ which coincides with the representation (4-20), where $\psi(z)$ is an analytic function. Whence it follows that the point $z_0$ is a pole of order $m$ of the function $f(z)$. The theorem is proved.

Observe that the po nt $z_0$, which is a zero of order $m$ of the analytic function $g(z)$, is a pole of the same order $m$ of the function $f(z) = \frac{1}{g(z)}$, and vice versa. This establishes a very simple relationship between the zeros and poles of analytic functions.

(3) The Laurent series of the function $f(z)$ has in the neighbourhood of its isolated singularity $z_0$ an infinite number of terms involving negative powers of the difference $(z-z_0)$, i.e. $f(z) = \sum_{n=-\infty}^{\infty} c_n(z-z_0)^n$. In this case, the point $z_0$ is called an *essential singularity of the function $f(z)$*. The behaviour of an analytic function in the neighbourhood of its essential singularity is described by the following theorem.

***Theorem 4.6 (Theorem of Sokhotsky and Weierstrass).*** *No matter what $\varepsilon > 0$, in any neighbourhood of an essential singularity $z_0$ of the function $f(z)$ there will be at least one point $z_1$ at which the value of the function $f(z)$ differs from an arbitrarily specified complex number $B$ by less than ε.*

*Proof*. Suppose that the theorem is not true; that is, for a given complex number $B$ and a specified $\varepsilon > 0$, there is an $\eta_0 > 0$ such that at all points $z$ of the $\eta_0$-neighbourhood of the point $z_0$ the value of the function $f(z)$ differs from the given $B$ by more than $\varepsilon$:

$$|f(z) - B| > \varepsilon, \quad |z - z_0| < \eta_0 \tag{4-21}$$

Consider an auxiliary function $\psi(z) = \frac{1}{f(z) - B}$. By virtue of (4-21) the function $\psi(z)$ is defined and bounded in the $\eta_0$-neighbourhood of the point $z_0$. Hence, by Theorem 4.3, $z_0$ is a removable singularity of the function $\psi(z)$. This implies that the expansion of the function $\psi(z)$ about the point $z_0$ is of the form

$$\psi(z) = (z - z_0)^m \tilde{\varphi}(z), \quad \tilde{\varphi}(z_0) \neq 0$$

Then, by the definition of the function $\psi(z)$, the following expansion of $f(z)$ is valid in the given neighbourhood of $z_0$:

$$f(z) = (z - z_0)^{-m} \varphi(z) + B \tag{4-22}$$

where the analytic function $\varphi(z) = \frac{1}{\tilde{\varphi}(z)}$ is bounded in the $\eta_0$-neighbourhood of $z_0$. But the expansion (4-22) signifies that the point $z_0$ is either a pole of order $m$, or, for $m = 0$, a regular point of the function $f(z)$, and the Laurent-series expansion of the latter must have only a finite number of terms, which contradicts the statement of the theorem. This contradiction proves the theorem.

Theorem 4.6 describes the behaviour of an analytic function in the neighbourhood $|z - z_0| < \eta_0$ of an essential singularity as follows: at an essential singularity $z_0$ there does not exist a finite or infinite limiting value of the analytic function. Depending on the choice of a sequence of points converging to the point $z_0$, we can obtain sequences of the values of the function that are convergent to different limits. It is always possible to choose a sequence that is convergent to any preassigned complex number, including $\infty$.

There is clearly no necessity to prove the converse of Theorem 4.6, since if, as $z \to z_0$, there does not exist either a finite or infinite limit of the function $f(z)$, then by Theorems 4.2 and 4.4 the point $z_0$ cannot be either removable or a pole.

Also note that if the point $z_0$ is an essential singularity of the function $f(z)$ with $f(z) \neq 0$ in some neighbourhood of $z_0$, then $z_0$ is an essential singularity for the function $g(z) = 1/f(z)$ as well.

The three cases that we have examined exhaust the possible types of Laurent-series expansion of an analytic function in the neighbourhood of its isolated singularity and are of decisive importance for investigating the general course of variation of an analytic function in the neighbourhood of its singular points.

From the foregoing it follows that there are two possible different viewpoints (each leading to the same results) concerning the classification of isolated singular points of a single-valued analytic function. We proceeded from the analytic point of view based on the nature of the Laurent-series expansion of the function and established the behaviour of the function itself as it approached a singular point. A different, geometric, approach is possible in which the classification is based on the behaviour of the function in the neighbourhood of an isolated singularity. Here, if the function is bounded in the neighbourhood of a singularity, then this point is termed removable and, as follows from Theorem 4.3, the Laurent-series expansion of the given function in the neighbourhood of that singular point does not involve negative powers. If in the approach to a singularity the function has an infinite limit, then the point is a pole and the Laurent-series expansion has a finite number of negative powers. Finally, if the function, in its approach to a singular point, does not have a finite or an infinite limit, then we have an essential singularity and the Laurent-series expansion contains an infinite number of negative powers.

To conclude this section, let us investigate the behaviour of an analytic function in the neighbourhood of the point at infinity. *The point at infinity of a complex plane is an isolated singular point of a single-valued analytic function $f(z)$ if a value of $R$ is indicated such that outside the circle $|z| > R$ the function $f(z)$ does not have any singularities at a finite distance from the point $z = 0$.* Since $f(z)$ is an analytic function in the annulus $R < |z| < \infty$, it may be expanded in the Laurent series

$$f(z) = \sum_{n=-\infty}^{\infty} c_n z^n, \quad R < |z| < \infty \tag{4-23}$$

convergent to $f(z)$ in the given annulus. As in the case of the finite isolated singularity $z_0$, there are three possible cases:

(1) The point $z = \infty$ is called a *removable singularity* of the function $f(z)$ if the expansion (4-23) does not have terms involving positive powers of $z$, i.e. $f(z) = \sum_{n=0}^{\infty} \frac{c_{-n}}{z^n} = c_0 + \sum_{n=1}^{\infty} \frac{c_{-n}}{z^n}$ or if as $z \to \infty$ there exists a finite limiting value of the function $f(z)$ that is independent of the approach path to the limit. If $c_0 = c_{-1} = \ldots = c_{-m+1} = 0$, $c_{-m} \neq 0$, then the point at infinity is a zero of order $m$ of the function $f(z)$.

(2) The point $z = \infty$ is a *pole of order m* of the function $f(z)$ if the expansion (4-23) contains a finite number $m$ of terms involving positive powers of $z$, that is, $f(z) = \sum_{n=-\infty}^{m} c_n z^n$ $(m > 0)$ or if the func-

tion increases indefinitely in absolute value as $z \to \infty$ irrespective of the kind of limit process.

(3) The point $z = \infty$ is called an *essential singularity* of the function $f(z)$ if the expansion (4-23) contains an infinite number of terms involving positive powers of $z$, i.e. $f(z) = \sum_{n=-\infty}^{\infty} c_n z^n$ or if, depending upon the choice of the sequence $\{z_n\} \to \infty$, it is possible to obtain a sequence of values $\{f(z_n)\}$ convergent to any prescribed limit.

It is obvious that the equivalence of all the foregoing definitions of the nature of the isolated singularity $z = \infty$ may be proved in the same way as for the case of a finite isolated singularity. Besides, as is readily seen, the transformation $z = \frac{1}{\zeta}$ carries the point $\infty$ of the $z$-plane to the point $\zeta = 0$; the character of the singularity does not change in this transformation by virtue of the following general theorem.

***Theorem 4.7.*** *Let the point $z_0$ be an isolated singular point of the function $f(z)$ analytic in the domain $\mathfrak{G}$. Let the analytic function $\zeta = \psi(z)$ establish a one-to-one correspondence between the domain $\mathfrak{G}$ and the domain $\mathfrak{G}'$ of the complex $\zeta$-plane in which the inverse function $z = \varphi(\zeta)$ is defined. Then the point $\zeta_0 = \psi(z_0)$ is an isolated singular point of the analytic function $F(\zeta) = f[\varphi(\zeta)]$, and the character of this singular point is the same as that of the point $z_0$.*

This theorem is an obvious consequence of the property of analytic functions that was established in Chapter 1, by virtue of which the analytic function of an analytic function is analytic, and also of the geometric properties of an analytic function in the neighbourhood of an isolated singular point.

*Example.* Consider the function $f(z) = \frac{1}{\sqrt{1+z^2}}$. This multiple-valued function has two branch points $z = \pm i$. The point $z = \infty$ is its regular point. Therefore, in the annulus $1 < |z| < \infty$ are defined two branches of the function; they are single-valued analytic functions in the given annulus. Choose the branch which is a direct analytic continuation of the real function $\frac{1}{\sqrt{1+x^2}}$ of the real variable $x > 1$, and construct its Laurent-series expansion about the point $z = \infty$. To do this, put $\zeta = \frac{1}{z}$ and map the given annulus onto a circle of unit radius in the $\zeta$-plane (then the point $z = \infty$ goes into the point $\zeta = 0$) and expand the function $\varphi(\zeta) = \frac{1}{\sqrt{1+\frac{1}{\zeta^2}}} = \frac{\zeta}{\sqrt{1+\zeta^2}}$ in a Taylor series in the neighbourhood of its regular point $\zeta = 0$. First note that the function $\varphi(\zeta)$ is a derivative of the function $\psi(\zeta) = \sqrt{1+\zeta^2}$. (Here our choice of the branch of the original function $f(z)$ determines the choice of that branch of the function $\psi(\zeta)$ for which $\psi(0) = +1$.) To expand the function $\psi(\zeta)$ in a Taylor series, put $w = \zeta^2$ and consider the function $\chi(w) = \sqrt{1+w}$. Computing the derivatives of the function $\chi(w)$, we get

$$\chi^{(n)}(w)\big|_{w=0} = \frac{1}{2}\left(\frac{1}{2}-1\right)\dots\left(\frac{1}{2}-n+1\right)(1+w)^{\frac{1}{2}-n}\Big|_{w=0} = (-1)^{n-1}\frac{(2n-2)!}{2^{2n-1}(n-1)!}$$

Then the expansion of the chosen branch of the function $\chi(w)$ in the annulus $|w| < 1$ is of the form

$$\chi(w) = \sqrt{1+w} = 1 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-2)!\,w^n}{2^{2n-1}(n-1)!\,n!}$$

Whence, for the function $\psi(\zeta)$ for $|\zeta| < 1$, we get

$$\psi(\zeta) = \sqrt{1+\zeta^2} = 1 + \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-2)!\,\zeta^{2n}}{2^{2n-1}(n-1)!\,n!}$$

and for the function $\varphi(\zeta)$

$$\varphi(\zeta) = \psi'(\zeta) = \frac{\zeta}{\sqrt{1+\zeta^2}} = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-2)!\,2n}{2^{2n-1}(n-1)!\,n!}\zeta^{2n-1}$$

$$= \sum_{n=1}^{\infty}(-1)^{n-1}\frac{(2n-2)!}{2^{2n-2}[(n-1)!]^2}\zeta^{2n-1} = \sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{2^{2k}(k!)^2}\zeta^{2k+1}$$

Finally, for the chosen branch of the function $f(z)$, in the annulus $1 < |z| < \infty$ we get the Laurent-series expansion

$$f(z) = \frac{1}{\sqrt{1+z^2}} = \sum_{k=0}^{\infty}(-1)^k\frac{(2k)!}{2^{2k}(k!)^2}\cdot\frac{1}{z^{2k+1}} \qquad (4\text{-}24)$$

# CHAPTER 5

# RESIDUES AND THEIR APPLICATIONS

## 5.1. The Residue of an Analytic Function at an Isolated Singularity

### *a. Definition of a residue. Formulas for evaluating residues*

We introduce the concept of the residue of a single-valued analytic function at an isolated singularity. It will be found to have extensive application.

Let a point $z_0$ be an isolated singularity of a single-valued analytic function $f(z)$. According to earlier investigations, in the neighbourhood of this point, $f(z)$ is capable of a unique expansion in a Laurent series:

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \tag{5-1}$$

where

$$c_n = \frac{1}{2\pi i} \int_C \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} d\zeta \tag{5-2}$$

and, in particular,

$$c_{-1} = \frac{1}{2\pi i} \int_C f(\zeta)\, d\zeta \tag{5-3}$$

*The residue of an analytic function $f(z)$ at an isolated singularity $z_0$ is a complex number equal to the value of the integral* $\frac{1}{2\pi i} \int_\gamma f(\zeta)\, d\zeta$ *taken in the positive direction around any closed contour $\gamma$ lying in the domain of analyticity of the function $f(z)$ and containing a unique singularity $z_0$ of the function $f(z)$.* For a residue we will use the designation $\text{Res}\,[f(z), z_0]$. It is clear that if the point $z_0$ is a regular point or a removable singularity of the function $f(z)$, the residue of $f(z)$ at this point is zero. To evaluate the residue of the function

$f(z)$ at an isolated singularity, we can use formula (5-3):

$$\operatorname{Res}[f(z), z_0] = \frac{1}{2\pi i}\int_C f(\zeta)\,d\zeta = c_{-1} \tag{5-4}$$

However, in a number of cases, a simpler method of computing a residue is possible; it reduces to differentiating the function $f(z)$ in the neighbourhood of the point $z_0$. Thus, the evaluation of the contour integral of an analytic function may be replaced by computing the derivatives of this function at certain points lying inside the contour of integration. This circumstance determines one of the basic applications of the calculus of residues. Let us examine such cases.

(1) Let the point $z_0$ be a first-order pole of the function $f(z)$. Then in the neighbourhood of this point we have the expansion

$$f(z) = c_{-1}(z-z_0)^{-1} + c_0 + c_1(z-z_0) + \ldots \tag{5-5}$$

Multiplying both sides of (5-5) by $(z - z_0)$ and passing to the limit as $z \to z_0$, we get

$$c_{-1} = \lim_{z\to z_0}(z-z_0)f(z) \tag{5-6}$$

Note that in the given case the function $f(z)$ may in the neighbourhood of $z_0$ be represented in the form of a ratio of two analytic functions:

$$f(z) = \frac{\varphi(z)}{\psi(z)} \tag{5-7}$$

$\varphi(z_0) \neq 0$ and the point $z_0$ is a first-order zero of the function $\psi(z)$, that is,

$$\psi(z) = (z-z_0)\psi'(z_0) + \frac{\psi''(z_0)}{2}(z-z_0)^2 + \ldots, \quad \psi'(z_0) \neq 0 \tag{5-8}$$

Then from (5-6) (5-8) we get the following formula.

*A formula for computing the residue at a first-order pole*:

$$\operatorname{Res}[f(z), z_0] = \frac{\varphi(z_0)}{\psi'(z_0)} \quad \left(f(z) = \frac{\varphi(z)}{\psi(z)}\right) \tag{5-9}$$

*Example 1*. Let $f(z) = \frac{z}{z^n - 1}$. The function $f(z)$ has the singular points $z_k = \sqrt[n]{1} = e^{i\frac{2\pi k}{n}}$ $(k = 0, 1, \ldots, n-1)$, and all these points are first-order poles. Let us find $\operatorname{Res}[f(z), z_k]$. According to formula (5-9), we get

$$\operatorname{Res}[f(z), z_k] = \frac{z_k}{nz_k^{n-1}} = \frac{1}{n}\cdot z_k^2 = \frac{1}{n}e^{i\frac{4\pi k}{n}} \quad (z_k^n = 1) \tag{5-10}$$

(2) Let the point $z_0$ be a pole of order $m$ of the function $f(z)$. From the foregoing, in the neighbourhood of this point we have the expansion

$$f(z) = c_{-m}(z - z_0)^{-m} + \ldots + c_{-1}(z - z_0)^{-1} + c_0 + c_1(z - z_0) + \ldots \quad (5\text{-}11)$$

Multiplying both sides of (5-11) by $(z - z_0)^m$, we get

$$(z - z_0)^m f(z) = c_{-m} + c_{-m+1}(z - z_0) + \ldots + c_{-1}(z - z_0)^{m-1} + \ldots \quad (5\text{-}12)$$

Taking the derivative of order $(m - 1)$ of both parts of this equality and passing to the limit as $z \to z_0$, we finally get the following formula.

*A formula for evaluating the residue at a pole of order m:*

$$\operatorname{Res}[f(z), z_0] = \frac{1}{(m-1)!} \lim_{z \to z_0} \frac{d^{m-1}}{dz^{m-1}} [(z - z_0)^m f(z)] \quad (5\text{-}13)$$

It is easy to see that formula (5-6) is a special case of this formula.

*Example 2.* Let $f(z) = \frac{1}{(1+z^2)^n}$. This function has the singular points $z_{1,2} = \pm i$; both of these points are poles of order $n$. Compute $\operatorname{Res}[f(z), i]$. According to (5-13) we have

$$\operatorname{Res}\left[\frac{1}{(1+z^2)^n}, i\right] = \frac{1}{(n-1)!} \lim_{z \to i} \frac{d^{n-1}}{dz^{n-1}} \left[(z - i)^n \frac{1}{(1+z^2)^n}\right]$$

$$= \frac{1}{(n-1)!} \lim_{z \to i} \frac{d^{n-1}}{dz^{n-1}} \left[\frac{1}{(z+i)^n}\right]$$

$$= (-1)^{n-1} \frac{n \cdot (n+1) \ldots (2n-2)}{(n-1)!} \cdot \frac{1}{(z+i)^{2n-1}}\bigg|_{z=i}$$

$$= (-1)^{n-1} \frac{(2n-2)!}{[(n-1)!]^2} \cdot \frac{1}{(2i)^{2n-1}} = -i \frac{(2n-2)!}{2^{2n-1}[(n-1)!]^2} \quad (5\text{-}14)$$

### b. The residue theorem

We now investigate the more important applications of the concepts we have introduced. The following theorem is very essential in numerous theoretical investigations and practical applications.

***Theorem 5.1 (Residue theorem).*** *Let the function $f(z)$ be analytic everywhere in a closed domain $\bar{\mathfrak{G}}$, except at a finite number of isolated singularities $z_k$ ($k = 1, \ldots, N$) lying inside the domain $\mathfrak{G}$. Then*

$$\int_{\Gamma+} f(\zeta)\, d\zeta = 2\pi i \sum_{k=1}^{N} \operatorname{Res}[f(z), z_k] \quad (5\text{-}15)$$

*where* $\Gamma^+$ *is the complete boundary of the domain* $\mathfrak{G}$ *traversed in the positive direction.*

*Proof.* Recall that if a function $f(z)$ is analytic in a closed domain $\overline{\mathfrak{G}}$, then all points of the boundary $\Gamma$ of that domain are regular points of $f(z)$. Isolate each of the singularities $z_k$ of the function $f(z)$ by a closed contour $\gamma_k$ not containing other singularities, except

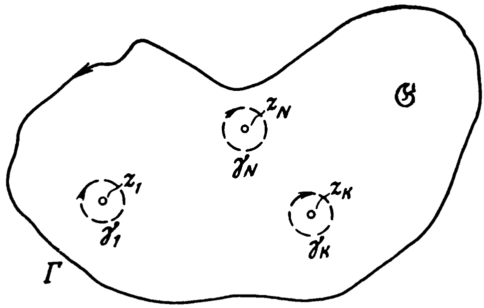


Fig. 5.1

the point $z_k$. Consider a closed multiply connected domain bounded by the contour $\Gamma$ and all the contours $\gamma_k$ (Fig. 5.1). The function $f(z)$ is analytic everywhere inside this domain. Therefore, by Cauchy's second theorem, we have

$$\int_{\Gamma^+} f(\zeta)\,d\zeta + \sum_{k=1}^{N} \int_{\gamma_k^-} f(\zeta)\,d\zeta = 0 \tag{5-16}$$

Transporting the second term in (5-16) to the right, we obtain by formula (5-4) the assertion of the theorem

$$\int_{\Gamma^+} f(\zeta)\,d\zeta = 2\pi i \sum_{k=1}^{N} \operatorname{Res}[f(z), z_k]$$

The great practical value of this formula lies in the fact that in many cases it turns out to be much simpler to evaluate the residues of the function $f(z)$ at singularities lying inside the domain of integration than to evaluate directly the integral in the left-hand member of (5-15). Later on we will consider a number of important applications of this formula. Let us now introduce one more concept—the residue at the point at infinity.

Let the point $z = \infty$ be an isolated singularity of the analytic function $f(z)$.

*The residue of the analytic function $f(z)$ at the point $z = \infty$ is a complex number equal to the value of the integral*

$$\frac{1}{2\pi i}\int_{C-} f(\zeta)\, d\zeta = -\frac{1}{2\pi i}\int_{C+} f(\zeta)\, d\zeta$$

*where the contour $C$ is an arbitrary closed contour, outside of which the function $f(z)$ is analytic and does not have any singular points different from $\infty$.* Clearly, by the definition of the coefficients of a Laurent series, we have the formula

$$\operatorname{Res}[f(z), \infty] = -\frac{1}{2\pi i}\int_{C+} f(\zeta)\, d\zeta = -c_{-1} \tag{5-17}$$

From this it follows, in particular, that if the point $z = \infty$ is a removable singularity of $f(z)$, then $\operatorname{Res}[f(z), \infty]$ may prove to be nonzero, whereas the residue at a finite removable singularity is always equal to zero.

The formulas (5-15) and (5-17) permit proving the following theorem.

**Theorem 5.2.** *Let the function $f(z)$ be analytic in the extended complex plane, with the exception of a finite number of isolated singular points $z_k$ $(k = 1, 2, \ldots, N)$, including also $z = \infty$ $(z_N = \infty)$. Then*

$$\sum_{k=1}^{N} \operatorname{Res}[f(z), z_k] = 0 \tag{5-18}$$

*Proof.* Indeed, consider the closed contour $C$ containing all $(N - 1)$ singularities $z_k$ located at a finite distance from the point $z = 0$. By Theorem 5.1

$$\frac{1}{2\pi i}\int_{C+} f(\zeta)\, d\zeta = \sum_{k=1}^{N-1} \operatorname{Res}[f(z), z_k]$$

But by virtue of (5-17), the integral on the left is equal to the residue (with sign reversed) of the function $f(z)$ at the point $z = \infty$. This asserts Theorem 5.2.

This theorem occasionally permits simplifying the computation of the integral of a function of a complex variable around a closed contour. Let $f(z)$ be a single-valued analytic function in the entire complex plane, except at a finite number of isolated singularities, and let it be required to compute the integral of $f(z)$ around some closed contour $\Gamma$. If inside $\Gamma$ there are many singularities of the function $f(z)$, then application of formula (5-15) may involve arduous calculations. It may turn out that outside $\Gamma$ the function $f(z)$ has only a few singularities $z_k$ $(k = 1, 2, \ldots, m)$, the values of the residues at which, and also the residue at the point at infinity, are determined in a simple enough manner. Then in place of a straightforward evaluation of the desired integral by formula (5-15) it is more convenient to take advantage of the obvious consequence of formulas (5-15) and (5-18):

$$\int_{\Gamma^+} f(\zeta)\,d\zeta = -2\pi i \sum_{k=1}^{m} \operatorname{Res}[f(z),\ z_k] - 2\pi i \operatorname{Res}[f(z),\ \infty] \qquad (5\text{-}19)$$

The formula (5-19) permits readily obtaining a generalization of the Cauchy formula [see Section 1.6, formulas (1-59), (1-60)] to the case of an unbounded domain. Let us consider a function $f(z)$ that is analytic outside a closed contour $\Gamma$, which is the boundary of a bounded domain $\mathfrak{G}$. Let all the points of $\Gamma$ be regular points of the function $f(z)$ and let the point $z = \infty$ be its removable singularity. Denote $\lim\limits_{z\to\infty} f(z) = f(\infty)$. Exterior to $\Gamma$ we construct the function $\varphi(z) = \frac{f(z)}{z - z_0}$, where $z_0$ is an arbitrary point of the complex plane. Clearly, $z = \infty$ is a removable singularity of the function $\varphi(z)$ as well, and $\operatorname{Res}[\varphi(z), \infty] = -f(\infty)$.

If the point $z_0$ lies inside $\Gamma$, then the function $\varphi(z)$ does not have any other singular points. If the point $z_0$ lies outside $\Gamma$, then $z = z_0$ is a pole, not exceeding the first order, of the function $\varphi(z)$, and $\operatorname{Res}[\varphi(z), z_0] = f(z_0)$.

Let us consider the integral $\int_{\Gamma^+} \varphi(\zeta)\,d\zeta = \int_{\Gamma^+} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta$ in which the contour $\Gamma$ is traversed so that the domain $\mathfrak{G}$ remains on the left-hand side. By formula (5-19) we obtain

$$\frac{1}{2\pi i}\int_{\Gamma^+} \frac{f(\zeta)}{\zeta - z_0}\,d\zeta = \begin{cases} f(\infty), & z_0 \text{ inside } \Gamma \\ f(\infty) - f(z_0), & z_0 \text{ outside } \Gamma \end{cases} \qquad (5.20)$$

The formula (5-20) is precisely the generalization of Cauchy's integral formula to the case of $f(z)$ analytic in an unbounded domain.

## 5.2. Evaluation of Definite Integrals by Means of Residues

The theorems of Section 5.1 find numerous applications not only in evaluating the integrals of functions of a complex variable, but also in evaluating various definite integrals of functions of a real variable. Very often one is able to obtain an answer in cases where the use of other methods of analysis proves complicated. Let us consider a number of typical cases.

*a. Integrals of the form* $\int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta$

We consider the integral

$$I = \int_0^{2\pi} R(\cos\theta, \sin\theta)\, d\theta \tag{5-21}$$

where $R$ is a rational function of its arguments. Type (5-21) integrals can easily be reduced to integrals of an analytic function of a complex variable over a closed contour. To do this, make the change of variable of integration, introducing the complex variable $z$, which is connected with the variable $\theta$ by the relation $z = e^{i\theta}$. It is obvious that

$$d\theta = \frac{1}{i}\frac{dz}{z}, \quad \cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$= \frac{1}{2}\left(z + \frac{1}{z}\right), \quad \sin\theta = \frac{1}{2i}\left(z - \frac{1}{z}\right)$$

As $\theta$ varies between 0 and $2\pi$, the complex variable $z$ runs over the closed contour, the circle $|z| = 1$, in the positive direction. Thus, the integral (5-21) is transformed into an integral around the closed contour of a function of a complex variable:

$$I = \frac{1}{i} \int_{|z|=1} R\left[z + \frac{1}{z},\ z - \frac{1}{z}\right]\frac{dz}{z} \tag{5-22}$$

By virtue of the general properties of analytic functions, the integrand in (5-22), which is obviously a rational function,

$$\tilde{R}(z) = \frac{a_0 + a_1 z + \ldots + a_n z^n}{b_0 + b_1 z + \ldots + b_m z^m} \tag{5-23}$$

is a function analytic inside the circle $|z| = 1$ everywhere except at a finite number $N \leqslant m$ of singular points $z_k$, which are the zeros of the denominator in (5-23). Therefore, by Theorem 5.1,

$$I = 2\pi \sum_{k=1}^{N} \operatorname{Res}[\tilde{R}(z), z_k] \tag{5-24}$$

The points $z_k$ are poles of the function $\tilde{R}(z)$. Let $\alpha_k$ be the order of the pole $z_k$ (clearly, $\sum_{k=1}^{N} \alpha_k \leqslant m$). Then on the basis of formula (5-13) we can rewrite (5-24) as

$$I = 2\pi \sum_{k=1}^{N} \frac{1}{(\alpha_k - 1)!} \lim_{z \to z_k} \frac{d^{\alpha_k - 1}}{dz^{\alpha_k - 1}} [(z - z_k)^{\alpha_k} \widetilde{R}(z)] \qquad (5\text{-}25)$$

*Example 1.* Evaluate the integral

$$I = \int_0^{2\pi} \frac{d\theta}{1 + a\cos\theta}, \quad |a| < 1 \qquad (5\text{-}26)$$

Putting $z = e^{i\theta}$, we get

$$I = \frac{1}{i} \int_{|z|=1} \frac{1}{1 + \frac{a}{2}\left(z + \frac{1}{z}\right)} \cdot \frac{dz}{z} = \frac{2}{i} \int_{|z|=1} \frac{dz}{az^2 + 2z + a} \qquad (5\text{-}27)$$

The zeros of the denominator $z_{1,2} = -\frac{1}{a} \pm \sqrt{\frac{1}{a^2} - 1}$ are singularities of the integrand function. They are poles of order one. Since $z_1 \cdot z_2 = 1$, only one of these points lies inside the circle $|z| = 1$. As is readily seen, this is the point $z_1 = -\frac{1}{a} + \sqrt{\frac{1}{a^2} - 1}$. Therefore, by Theorem 5.1,

$$I = 4\pi \operatorname{Res}\left[\frac{1}{az^2 + 2z + a}, \ z_1\right] = 4\pi \frac{1}{a(z - z_2)}\bigg|_{z=z_1} = \frac{2\pi}{\sqrt{1 - a^2}} \qquad (5\text{-}28)$$

### b. Integrals of the form $\int_{-\infty}^{\infty} f(x)\,dx$

We now consider applying the calculus of residues to evaluating improper integrals of the first kind of the form $\int_{-\infty}^{\infty} f(x)\,dx$. We will consider the case when the function $f(x)$ is specified on the entire real axis and may be analytically continued into the upper half-plane so that its continuation satisfies certain supplementary conditions. These conditions will be formulated below in Theorem 5.3.

For what follows we will need some auxiliary propositions.

*Lemma 1. Let the function $f(z)$ be analytic everywhere in the upper half-plane* $\operatorname{Im} z > 0$, *with the exception of a finite number of isolated singular points, and let there exist positive numbers $R_0$, $M$ and $\delta$ such that for all points of the upper half-plane which satisfy the condition $|z| > R_0$ we have the evaluation*

$$|f(z)| < \frac{M}{|z|^{1+\delta}}, \quad |z| > R_0 \qquad (5\text{-}29)$$

*Then*

$$\lim_{R\to\infty} \int\limits_{C'_R} f(\zeta)\, d\zeta = 0 \tag{5-30}$$

*where the contour of integration $C'_R$ is a semicircle* $|z| = R$, Im $z > 0$ *in the upper half of the z-plane* (Fig. 5.2).

Indeed, by virtue of (1-41) and the conditions of the lemma, for $R > R_0$,

$$\left|\int\limits_{C'_R} f(\zeta)\, d\zeta\right| \leqslant \int\limits_{C'_R} |f(\zeta)|\, ds < \frac{M\pi R}{R^{1+\delta}} = \frac{\pi M}{R^{\delta}} \xrightarrow[R\to\infty]{} 0$$

which proves the lemma.

*Note 1.* If the conditions of the lemma are fulfilled in some sector $\varphi_1 < \arg z < \varphi_2$ of the $z$-plane, then formula (5-30) is valid in integration along the arc $C'_R$ of a circle lying in the given sector.

*Note 2.* The conditions of the lemma will obviously be fulfilled if the function $f(z)$ is analytic in the neighbourhood of the point

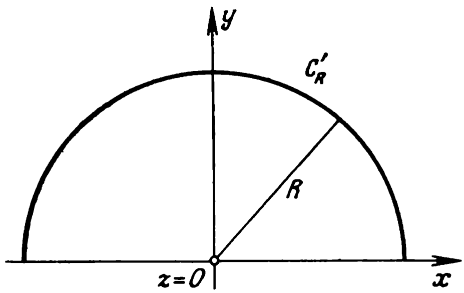


**Fig. 5.2**

at infinity and the point $z = \infty$ is a zero of order not below second of the function $f(z)$. Indeed, in this case, a Laurent-series expansion of $f(z)$ in the neighbourhood of $z = \infty$ is of the form

$$f(z) = \frac{c_{-2}}{z^2} + \frac{c_{-3}}{z^3} + \ldots = \frac{\psi(z)}{z^2}$$

and $|\psi(z)| < M$, whence we get the evaluation (5-29) for $\delta = 1$.

Lemma 1 finds broad application in computing a number of improper integrals of the form $\int\limits_{-\infty}^{\infty} f(x)\, dx$.

***Theorem 5.3.*** *Let it be possible for the function $f(x)$ specified on the entire real axis $-\infty < x < \infty$ to be analytically continued into the upper half-plane* Im $z \geqslant 0$; *its analytic continuation, the function*

*$f(z)$, satisfies the conditions of Lemma 1 and does not have singularities on the real axis. Then the improper integral of the first kind $\int_{-\infty}^{\infty} f(x)\,dx$ exists and*

$$\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{k=1}^{N} \operatorname{Res}[f(z), z_k] \tag{5-31}$$

*where $z_k$ are singularities of the function $f(z)$ in the upper half-plane.*

*Proof.* By hypothesis, the function $f(z)$ in the upper half-plane has a finite number of singularities $z_k$, and they all satisfy the condition $|z_k| < R_0$. Consider in the upper half-plane a closed contour consisting of a segment of the real axis $-R \leqslant x \leqslant R$ $(R > R_0)$ and the semicircle $C'_R$, $|z| = R$. By the residue theorem

$$\int_{-R}^{R} f(x)\,dx + \int_{C'_R} f(z)\,dz = 2\pi i \sum_{k=1}^{N} \operatorname{Res}[f(z), z_k] \tag{5-32}$$

Since the conditions of Lemma 1 are fulfilled, the limit of the second term on the left of (5-32) is zero as $R \to \infty$; the right side of (5-32) is independent of $R$ for $R > R_0$. Whence it follows that the limit of the first term exists and its value is defined by formula (5-31). The theorem is proved.

*Example 2.* Compute the integral

$$I = \int_{-\infty}^{\infty} \frac{dx}{x^4+1} \tag{5-33}$$

Analytic continuation of the integrand function into the upper half-plane, the function $f(z) = \frac{1}{z^4+1}$, obviously satisfies the conditions of Theorem 5.3. Its singular points in the upper half-plane are the points $z_{0,1} = e^{i\frac{\pi+2\pi k}{4}}$ $(k = 0, 1)$; both of these points are first-order poles. Therefore

$$I = 2\pi i \left\{ \operatorname{Res}\left[\frac{1}{1+z^4}, e^{i\frac{\pi}{4}}\right] + \operatorname{Res}\left[\frac{1}{1+z^4}, e^{i\frac{3\pi}{4}}\right] \right\}$$

$$= 2\pi i \left\{ \frac{1}{4z^3}\bigg|_{z=e^{i\frac{\pi}{4}}} + \frac{1}{4z^3}\bigg|_{z=e^{i\frac{3\pi}{4}}} \right\} = \frac{\pi\sqrt{2}}{2} \tag{5-34}$$

*Note 1*. If the function $f(x)$ is even and satisfies the conditions of Theorem 5.3, then

$$\int_0^\infty f(x)\,dx = \pi i \sum_{k=1}^N \text{Res}\,[f(z),\ z_k] \tag{5-35}$$

Indeed, if $f(x)$ is an even function, then

$$\int_0^\infty f(x)\,dx = \frac{1}{2}\int_{-\infty}^\infty f(x)\,dx$$

whence follows formula (5-35).

*Note 2*. Obviously, a similar theorem holds true also in the case when the analytic continuation of the function $f(x)$ into the lower half-plane satisfies conditions analogous to those of Lemma 1.

### c. *Integrals of the form* $\int_{-\infty}^{\infty} e^{iax} f(x)\,dx$. *Jordan's lemma*

Evaluation of the following important class of improper integrals by means of the calculus of residues is based on the use of the so-called Jordan lemma, which we will now prove.

***Lemma 2 (Jordan's lemma).*** *Let the function* $f(z)$ *be analytic in the upper half-plane* $\text{Im}\, z > 0$, *with the exception of a finite number of isolated singularities, and let it tend uniformly to zero in* $\arg z$ $(0 \leqslant \arg z \leqslant \pi)$ *as* $|z| \to \infty$. *Then for* $a > 0$

$$\lim_{R\to\infty} \int_{C'_R} e^{ia\zeta} f(\zeta)\,d\zeta = 0 \tag{5-36}$$

*where* $C'_R$ *is a semicircular arc* $|z| = R$ *in the upper half of the z-plane.*

*Proof*. The condition of uniform approach of $f(z)$ to zero implies that for $|z| = R$ we have the evaluation

$$|f(z)| < \mu_R, \quad |z| = R \tag{5-37}$$

where $\mu_R \to 0$ as $R \to \infty$. Using the relation (5-37) we evaluate the desired integral. Make a change of variable, putting $\zeta = Re^{i\varphi}$, and take advantage of the obvious relation

$$\sin\varphi \geqslant \frac{2}{\pi}\varphi \quad \text{for} \quad 0 \leqslant \varphi \leqslant \frac{\pi}{2} \tag{5-38}$$

We then obtain

$$\left|\int_{C_R'} e^{ia\zeta} f(\zeta)\, d\zeta\right| \leqslant \mu_R \cdot R \int_0^{\pi} |e^{ia\zeta}|\, d\varphi$$

$$= \mu_R \cdot R \int_0^{\pi} e^{-aR\sin\varphi}\, d\varphi = 2\mu_R \cdot R \int_0^{\pi/2} e^{-aR\sin\varphi}\, d\varphi$$

$$< 2\mu_R \cdot R \int_0^{\pi/2} e^{-\frac{2aR}{\pi}\varphi}\, d\varphi = \frac{\pi}{a}\mu_R (1 - e^{-aR}) \underset{R\to\infty}{\to} 0 \qquad (5\text{-}39)$$

which proves the lemma.

*Note 1*. If $a < 0$ and the function $f(z)$ satisfies the conditions of Jordan's lemma in the lower half-plane $\operatorname{Im} z \leqslant 0$, then formula (5-36) is valid in integration around the semicircular arc $C_R'$ in the lower half of the $z$-plane. Similar assertions hold for $a = \pm i\alpha$ $(\alpha > 0)$ as well when integrating, respectively, in the right ($\operatorname{Re} z \geqslant 0$, Fig. 5.3) or left ($\operatorname{Re} z \leqslant 0$) half of the $z$-plane. The proofs of these statements are carried out in an exactly similar manner, so we leave them to the reader. The following form of Jordan's lemma, which refers to integration in the right half-plane, will be needed in future applications:

$$\lim_{R\to\infty} \int_{C_R'} e^{-\alpha\zeta} f(\zeta)\, d\zeta = 0, \quad \alpha > 0 \qquad (5\text{-}40)$$

where $C_R'$ is the semicircular arc $|z| = R$ in the right half-plane $\operatorname{Re} z \geqslant 0$. Formula (5-40) and a number of others that follow will be extensively used in Chapter 8 for evaluating various integrals that play an important role in operational calculus.

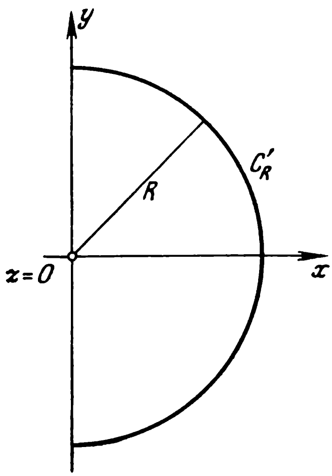

Fig. 5.3

*Note 2*. Jordan's lemma holds true for the case when the function $f(z)$ satisfies the above-formulated conditions in the half-plane $\operatorname{Im} z \geqslant y_0$ ($y_0$ is a fixed number which may be either positive or negative), and the integration is carried out along the semicircular arc $|z - iy_0| = R$ in the half-plane $\operatorname{Im} z \geqslant y_0$. The proof is similar to the previous proof. When evaluating the integral, make a change of the variable of integration: $\zeta = Re^{i\varphi} + iy_0$.

*Note 3*. Jordan's lemma holds true also for relaxed conditions imposed on the function $f(z)$. Let the function $f(z)$ in the upper half-plane $\operatorname{Im} z > y_0$ for $|z| > R_0$ tend to zero uniformly in the argument $z - iy_1$ as $|z| \to \infty$ in the sectors $-\varphi_0 \leqslant \arg(z - iy_1) \leqslant \varphi_1$, $\pi - \varphi_2 \leqslant \arg(z - iy_1) \leqslant \pi + \varphi_0$ and let it be uniformly bounded in the sector $\varphi_1 \leqslant \arg(z - iy_1) \leqslant \pi - \varphi_2$, where $\varphi_0$, $\varphi_1$ and $\varphi_2$ are specified positive numbers $0 \leqslant \varphi_0, \varphi_1, \varphi_2 \leqslant \frac{\pi}{2}$ and $y_1 > y_0$. Then the integral $\int\limits_{C_R} e^{ia\zeta} f(\zeta)\, d\zeta$ tends to zero along the arc $C_R$ of the circle $|z - iy_1| = R$, $\operatorname{Im} z \geqslant y_0$ for $a > 0$ and as $R \to \infty$.

To prove this, split the integral into the sum $I = I_1 + I_2 + I_3 + I_4 + I_5$ of integrals along the arcs $C_R^{(1)}$ $(y_1 > \operatorname{Im} z > y_0$, $\arg(z - iy_1) < 0)$, $C_R^{(2)}$ $(0 < \arg(z - iy_1) < \varphi_1)$, $C_R^{(3)}$ $(\varphi_1 < \arg(z - iy_1) < \pi - \varphi_2)$, $C_R^{(4)}$ $(\pi - \varphi_2 < \arg(z - iy_1) < \pi)$ and $C_R^{(5)}$ $(y_1 > \operatorname{Im} z > y_0$, $\arg(z - iy_1) > \pi)$ and prove the convergence, to zero, of each integral separately. For the integral $I_1$ we get $|I_1| \leqslant \mu_R e^{-ay_0} L_R^{(1)}$, where $L_R^{(1)}$ is the length of the curve $C_R^{(1)}$. As $R \to \infty$ the quantity $L_R^{(1)}$ remains bounded and tends to the value $y_1 - y_0$. Therefore $|I_1| \to 0$ as $R \to \infty$. Analogously, $I_5 \to 0$. The convergence to zero of the integrals $I_2$ and $I_4$ is established by a technique used in the proof of Jordan's lemma. For the integral $I_3$ it is easy to obtain the estimate $|I_3| < Ce^{-aR \sin \varphi^*} R (\pi - \varphi_1 - \varphi_2)$, where $|f(\zeta)| < C$ and $\varphi^* = \min\{\varphi_1, \varphi_2\}$, from which it follows that $I_3 \to 0$ as $R \to \infty$.

Thus, the Jordan lemma holds under considerably weaker restrictions imposed on the function $f(z)$ than in the case of Lemma 1. This is connected with the presence, in the integrand function, of an additional factor $e^{ia\zeta}$, which, for $a > 0$, ensures a sufficiently rapid decrease of the integrand function in the sector $0 < \varphi_1 \leqslant \arg(z - iy_1) \leqslant \pi - \varphi_2$ as $|z| \to \infty$.

Jordan's lemma finds numerous applications in the calculation of a broad class of improper integrals.

***Theorem 5.4.*** *Let it be possible for a function $f(x)$ given on the entire real axis $-\infty < x < \infty$ to be continued into the upper half-plane $\operatorname{Im} z \geqslant 0$, and let its analytic continuation $f(z)$ in the upper half-plane satisfy the conditions of Jordan's lemma and have no singularities on the real axis. Then the integral $\int\limits_{-\infty}^{\infty} e^{iax} f(x)\, dx$, $a > 0$, exists and is equal to*

$$\int\limits_{-\infty}^{\infty} e^{iax} f(x)\, dx = 2\pi i \sum_{k=1}^{n} \operatorname{Res}[e^{iaz} f(z), z_k] \tag{5-41}$$

*where $z_k$ are singularities of the function $f(z)$ in the upper half of the z-plane.*

*Proof*. By hypothesis, the singular points $z_k$ of the function $f(z)$ in the upper half-plane satisfy the condition $|z_k| < R_0$. Consider, in the upper half of the $z$-plane, a closed contour consisting of a segment of the real axis $-R \leqslant x \leqslant R$, $R > R_0$ and of the arc $C'_R$ of the semicircle $|z| = R$ in the upper half of the $z$-plane. By the residue theorem

$$\int_{-R}^{R} e^{i\alpha x} f(x)\, dx + \int_{C'_R} e^{i\alpha\zeta} f(\zeta)\, d\zeta = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\,[e^{i\alpha z} f(z),\ z_k] \quad (5\text{-}42)$$

By Jordan's lemma, the limit of the second term on the left of (5-42) is zero as $R \to \infty$. This asserts the theorem.

*Example 3*. Evaluate the integral

$$I = \int_{-\infty}^{\infty} \frac{\cos \alpha x}{x^2 + a^2}\, dx, \quad \alpha > 0, \quad a > 0 \quad (5\text{-}43)$$

In order to be able to take advantage of Jordan's lemma, note that by Euler's formula

$$I = \operatorname{Re} I_1 = \operatorname{Re} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x^2 + a^2}\, dx \quad (5\text{-}44)$$

The analytic continuation of the integrand of the integral $I_1$, the function $e^{i\alpha z} \frac{1}{z^2 + a^2}$, satisfies the conditions of Theorem 5.4 and in the upper half-plane has a unique singularity $z_1 = ia$, which is a first-order pole. Therefore,

$$I_1 = 2\pi i \operatorname{Res}\left[\frac{e^{i\alpha z}}{z^2 + a^2},\ ia\right] = 2\pi i \frac{e^{-a\alpha}}{2ia} = \frac{\pi}{a} e^{-a\alpha}$$

Whence

$$I = \operatorname{Re} I_1 = \frac{\pi}{a} e^{-a\alpha} \quad (5\text{-}45)$$

*Note 1*. If $f(x)$ is an even function that satisfies the conditions of Theorem 5.4, then for $\alpha > 0$

$$\int_{0}^{\infty} f(x) \cos \alpha x\, dx = \pi \operatorname{Re} i \sum_{k=1}^{n} \operatorname{Res}\,[e^{i\alpha z} f(z),\ z_k]$$

$$= -\pi \operatorname{Im} \sum_{k=1}^{n} \operatorname{Res}\,[e^{i\alpha z} f(z),\ z_k] \quad (5\text{-}46)$$

*Note 2.* If $f(x)$ is an odd function that satisfies the conditions of Theorem 5.4, then for $a > 0$

$$\int_0^\infty f(x)\sin ax\,dx = \pi \operatorname{Re} \sum_{k=1}^{n} \operatorname{Res}\,[e^{iaz} f(z),\ z_k] \tag{5-47}$$

We proved Lemma 1 and Lemma 2 on the assumption that the function $f(x)$ has only a finite number of singularities in the upper half-plane. However, as will readily be seen, a slight change in the statements of these lemmas makes them hold true for the case of an infinite number of isolated singularities of the function $f(z)$. Let us require that there be a sequence of numbers $R_n$ increasing indefinitely as $n \to \infty$, such that on the semicircular arcs $C'_{R_n}$ in the upper half-plane the conditions (5-29) or (5-37) are fulfilled. Then the assertions (5-30) or, respectively, (5-36) of Lemma 1 and Lemma 2 will hold true provided that the limit process in the integrals under consideration occurs with respect to the sequence of arcs $C'_{R_n}$ as $n \to \infty$. It is also clear that if the appropriate integrals exist we can extend the integration methods at hand to the case of functions with an infinite number of isolated singularities. The *meromorphic* functions constitute an important class of such functions.

*The function $f(z)$ of a complex variable is meromorphic if it is defined on the entire complex plane and if in a finite portion of the plane it does not have singularities different from poles.* It is easy to see that in any bounded domain of the complex plane, a meromorphic function has a finite number of singularities. Indeed, if the number of singularities in a bounded domain were infinite, then by Theorem 1.2, in this domain there would be a limit point of the given set which would thus not be an isolated singular point. This contradicts the condition. Examples of meromorphic functions are fractional-rational functions, trigonometric functions $\tan z$, $\sec z$.

In proving Theorems 5.3 and 5.4 we assumed that the function $f(x)$ does not have singularities on the real axis. However, slight supplementary considerations enable one to use these theorems to evaluate improper integrals even when the function $f(x)$ has several singularities on the real axis.

Let us illustrate this by a simple example.

*Example 4.* Evaluate the integral

$$I = \int_0^\infty \frac{\sin \alpha x}{x}\,dx, \quad \alpha > 0 \tag{5-48}$$

Taking advantage of Euler's formula and the evenness property of the integrand function, perform the formal transformation

$$I = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\,dx = \frac{1}{2} \operatorname{Im} I_1 \tag{5-49}$$

Note that the integral $I_1$ is meaningful only as the principal value of an improper integral of the second kind:

$$I_1 = \text{V.p.} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x}\, dx = \lim_{\substack{\rho \to 0 \\ R \to \infty}} \left\{ \int_{-R}^{-\rho} \frac{e^{i\alpha x}}{x}\, dx + \int_{\rho}^{R} \frac{e^{i\alpha x}}{x}\, dx \right\} \tag{5-50}$$

In the upper half-plane $\operatorname{Im} z \geqslant 0$, consider a closed contour $\Gamma$ consisting of segments of the real axis $[-R, -\rho]$, $[\rho, R]$ and semicircular arcs $C'_\rho$, $|z| = \rho$, and $C'_R$, $|z| = R$ (Fig. 5.4). The func-

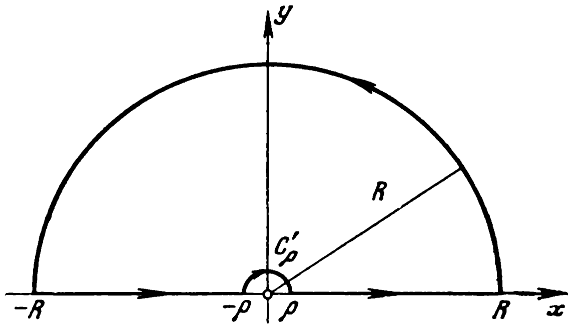


Fig. 5.4

tion $\frac{e^{i\alpha z}}{z}$, which is an analytic continuation into the upper half-plane $\operatorname{Im} z > 0$ of the function $\frac{e^{i\alpha x}}{x}$ specified on the positive real axis $0 < x < \infty$, does not have any singularities in the domain bounded by the contour $\Gamma$. Therefore, on the basis of Cauchy's theorem

$$\int_{\Gamma} f(\zeta)\, d\zeta = \int_{-R}^{-\rho} \frac{e^{i\alpha x}}{x}\, dx + \int_{\rho}^{R} \frac{e^{i\alpha x}}{x}\, dx + \int_{C'^{-}_{\rho}} \frac{e^{i\alpha\zeta}}{\zeta}\, d\zeta + \int_{C'^{+}_{R}} \frac{e^{i\alpha\zeta}}{\zeta}\, d\zeta = 0 \tag{5-51}$$

By Jordan's lemma, the last term on the left of (5-51) approaches zero as $R \to \infty$. Consider the third term. Noting that in this integral the semicircle $C'_\rho$ is traversed in the negative sense (clockwise) and making the change of integration variable $\zeta = \rho e^{i\varphi}$, we get

$$I_3 = \int_{C'^{-}_{\rho}} \frac{e^{i\alpha\zeta}}{\zeta}\, d\zeta = i \int_{\pi}^{0} e^{i\alpha\rho(\cos\varphi + i\sin\varphi)}\, d\varphi \tag{5-52}$$

The integrand in (5-52) is a continuous function of the parameter $\rho$ and its limit is 1 as $\rho \to 0$. Therefore,

$$\lim_{\rho\to 0} I_3 = -i\pi \tag{5-53}$$

Passing to the limit in (5-51) as $\rho \to 0$ and $R \to \infty$, we get, according to (5-50) and (5-53),

$$I_1 = \text{V.p.} \int_{-\infty}^{\infty} \frac{e^{i\alpha x}}{x} dx = i\pi, \quad \alpha > 0 \tag{5-54}$$

whence

$$\int_0^{\infty} \frac{\sin \alpha x}{x} dx = \frac{\pi}{2}, \quad \alpha > 0 \tag{5-55}$$

For $\alpha < 0$ we have the formula

$$\int_0^{\infty} \frac{\sin \alpha x}{x} dx = -\frac{\pi}{2}, \quad \alpha < 0 \tag{5-56}$$

This becomes obvious if we change the sign of $\alpha$ in (5-55).

### *d. The case of multiple-valued functions*

In all the preceding considerations we proceeded from Cauchy's formula, which is valid for a single-valued analytic function. Hence, these methods are applicable only when the analytic continuation $f(z)$ of the function $f(x)$ from the real axis into a domain bounded by the contour of integration is a single-valued analytic function. In those cases when the complete analytic function $F(z)$ is multiple-valued in the extended complex $z$-plane, the contour of integration must be chosen so that there are no branch points of the function $F(z)$ inside it, and we have to consider only the single-valued branch $f(z)$ of the complete analytic function $F(z)$, which is a direct analytic continuation of the function $f(x)$ into the complex domain. This reasoning enables us to extend the foregoing methods to a number of improper integrals that are frequently encountered in applications. We consider a few typical cases.

(1) Integrals of the form

$$I = \int_0^{\infty} x^{\alpha-1} f(x)\, dx, \quad 0 < \alpha < 1 \tag{5-57}$$

Let it be possible for the function $f(x)$ specified on the positive real axis to be analytically continued into the entire complex plane.

Let its analytic continuation $f(z)$ be a single-valued analytic function, except for a finite number of isolated singularities $z_k$ ($k = 1, \ldots, n$) not lying on the positive real axis, and let $z = \infty$ be a zero of order not lower than first of the function $f(z)$, and let the point $z = 0$ be a removable singularity. The function

$$\varphi(z) = z^{\alpha-1} f(z) \tag{5-58}$$

in the domain $\mathfrak{G}$ $[0 < \arg z < 2\pi]$, which is the $z$-plane cut along the positive real axis, is obviously the analytic continuation of the integrand function, which coincides with it on the upper lip of the cut ($\arg z = 0$). The function $\varphi(z)$ is a single-valued function in the domain $\mathfrak{G}$, and its singular points coincide with the singularities $z_k$ of the function $f(z)$. In the domain $\mathfrak{G}$ we consider a closed contour $\Gamma$ composed of segments of the real axis $[\rho, R]$ on the upper and lower lips of the cut, and of open circles $C_\rho$, $|z| = \rho$, and $C_R$, $|z| = R$ (Fig. 5.5). By the residue theorem

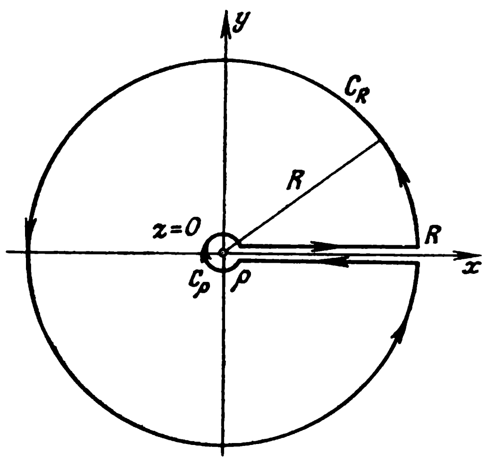


Fig. 5.5

$$\int_\Gamma \varphi(\zeta)\,d\zeta = \int_\rho^R x^{\alpha-1} f(x)\,dx + \int_{C_R^+} \zeta^{\alpha-1} f(\zeta)\,d\zeta + \int_R^\rho \zeta^{\alpha-1} f(\zeta)\,d\zeta$$

$$+ \int_{C_\rho^-} \zeta^{\alpha-1} f(\zeta)\,d\zeta = 2\pi i \sum_{k=1}^{n} \operatorname{Res}\,[z^{\alpha-1} f(z),\ z_k] \tag{5-59}$$

Let us consider each of the terms on the left-hand side of (5-59).

$$|I_2|=\left|\int\limits_{C_R^+}\zeta^{\alpha-1}f(\zeta)\,d\zeta\right|\leqslant\frac{MR^{\alpha-1}}{R}2\pi R=2\pi MR^{\alpha-1}\xrightarrow[R\to\infty]{}0 \quad (5\text{-}60)$$

since by hypothesis in the neighbourhood of the point $z=\infty$ we have for the function $f(z)$ the evaluation $|f(z)|<\frac{M}{|z|}$. The third term in (5-59) is an integral over the lower lip of the cut, where $\arg z=2\pi$, that is, $z=x\cdot e^{i2\pi}$ $(x>0)$ and $z^{\alpha-1}=x^{\alpha-1}\cdot e^{i2\pi(\alpha-1)}$. Therefore,

$$\int\limits_R^\rho \zeta^{\alpha-1}f(\zeta)\,d\zeta=-e^{i2\pi(\alpha-1)}\int\limits_\rho^R x^{\alpha-1}f(x)\,dx \quad (5\text{-}61)$$

Finally,

$$\left|\int\limits_{C_\rho^-}\zeta^{\alpha-1}f(\zeta)\,d\zeta\right|<M_1\rho^{\alpha-1}2\pi\rho\xrightarrow[\rho\to 0]{}0 \quad (5\text{-}62)$$

since in the neighbourhood of the point $z=0$ we have the evaluation $|f(z)|<M_1$ and $\alpha>0$.

Taking the limit in (5-59) as $\rho\to 0$ and $R\to\infty$, we finally get [on the basis of (5-60) to (5-62)]

$$\int\limits_0^\infty x^{\alpha-1}f(x)\,dx=\frac{2\pi i}{1-e^{i2\pi\alpha}}\sum_{k=1}^N \text{Res}\,[z^{\alpha-1}f(z),\ z_k] \quad (5\text{-}63)$$

*Example 5*. Evaluate the integral

$$I=\int\limits_0^\infty \frac{x^{\alpha-1}}{1+x}\,dx,\quad 0<\alpha<1 \quad (5\text{-}64)$$

The integrand function in (5-64) satisfies all the conditions enumerated above. Therefore

$$I=\frac{2\pi i}{1-e^{i2\pi\alpha}}\text{Res}\left[\frac{z^{\alpha-1}}{1+z},\ -1\right]=\frac{2\pi i e^{i\pi(\alpha-1)}}{1-e^{i2\pi\alpha}}=\frac{\pi}{\sin\alpha\pi} \quad (5\text{-}65)$$

(2) Integrals of the form *

$$\int\limits_0^1 x^{\alpha-1}(1-x)^{-\alpha}f(x)\,dx,\quad 0<\alpha<1 \quad (5\text{-}66)$$

* It is readily seen that this integral can be reduced to an integral of the type (5-57) by the substitution $y=\frac{x}{1-x}$. However, in a number of cases it is simple to evaluate the integral (5-66) directly. That is what is done here.

Let it be possible for the function $f(x)$, specified on the interval $(0, 1)$ of the real axis, to be analytically continued over the entire complex plane. Let its analytic continuation be a single-valued analytic function, except for a finite number of isolated singularities $z_k$ $(k = 1, 2, \ldots, N)$ not lying on the interval $0 \leqslant x \leqslant 1$, and let the point $z = \infty$ be a removable singularity of the function $f(z)$. Then the integral (5-66) can readily be evaluated by methods similar to those investigated above. Note that the analytic continuation of the integrand function $\Phi(z) = z^{\alpha-1}(1-z)^{-\alpha} f(z)$ has two branch points: $z = 0$ and $z = 1$. The point $z = \infty$ is a removable singularity of the function $\Phi(z)$. Indeed, a complete circuit around a circle of sufficiently large radius containing both branch points $z = 0$ and $z = 1$ does not change the value of the function $\Phi(z)$. We consider the domain $\mathfrak{G}$, which is the extended $z$-plane cut along the real-axis interval $[0, 1]$. The branch of the function $\Phi(z)$ that coincides, on the upper lip of the cut, with the integrand function (5-66) of the real variable $x$ is a single-valued analytic function in $\mathfrak{G}$. In $\mathfrak{G}$ we choose a closed contour $\Gamma$ consisting of both lips of the cut $[0, 1]$, the circles that close them $C'_\rho$, $|z| = \rho$, and $C''_\rho$, $|z-1| = \rho$, of sufficiently small radius $\rho$, and the circle $C_R$, $|z| = R$, containing the interval $[0, 1]$ and all the singularities of the function $f(z)$ (Fig. 5.6). By the residue theorem

$$\int_\Gamma \Phi(\zeta)\, d\zeta = \int_\rho^{1-\rho} x^{\alpha-1}(1-x)^{-\alpha} f(x)\, dx + \int_{C''^-_\rho} \Phi(\zeta)\, d\zeta$$

$$+ \int_{1-\rho}^{\rho} \Phi(\zeta)\, d\zeta + \int_{C'^-_\rho} \Phi(\zeta)\, d\zeta + \int_{C^+_R} \Phi(\zeta)\, d\zeta$$

$$= 2\pi i \sum_{k=1}^{N} \operatorname{Res}\,[z^{\alpha-1}(1-z)^{-\alpha} f(z),\ z_k] \qquad (5\text{-}67)$$

Consider each term in the left member of the equality (5-67). It is given that $z = \infty$ is a removable singularity of $f(z)$, i.e. in the neighbourhood of $z = \infty$ we have the expansion

$$f(z) = a_0 + \frac{a_{-1}}{z} + \ldots \qquad (5\text{-}68)$$

where $a_0 = \lim\limits_{z\to\infty} f(z)$.

Consider the function

$$\varphi(z) = z^{\alpha-1}(1-z)^{-\alpha} = \frac{1}{z}\left(\frac{z}{1-z}\right)^{\alpha} \qquad (5\text{-}69)$$

which is the above-indicated branch of the function $\Phi(z)/f(z)$. The point $z = \infty$ is a regular point of the chosen branch of the

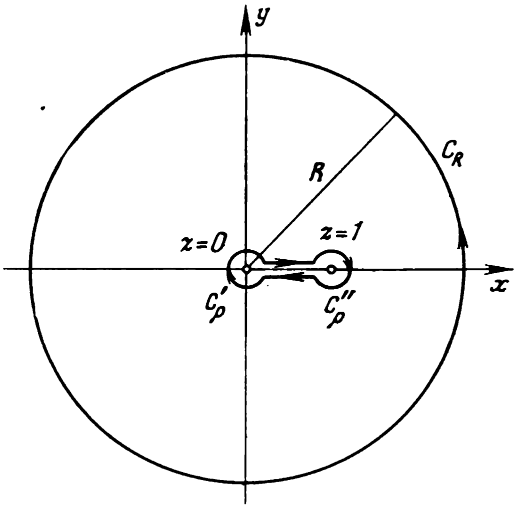


Fig. 5.6

function $\varphi(z)$; therefore, the function $\varphi(z)$ in the neighbourhood of $z = \infty$ may be represented in the form

$$\varphi(z) = \frac{e^{i\pi\alpha}}{z} + \frac{\psi_1(z)}{z^2} \qquad (5\text{-}70)$$

where $\psi_1(z)$ is a bounded analytic function in the neighbourhood of the point $z = \infty$. Whence, for a Laurent-series expansion of the function $\Phi(z)$ about the point $z = \infty$ we get the expression

$$\Phi(z) = a_0 \frac{e^{i\pi\alpha}}{z} + \frac{\psi(z)}{z^2} \qquad (5\text{-}71)$$

where $\psi(z)$ is a bounded analytic function in the neighbourhood of the point $z = \infty$. From (5-71) we find

$$\text{Res}\,[\Phi(z), \infty] = -a_0 e^{i\pi\alpha} \qquad (5\text{-}72)$$

Therefore, by formula (5-17),

$$\int_{C_R^+} \Phi(\zeta)\, d\zeta = 2\pi i a_0 e^{i\pi\alpha} \qquad (5\text{-}73)$$

Since, when we encircle the point $z = 1$ clockwise, the argument of the expression $(1 - z)$ varies by $-2\pi$, the argument of the function $\Phi(z)$ is greater on the lower lip of the cut than the argument on the upper lip of the cut by $2\pi\alpha$. Therefore,

$$\int_{1-\rho}^{\rho} \Phi(\zeta)\, d\zeta = -e^{i2\pi\alpha} \int_{\rho}^{1-\rho} \Phi(x)\, dx \qquad (5\text{-}74)$$

As may readily be demonstrated with the aid of evaluations similar to (5-62), for $0 < \alpha < 1$ the integrals over the small circles $C'_\rho$ and $C''_\rho$ tend to zero as $\rho \to 0$. Then, taking the limit in (5-67), as $\rho \to 0$, we obtain

$$(1 - e^{i2\pi\alpha}) I + 2\pi i e^{i\pi\alpha} a_0 = 2\pi i \sum_{k=1}^{N} \operatorname{Res}\,[z^{\alpha-1}(1-z)^{-\alpha} f(z),\ z_k]$$

whence

$$I = \frac{\pi a_0}{\sin \pi\alpha} + \frac{2\pi i}{1 - e^{i2\pi\alpha}} \sum_{k=1}^{N} \operatorname{Res}\,[z^{\alpha-1}(1-z)^{-\alpha} f(z),\ z_k] \qquad (5\text{-}75)$$

where $a_0 = \lim\limits_{z\to\infty} f(z)$.

*Example 6*. Evaluate the integral*

$$I = \int_0^1 x^{\alpha-1}(1-x)^{-\alpha}\, dx, \quad 0 < \alpha < 1 \qquad (5\text{-}76)$$

Since all the earlier formulated conditions are fulfilled and $a_0 = 1$, it follows that

$$I = \frac{\pi}{\sin \pi\alpha} \qquad (5\text{-}77)$$

(3) Integrals of the form

$$I = \int_0^\infty f(x) \ln x\, dx \qquad (5\text{-}78)$$

Let $f(x)$ be an even function which may be analytically continued onto the upper half-plane $\operatorname{Im} z > 0$, and let its analytic continuation satisfy the conditions of Lemma 1. Consider, in the upper half-plane, the closed contour $\Gamma$ consisting of segments $[-R, -\rho]$, $[\rho, R]$ of the real axis and the semicircles $C'_\rho$, $|z| = \rho$, and $C'_R$, $|z| = R$, connecting them. The function $\Phi(z)$, which is a branch of the complete analytic function and coincides with $f(x) \ln x$ on the positive real axis ($x > 0$), on the negative real axis, for $z = |z| e^{i\pi} = xe^{i\pi} = -x$ ($x > 0$), takes the value

$$\Phi(z)\,|_{z = xe^{i\pi}} = f(x) \ln (xe^{i\pi}) = f(x)\,[\ln x + i\pi]$$

* Note that the integral at hand is a particular case of the $B$ function:

$$B(p, q) + \int_0^1 x^{p-1}(1-x)^{q-1}\, dx$$

Therefore,

$$\int_{\Gamma} \Phi(\zeta)\, d\zeta = \int_{\rho}^{R} f(x) \ln x\, dx + \int_{C'^{+}_{R}} \Phi(\zeta)\, d\zeta + \int_{\rho}^{R} f(x)[\ln x + i\pi]\, dx$$

$$+ \int_{C'^{-}_{\rho}} \Phi(\zeta)\, d\zeta = 2\pi i \sum_{k=1}^{N} \operatorname{Res}[f(z) \ln z,\ z_k] \quad (5\text{-}79)$$

Consider the second term on the left of (5-79):

$$\left| \int_{C'^{+}_{R}} \Phi(\zeta)\, d\zeta \right| \leqslant \frac{M}{R^{1+\delta}} \int_{0}^{\pi} |\ln \zeta|\, ds = \frac{M}{R^{1+\delta}} \int_{0}^{\pi} |\ln R + i \arg \zeta|\, ds$$

$$\leqslant \frac{M\pi}{E^{\delta}} \sqrt{\ln^2 R + \pi^2} \underset{R\to\infty}{\longrightarrow} 0 \quad (5\text{-}80)$$

By carrying out similar evaluations, it is easy to show that the last term in the left member of (5-79) also approaches zero as $\rho \to 0$. Finally, the improper integral $\int_0^\infty f(x)\, dx$ exists and, by (5-35), is equal to

$$\int_{0}^{\infty} f(x)\, dx = \pi i \sum_{k=1}^{N} \operatorname{Res}[f(z),\ z_k] \quad (5\text{-}81)$$

Therefore, taking the limit in (5-79) as $\rho \to 0$ and $R \to \infty$, we get

$$I = \int_{0}^{\infty} f(x) \ln x\, dx = \pi i \sum_{k=1}^{N} \operatorname{Res}\left[f(z)\left(\ln z - \frac{i\pi}{2}\right),\ z_k\right] \quad (5\text{-}82)$$

*Example 7.* Evaluate the integral

$$I = \int_{0}^{\infty} \frac{\ln x}{(1+x^2)^2}\, dx \quad (5\text{-}83)$$

According to the foregoing reasoning,

$$I = \pi i \operatorname{Res}\left[\frac{1}{(1+z^2)^2}\left(\ln z - \frac{i\pi}{2}\right),\ i\right] = -\frac{\pi}{4} \quad (5\text{-}84)$$

## 5.3. Logarithmic Residue

### *a. The concept of a logarithmic residue*

Let there be given in a domain $\mathfrak{G}$ a single-valued function $f(z)$ analytic everywhere in $\mathfrak{G}$ except at a finite number of isolated singularities $z_k$ ($k = 1, \ldots, p$), all $z_k$ being poles. Assume that on the boundary $\Gamma$ of the domain $\mathfrak{G}$ there are no zeros and no singularities of the function $f(z)$; we consider the auxiliary function

$$\varphi(z)=\frac{f'(z)}{f(z)} \tag{5-85}$$

The function $\varphi(z)$ is often called the *logarithmic derivative* of the function $f(z)$, and the residues of the function $\varphi(z)$ at its singularities $z_m$ $(m = 1, \ldots, M)$ are called the *logarithmic residues* of the function $f(z)$. Define the singularities of $\varphi(z)$ in $\mathfrak{G}$. By virtue of the general properties of analytic functions it is clear that the zeros $\tilde{z}_k$ $(k = 1, \ldots, n)$ and the poles $z_k$ $(k = 1, \ldots, p)$ of the function $f(z)$ will be the singularities of the function $\varphi(z)$. Let us find the value of the residue of the function $\varphi(z)$ at each of its singular points. Let the point $z = \tilde{z}_k$ be a zero of order $n_k$ of the function $f(z)$. Then the function $f(z)$ in the neighbourhood of this point is of the form

$$f(z) = (z-\tilde{z}_k)^{n_k} f_1(z), \quad f_1(\tilde{z}_k) \neq 0 \tag{5-86}$$

and the point $\tilde{z}_k$ is a regular point of the function $f_1(z)$. Evaluating the function $\varphi(z)$ in the neighbourhood of the point $z = \tilde{z}_k$ by formula (5-85), we obtain

$$\varphi(z)=(\ln f(z))'=n_k(\ln(z-\tilde{z}_k))'+(\ln f_1)'=\frac{n_k}{z-\tilde{z}_k}+\frac{f_1'(z)}{f_1(z)}$$

Whence it follows that the point $\tilde{z}_k$ is a pole of order one of the function $\varphi(z)$, and the residue of the function $\varphi(z)$ at this point is equal to $n_k$. Thus, at a zero of order $n_k$ of the function $f(z)$, its logarithmic residue is equal to $n_k$, that is, to the order of the zero:

$$\operatorname{Res}\left[\frac{f'(z)}{f(z)},\ \tilde{z}_k\right]=n_k \tag{5-87}$$

Let the point $z_k$ be a pole of order $p_k$ of the function $f(z)$. Then in the neighbourhood of this point the function $f(z)$ is of the form

$$f(z)=\frac{f_1(z)}{(z-z_k)^{p_k}}, \quad f_1(z_k)\neq 0 \tag{5-88}$$

and the point $z_k$ is a regular point of the function $f_1(z)$. Therefore, for the logarithmic derivative of the function $f(z)$ in the neighbourhood of the point $z = z_k$ we get the expression

$$\varphi(z)=-\frac{p_k}{z-z_k}+\frac{f_1'(z)}{f_1(z)}$$

From this it follows that the point $z_k$ is also a first-order pole of the function $\varphi(z)$, and the residue at this point is $-p_k$. Thus, at a pole of order $p_k$ of the function $f(z)$ the logarithmic residue of the function is equal to the order of the pole with a minus sign:

$$\operatorname{Res}\left[\frac{f'(z)}{f(z)},\ z_k\right] = -p_k \tag{5-89}$$

*b. Counting the number of zeros of an analytic function*

The results obtained enable us to prove the following important theorem.

***Theorem 5.5.*** *Let a function $f(z)$ be analytic everywhere in a closed domain $\overline{\mathfrak{G}}$ except at a finite number of isolated singularities $z_k$ inside $\mathfrak{G}$, which are all poles, and let $f(z)$ be nonvanishing at any point of the boundary $\Gamma$ of the domain $\mathfrak{G}$. Then the difference between the total number of zeros and the total number of poles of $f(z)$ in the domain $\mathfrak{G}$ is determined by the expression*

$$N - P = \frac{1}{2\pi i}\int_{\Gamma+} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta \tag{5-90}$$

By the total number of zeros (poles) is meant the number of zeros $N$ (poles $P$) counting multiplicities:

$$N = \sum_{k=1}^{n} n_k, \quad P = \sum_{k=1}^{p} p_k \tag{5-91}$$

*Proof.* To prove the theorem, note that the integral over $\Gamma$ of the function $\varphi(z) = \frac{f'(z)}{f(z)}$ may be evaluated by means of the residue theorem; and since all the singularities of the function $\varphi(z)$ are zeros and poles of $f(z)$, and the residues at these points are determined by the formulas (5-87) and (5-89), it follows that

$$\int_{\Gamma+} \varphi(\zeta)\, d\zeta = 2\pi i \sum_{m=1}^{M} \operatorname{Res}[\varphi(z), z_m]$$

$$= 2\pi i \left\{\sum_{k=1}^{n} n_k - \sum_{k=1}^{p} p_k\right\} = 2\pi i (N - P)$$

which proves the theorem.

To see the simple geometric meaning of this theorem, transform the integral on the right of (5-90):

$$\frac{1}{2\pi i}\int_{\Gamma+} \frac{f'(\zeta)}{f(\zeta)}\, d\zeta = \frac{1}{2\pi i}\int_{\Gamma+} d\ln f(\zeta) = \frac{1}{2\pi i}\int_{\Gamma+} d\{\ln|f(\zeta)| + i\arg f(\zeta)\}$$

$$= \frac{1}{2\pi i}\int_{\Gamma+} d\ln|f(\zeta)| + \frac{1}{2\pi}\int_{\Gamma+} d\arg f(\zeta) \tag{5-92}$$

The real function $\ln |f(\zeta)|$ is a single-valued function; therefore, its variation, as the point $\zeta$ traverses the closed contour $\Gamma$, is zero. Hence, the first term on the right of (5-92) is zero. The second term is the total variation of the argument of the function $f(\zeta)$, as the point $\zeta$ traverses the closed contour $\Gamma$, divided by $2\pi$. Thus,

$$N-P=\frac{1}{2\pi}\operatorname{Var}[\arg f(z)]_{\Gamma+} \tag{5-93}$$

Let us depict the values of the function $w = f(z)$ using points in the complex $w$-plane. Since the function $f(z)$ is continuous on the contour $\Gamma$, for a complete traversal of the contour $\Gamma$ by $z$ in the $z$-plane, the corresponding point in the $w$-plane describes a certain closed contour $C$. The point $w = 0$ may lie either inside or outside the domain bounded by the contour $C$. In the former case, the variation of the argument $w$ in a complete traversal of $C$ is obviously equal to zero. In the latter case, the variation of the argument $w$ is determined by the number of total circuits about the point $w = 0$ performed by $w$ in its motion along the contour $C$. The point $w$ can encircle the point $w = 0$ either counterclockwise (in a positive sense) or clockwise (in a negative sense). And so the difference between the total number of zeros and poles of the function $f(z)$ in the domain $\mathfrak{G}$ is determined by the number of circuits performed by the point $w = f(z)$ about $w = 0$ as the point $z$ traverses the contour $\Gamma$ in a positive sense. This reasoning is often essential when counting the total number of zeros of an analytic function in a given domain. In many cases, the computations can be appreciably simplified by the following theorem.

***Theorem 5.6 (Rouché's theorem).*** *Let the functions $f(z)$ and $\varphi(z)$ be analytic in a closed domain $\overline{\mathfrak{G}}$, with the following inequality valid on the boundary $\Gamma$ of $\mathfrak{G}$:*

$$|f(z)|_\Gamma > |\varphi(z)|_\Gamma \tag{5-94}$$

*Then the total number of zeros in $\mathfrak{G}$ of the function $F(z) = f(z) + \varphi(z)$ is equal to the total number of zeros of the function $f(z)$.*

*Proof.* All the conditions of Theorem 5.5 are fulfilled for the functions $f(z)$ and $F(z) = f(z) + \varphi(z)$. Indeed, $f(z)$ does not have singularities on $\Gamma$ (it is analytic in $\overline{\mathfrak{G}}$) and does not vanish on $\Gamma$ by virtue of (5-94). These conditions are also fulfilled for the function $F(z)$, since $|F(z)|_\Gamma = |f(z) + \varphi(z)| \geqslant |f(z)|_\Gamma - |\varphi(z)|_\Gamma > 0$. Therefore, by formula (5-93), we get

$$N[f(z)+\varphi(z)]=\frac{1}{2\pi}\operatorname{Var}[\arg(f+\varphi)]_\Gamma$$

and

$$N[f(z)]=\frac{1}{2\pi}\operatorname{Var}[\arg f(z)]_\Gamma$$

Consider the difference

$$N[f(z)+\varphi(z)]-N[f(z)]$$
$$=\frac{1}{2\pi}\operatorname{Var}[\arg(f+\varphi)-\arg f]_\Gamma=\frac{1}{2\pi}\operatorname{Var}\left[\arg\left(1+\frac{\varphi}{f}\right)\right]_\Gamma$$
$$\left(\arg(f+\varphi)-\arg f=\arg\frac{f+\varphi}{f}\right)$$

We introduce the function $w=1+\frac{\varphi(z)}{f(z)}$. It will readily be seen that as the point $z$ traverses the contour $\Gamma$ the corresponding point $w$ describes a closed curve $C$, which by virtue of the condition (5-94)

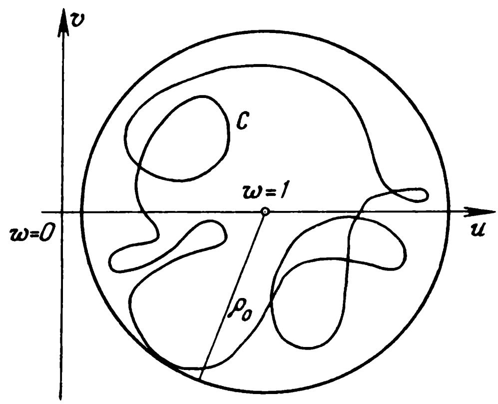


Fig. 5.7

will lie entirely inside some circle $|w-1|\leqslant\rho_0<1$ (Fig. 5.7). And so the point $w=0$ lies outside the curve $C$. Consequently, $\operatorname{Var}[\arg w]_\Gamma=0$, which proves the theorem.

*Example.* Find the total number of zeros of the function $F(z)=$ $=z^8-5z^5-2z+1$ inside the unit circle $|z|<1$. Represent the function $F(z)$ in the form $F(z)=f(z)+\varphi(z)$ putting $f(z)=$ $=-5z^5+1$ and $\varphi(z)=z^8-2z$. Then

$$|f(z)|_{|z|=1}\geqslant|-5z^5|_{|z|=1}-1=4$$
$$|\varphi(z)|_{|z|=1}\leqslant|z^8|_{|z|=1}+|2z|_{|z|=1}=3$$

whence $|f(z)|_{|z|=1}>|\varphi(z)|_{|z|=1}>0$. Hence, the total number of zeros in the domain $|z|<1$ of the function $F(z)$ is equal to the total number of zeros of the function $f(z)$, but the latter clearly has exactly five zeros:

$$z_k = \sqrt[5]{\frac{1}{5}}\, e^{i\frac{2\pi k}{5}}$$

$$(k = 0,\ 1,\ \ldots,\ 4)$$

An important fundamental corollary to Rouché's theorem is

***Theorem 5.7*** **(*The fundamental theorem of algebra*).** *Every polynomial of degree n in the complex plane has n zeros, counting multiplicities.*

*Proof.* Represent the polynomial $F(z) = a_0z^n + a_1z^{n-1} + \ldots$ $\ldots + a_n$ as $F(z) = f(z) + \varphi(z)$, putting $f(z) = a_0z^n$, $\varphi(z) =$ $= a_1z^{n-1} + \ldots + a_n$. Form the ratio $\frac{\varphi(z)}{f(z)} = \frac{a_1}{a_0}\cdot\frac{1}{z} + \ldots$ $\ldots + \frac{a_n}{a_0}\cdot\frac{1}{z^n}$. It will then be seen that for any specified values of the coefficients $a_0, a_1, \ldots, a_n$ there will always be a value $R_0$, such that for all the values $|z| = R > R_0$ the following inequality holds:

$$0 < \left|\frac{\varphi(z)}{f(z)}\right|_{|z|=R} < 1 \qquad (5\text{-}95)$$

By Rouché's theorem, from (5-95) it follows that the total number of zeros of the function $F(z)$ in the circle $|z| = R$ is equal to the number of zeros in this circle of the function $f(z) = a_0z^n$. But the function $f(z) = a_0z^n$ in the entire complex plane has a unique $n$-fold zero—the point $z = 0$. The assertion of the theorem follows by virtue of the arbitrariness of $R \geqslant R_0$.

# CHAPTER 6

# CONFORMAL MAPPING

A study of the geometric properties of conformal mappings carried out by analytic functions is of great importance both in constructing the general theory of functions of a complex variable and in its numerous applications. In Chapter 1 we introduced the concept of a conformal mapping under which angles are preserved and stretching is invariant. The fundamental task of the theory of conformal mappings is the following. Given two domains of a complex plane, it is required to find the function that accomplishes a one-to-one and conformal mapping of one domain onto another. There naturally arise questions of the conditions of existence and unique definition of such a function.

In this chapter we will briefly discuss the basic concepts of the theory of conformal mapping. We will also consider some geometric properties of mappings carried out by a number of analytic functions in practical applications.

## 6.1. General Properties

### *a. Definition of a conformal mapping*

The concept of a conformal mapping was introduced in Chapter 1 when we considered the geometric meaning of the modulus and argument of a derivative. It was shown that if a function $w = f(z)$ is single-valued and analytic in the neighbourhood of some point $z_0$ and $f'(z_0) \neq 0$, then the mapping accomplished by the given function preserves angles and invariance of stretching at the point $z_0$. That is, the angle between any two smooth curves intersecting at the point $z_0$ is equal in magnitude and sense to the angle between their images in the $w$-plane at the point $w_0 = f(z_0)$, and infinitesimal line elements emanating from the point $z_0$ are transformed in similar fashion. This means that in such mapping any infinitely small triangle with vertex at $z_0$ is transformed into a similar infinitely small triangle with vertex at the point $w_0$. Note that by virtue of the general properties of analytic functions* the analytic function

* See Chapter 1, page 33.

$z = \varphi(w)$ is defined in the neighbourhood of the point $w_0$. In this way, a one-to-one correspondence is set up between the neighbourhoods of the points $z_0$ and $w_0$. We introduce the following fundamental definition.

*A one-to-one mapping of a domain* $\mathfrak{G}$ *of the complex z-plane onto a domain G of the complex w-plane is called conformal if at all points* $z \in \mathfrak{G}$ *the mapping preserves angles and invariance of stretching.* Let it be stressed that this definition tacitly implies continuity of the mapping.

From the foregoing it is clear that in the conformal mapping of a domain $\mathfrak{G}$ into a domain $G$, infinitely small plane figures of the domain $\mathfrak{G}$ are transformed into similar infinitely small figures of the domain $G$. It is also evident that in a conformal mapping the property of mutual orthogonality of the system of curves in the plane is preserved. Indeed, let there be given two mutually orthogonal one-parameter families of curves $\varphi(x, y) = c$ and $\psi(x, y) = c$ in the domain $\mathfrak{G}$ of the $z$-plane ($z = x + iy$) and through any point of $\mathfrak{G}$ there passes one line of each family. Then, in a conformal mapping of the domain $\mathfrak{G}$ onto a domain $G$ of the $w$-plane ($w = u + iv$), the images of the given curves in the $w$-plane—the curves $\Phi(u, v) = c$ and $\Psi(u, v) = c$—will be mutually orthogonal on the basis of the property of preservation of angles. This means that if in $\mathfrak{G}$ we introduce some orthogonal curvilinear system of coordinates, then in a conformal mapping this system of coordinates transforms into an orthogonal system.

Let us now see what properties a function of a complex variable must have so as to effect a conformal mapping.

We have the following theorem.

***Theorem 6.1.*** *Let $f(z)$ be a single-valued and univalent analytic function in the domain* $\mathfrak{G}$ *and $f'(z) \neq 0$ for $z \in \mathfrak{G}$. Then $f(z)$ maps the domain* $\mathfrak{G}$ *conformally onto the domain G of the complex w-plane, which is the range of values of the function $w = f(z)$ for $z \in \mathfrak{G}$.*

*Proof.* Indeed, by virtue of the condition $f'z \neq 0$, for $z \in \mathfrak{G}$ the mapping, by the function $f(z)$, possesses at all points of $\mathfrak{G}$ the properties of preservation of angles and invariance of stretching, which proves the theorem.

To summarize, then, the conditions of analyticity and univalence, and the fact that the derivative of a function of a complex variable is nonzero are sufficient conditions for the function to map conformally. The natural question to ask is whether these conditions are necessary. The following theorem gives the answer.

***Theorem 6.2.*** *Let the function $f(z)$ map a domain* $\mathfrak{G}$ *of the complex z-plane conformally onto the domain G of the complex w-plane and let it be bounded in* $\mathfrak{G}$. *Then the function $f(z)$ is univalent and analytic in the domain* $\mathfrak{G}$, *and $f'(z) \neq 0$ for $z \in \mathfrak{G}$.*

*Proof*. Since the function $f(z)$ maps conformally, the mapping is one-to-one, and at any point $z_0 \in \mathfrak{G}$ the angles are preserved and stretching is invariant. Hence, for any points $z_1$ and $z_2$ belonging to the neighbourhood of the point $z_0$, the following relations hold to within infinitesimals:

$$\arg \Delta w_2 - \arg \Delta w_1 = \arg \Delta z_2 - \arg \Delta z_1 \tag{6-1}$$

and

$$\frac{|\Delta w_2|}{|\Delta z_2|} = \frac{|\Delta w_1|}{|\Delta z_1|} = k \neq 0 \tag{6-2}$$

where $\Delta z_1 = z_1 - z_0$ and $\Delta z_2 = z_2 - z_0$ are infinitely small line elements emanating from the point $z_0$, and $\Delta w_1$ and $\Delta w_2$ are their

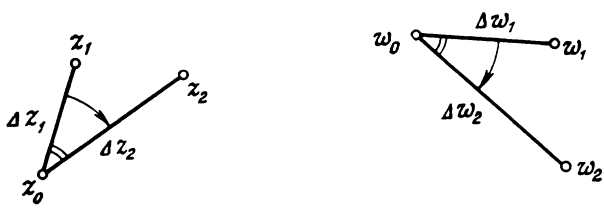


**Fig. 6.1**

images (Fig. 6.1). Note that by (6-1) the corresponding angles at the points $z_0$ and $w_0$ are not only equal in magnitude but in sense as well. Denoting $\arg \frac{\Delta w_2}{\Delta z_2}$ by $\alpha$, we find from (6-1) that $\arg \frac{\Delta w_1}{\Delta z_1} = \alpha$ as well. Indeed

$$\arg \frac{\Delta w_2}{\Delta z_2} = \arg \Delta w_2 - \arg \Delta z_2 = \arg \Delta w_1 - \arg \Delta z_1 = \arg \frac{\Delta w_1}{\Delta z_1} = \alpha \tag{6-3}$$

From (6-2) and (6-3) we find that, to within infinitesimals, the relation

$$\frac{\Delta w_2}{\Delta z_2} = \frac{\Delta w_1}{\Delta z_1} = ke^{i\alpha} \tag{6-4}$$

holds true. Since the choice of points $z_1$ and $z_2$ in the neighbourhood of the point $z_0$ is arbitrary, the relation (6-4) implies that there is a limit of the difference quotient $\frac{\Delta w}{\Delta z}$ as $\Delta z \to 0$. By definition, this limit is the derivative of the function $f(z)$ at the point $z_0$. Since $k \neq 0$, this derivative is different from zero:

$$\lim_{\Delta z \to 0} \frac{\Delta w}{\Delta z} = f'(z_0) \neq 0 \tag{6-5}$$

The point $z_0$ is an arbitrary point of the domain $\mathfrak{G}$; therefore, it follows from (6-5) that $f(z)$ is an analytic function* in $\mathfrak{G}$ and $f'(z) \neq 0$ for $z \in \mathfrak{G}$. The univalence of $f(z)$ follows from the one-to-one nature of the mapping. The theorem is proved.

Thus, *a conformal mapping of a domain $\mathfrak{G}$ of the complex z-plane onto a domain G of the complex w-plane is effected only by univalent analytic functions of a complex variable with derivative different from zero at all points of the domain $\mathfrak{G}$.*

Note that the condition $f'(z) \neq 0$ everywhere in the domain $\mathfrak{G}$ is a necessary but not sufficient condition for the mapping of the domain $\mathfrak{G}$ onto the domain $G$ performed by the function $f(z)$ to be conformal. Clearly, if the function $f(z)$ is analytic in the domain $\mathfrak{G}$ and $f'(z) \neq 0$ everywhere in $\mathfrak{G}$, but the function $f(z)$ is not univalent in $\mathfrak{G}$, then the mapping effected by this function will not be one-to-one, and thus will not be conformal. An elementary case is the function $w = z^4$ specified in the semi-annular region $1 \leqslant |z| \leqslant 2$, $0 \leqslant \arg z \leqslant \pi$. This function is analytic in the given domain, and $w' = 4z^3 \neq 0$ everywhere in the given semi-annular region. However, this function maps the given semi-annular region onto the domain $1 \leqslant |w| \leqslant 16$, $0 \leqslant \arg w \leqslant 4\pi$, that is, a domain that twice covers the corresponding annulus in the $w$-plane; but this violates one-to-one correspondence.

Thus, univalence of a single-valued analytic function in a domain $\mathfrak{G}$ is an extremely important condition for a conformal mapping. As will be shown later on (see Theorem 6.3—the principle of one-to-one correspondence), this condition is necessary and sufficient for a mapping to be conformal.

As has already been pointed out, the property of preserving angles means that not only the magnitude of the angles is preserved between curves intersecting at a point $z_0$ and their images but the direction of the angles is preserved as well. A mapping in which the magnitudes of the angles between curves and their images are preserved but the sense of the angles is reversed is termed a *conformal mapping of the second kind*. The mapping considered above is called a *conformal mapping of the first kind*.

It is easy to show that a conformal mapping of the second kind is accomplished by functions of a complex variable which are complex conjugate to analytic functions with nonzero derivatives. Indeed, let the function $w = f(z)$ perform a conformal mapping of the second kind of some domain $\mathfrak{G}$ of the complex $z$-plane onto a domain $G$ of the complex $w$-plane. Let us consider the function $w_1 = \bar{w}$ which maps $G$ onto $G^*$ of the complex $w_1$-plane. Clearly, the geometric meaning of the latter mapping consists in a mirror reflection of $G$ about the real $u$-axis of the $w$-plane. But in a mirror reflection

* See footnote on page 32.

the angles preserve magnitude but the sense is reversed. This means that the mapping by the function

$$\varphi(z) = w_1 = \bar{w} = \overline{f(z)}, \quad z \in \mathfrak{G} \tag{6-6}$$

of the domain $\mathfrak{G}$ onto the domain $G^*$ is a conformal mapping of the first kind. Thus, the function $\varphi(z)$ must be analytic in the domain $\mathfrak{G}$, and $\varphi'(z) \neq 0$, $z \in \mathfrak{G}$. But from (6-6) it follows that $f(z) = \overline{\varphi(z)}$. This proves the assertion. Up to now we have assumed that a bounded domain $\mathfrak{G}$ is mapped conformally onto a bounded domain $G$. In some cases, one has to consider the mapping of the neighbourhood of a point $z_0$ onto the neighbourhood of the point $w = \infty$ (or vice versa). We will then call the mapping conformal if the neighbourhood of the point $z_0$ is conformally mapped onto the neighbourhood of the point $\zeta = 0$, where $\zeta = \frac{1}{w}$. In similar fashion we define the conformal mapping of the neighbourhood of the point $z = \infty$ onto the neighbourhood of the point $w = \infty$.

## b. Elementary examples

In the previous chapters we have already considered a number of geometric properties of mappings accomplished by a variety of elementary functions. Let us now see if these mappings are conformal, and if they are, then in what domains.

It is readily seen that the linear function $w = f(z) = az + b$ ($a \neq 0$ and $b$ are arbitrary complex constants) maps conformally the extended complex $z$-plane onto the extended $w$-plane, since this function is univalent and its derivative $f'(z) = a$ is nonzero at all points of the $z$-plane. To be satisfied of the conformal nature of the mapping of the neighbourhood of the point $z = \infty$ onto the neighbourhood of the point $w = \infty$, we put (in accord with the remark made above) $t = \frac{1}{z}$ and $\zeta = \frac{1}{w}$. The function $w = az + b$ is transformed into the function $\zeta = \frac{t}{a + bt}$, which maps the neighbourhood of the point $t = 0$ conformally onto the neighbourhood of the point $\zeta = 0$ (the point $t = 0$ is a regular point of this function, and $\zeta'(t)\,|_{t=0} = \frac{1}{a} \neq 0$).

We saw above that the geometric meaning of a mapping accomplished by a linear function consists in a similarity stretching and a translation of the $z$-plane. This function can therefore be used for constructing conformal mappings of similar figures.

*Example 1.* Construct a function to map conformally the circle $|z - 1 - i| \leqslant 2$ onto the unit circle $|w| \leqslant 1$.

Since the domains $\mathfrak{G}$ and $G$ are similar figures, the problem may be solved by means of a linear function which accomplishes similar stretching of the $z$-plane and translation of the coordinate origin. It is easy to see that the desired function is of the form

$$w = a(z - 1 - i)$$

where $|a| = \frac{1}{2}$ and the argument $a$ of the complex number can have any value and determines the rotation of the $w$-plane about the point $w = 0$.

We consider the power function $w = f(z) = z^n$, where $n > 1$ is an integer. It follows from the reasoning of Chapter 1 and Chapter 3 that the function carries out a one-to-one mapping of the domain of its univalence—the sector $\psi_0 < \arg z < \psi_0 + \frac{2\pi}{n}$ —onto the extended $w$-plane cut along the ray $\arg w = n\psi_0$. Its derivative

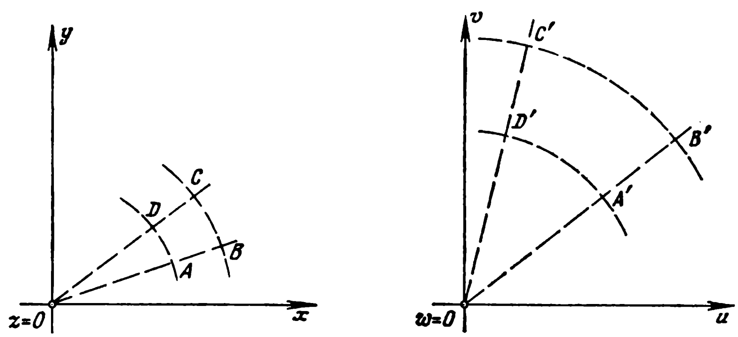


Fig. 6.2

$f'(z) = nz^{n-1}$ is nonzero and is bounded everywhere within the given sector and at points of its boundary, with the exception of $z = 0$ and $z = \infty$. Therefore, the given function maps conformally onto the cut $w$-plane a domain within the indicated sector. Any infinitely small plane figure lying inside the given sector is transformed into a similar infinitely small figure in the $w$-plane; for instance the parallelogram $ABCD$, whose sides are the coordinate lines of the polar system of coordinates (Fig. 6.2), will be transformed into a similar infinitely small parallelogram $A'B'C'D'$ whose sides are also the coordinate lines of the polar system of coordinates in the $w$-plane. However, at the boundary point $z = 0$ the conformality of the mapping is violated. Indeed, consider the curves $\gamma_1$ and $\gamma_2$ lying inside the given sector and intersecting at the point $z = 0$ at an angle $\varphi_0$ (Fig. 6.3). Clearly, the function $w = z^n$ transforms these curves into the curves $\Gamma_1$ and $\Gamma_2$ which intersect at the point $w = 0$ at an angle $\Phi_0 = n\varphi_0 \neq \varphi_0$. Thus, the given function will map an infinitely small triangle with vertex at the point $z = 0$ onto a triangle which is no longer similar to the original one. We note that at the point $z = 0$ where conformality of mapping is violated, the derivative of the function $f(z) = z^n$ is zero. Continuing our investigation, we readily see that the function $w = z^n$ maps conformally the domain of the complex $z$-plane, which is the extended $z$-plane, with the exception of the points $z = 0$ and $z = \infty$, onto an $n$-valent Riemann surface of the inverse function $z = \sqrt[n]{w}$. To the points $z = 0$ and $z = \infty$, at which conformality of mapping

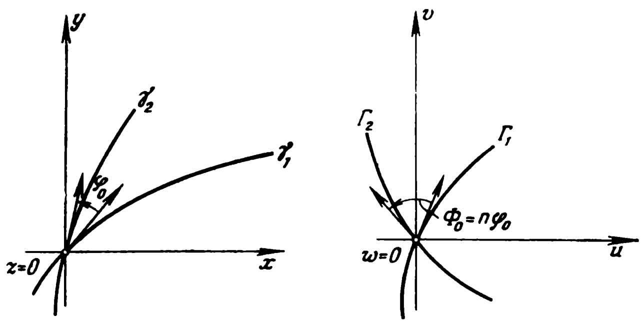


**Fig. 6.3**

is violated, there correspond the points $w = 0$ and $w = \infty$, which are branch points of the inverse function.

In the general case, the power function $w = f(z) = z^\alpha$, where $\alpha > 0$ is a given real number, maps the sector $\frac{2\pi}{\alpha}k < \arg z < \frac{2\pi}{\alpha}(k+1)$ $(k = 0, \pm 1, \ldots)$ of its Riemann surface (which is infinite-sheeted for irrational $\alpha$, finite-sheeted for rational $\alpha$, and the ordinary $z$-plane for integral $\alpha$) onto the extended $w$-plane (the ray $\arg z = \frac{2\pi}{\alpha}k$ is mapped onto the positive real axis). Its derivative $f'(z) = \alpha z^{\alpha-1}$ exists and is nonzero everywhere inside the given sector, except at the points $z = 0$ and $z = \infty$. Thus, this function too maps the given sector conformally onto the cut $w$-plane.

As in the case of the function $w = z^n$, the conformality of the mapping is violated at the points $z = 0$ and $z = \infty$.

*Example 2.* Construct a function that maps the first quadrant of the $z$-plane ($\operatorname{Re} z > 0$, $\operatorname{Im} z > 0$) conformally onto the upper half of the $w$-plane ($\operatorname{Im} w > 0$).

It is easy to see that the function

$$w = az^2 + b$$

where $a > 0$ and $b$ are arbitrary real constants, yields the solution to this problem. At the points $z = 0$ and $z = \infty$ conformality is violated.

In Chapter 3 we examined a mapping by the exponential function $w = f(z) = e^z$. It was shown that this function maps in one-to-one fashion any domain of univalence—the strip $y_0 < \operatorname{Im} z < y_0 + 2\pi$ of the $z$-plane—onto the extended $w$-plane cut along the ray $\arg w = y_0$. Since the derivative of the function at hand, $f'(z) = e^z$, is nonzero everywhere inside the given strip, the mapping is conformal. It is easy to see that in this mapping an orthogonal grid of Cartesian coordinates $x = C_1$, $y = C_2$ inside the given strip is transformed into an orthogonal grid of polar coordinates $|w| = e^{C_1}$, $\arg w = C_2$ in the $w$-plane. The complete analytic function $F(z) = e^z$, which is an entire function in the $z$-plane, maps the extended $z$-plane conformally onto the infinite-sheeted Riemann surface of the inverse function* $z = \operatorname{Ln} w$. Note that the conformal mapping breaks down in the neighbourhood of the points $w = 0$ and $w = \infty$ of the $w$-plane, which are branch points of the function $\operatorname{Ln} w$, where the mapping is not one-to-one.

*Example 3*. Construct a function that maps the strip $0 < \operatorname{Re} z < a$ conformally onto the upper half-plane $\operatorname{Im} w > 0$.

The function $z_1 = \frac{\pi}{a} z$ maps the original strip onto the strip $0 < \operatorname{Re} z_1 < \pi$. The function $z_2 = iz_1$ transforms the resulting strip into the strip $0 < \operatorname{Im} z_2 < \pi$. Finally, the function $w = e^{z_2}$ maps the given strip conformally onto the upper half-plane $\operatorname{Im} w > 0$. Therefore, the function which accomplishes the given conformal mapping may be taken in the form

$$w = e^{i \frac{\pi}{a} z}$$

## *c. Basic principles*

We considered a few elementary examples of functions that map conformally and, with their aid, we solved the basic problem of conformal mapping for a number of elementary domains. More complicated examples require the use of general principles of conformal mapping. Let us investigate these principles. In a number of cases we will confine ourselves solely to a statement of the appropriate propositions without substantiating them rigorously, for this would take us beyond the limits of our course.

(a) *One-to-one correspondence*. It has been pointed out that a conformal mapping of a domain $\mathfrak{G}$ of the complex $z$-plane onto a domain $G$

---

* For construction of the Riemann surface of the function Ln $w$, see Chapter 3, page 103.

of the $w$-plane, which is accomplished by a function $f(z)$ analytic in $\mathfrak{G}$, sets up a one-to-one correspondence between these domains. Thus, the condition of univalence of the function $f(z)$ in the domain $\mathfrak{G}$ is a necessary condition for the conformality of the mapping. It turns out that this condition is also sufficient.

***Theorem 6.3.*** *Let $f(z)$ be a single-valued analytic function in a domain $\mathfrak{G}$ which maps the domain $\mathfrak{G}$ one-to-one onto a domain $G$ of the complex $w$-plane. This mapping is then conformal.*

*Proof.* To prove the theorem it is evidently sufficient to demonstrate that if the conditions of the theorem are fulfilled the derivative of $f(z)$ is nonzero everywhere in the domain $\mathfrak{G}$. Suppose this is not the case, i.e. that in the domain $\mathfrak{G}$ there exists a point $z_0$ at which $f'(z_0) = 0$. Since $f(z)$ is analytic in $\mathfrak{G}$, then by virtue of the supposition its power-series expansion about the point $z_0$ must be of the form

$$f(z) = a_0 + a_k (z - z_0)^k + a_{k+1} (z - z_0)^{k+1} + \dots \tag{6-7}$$

and $k \geqslant 2$ and $a_k \neq 0$. If $f'(z) \not\equiv 0$, then $z_0$ cannot be a limit point of the zeros of the function $f'(z)$. This means that there is a value $\delta'$ such that $f'(z) \neq 0$ at all points $z \neq z_0$ inside the circle $|z - z_0| < \delta'$. Also, it is obviously possible to choose a value $\delta''$ such that we will have the inequality

$$\psi(z) = a_k + a_{k+1}(z - z_0) + \dots \neq 0$$

for $|z - z_0| < \delta''$.

Choosing $\delta = \min\{\delta', \delta''\}$, we have

$$\left.\begin{array}{l} f'(z) \neq 0 \text{ for } z \neq z_0 \\ \psi(z) = a_k + a_{k+1}(z - z_0) + \dots \neq 0 \end{array}\right\} \text{ for } |z - z_0| \leqslant \delta \tag{6-8}$$

From the latter relation, by virtue of the continuity of the function $\psi(z)$, it follows that

$$\min |(z - z_0)^k \psi(z)|_{|z - z_0| = \delta} = m > 0$$

Choose some complex number $\alpha$ which satisfies the condition $|\alpha| < m$. By Rouché's theorem, the analytic function

$$\varphi(z) = (z - z_0)^k \psi(z) - \alpha = f(z) - a_0 - \alpha \tag{6-9}$$

has inside the circle $|z - z_0| \leqslant \delta$ just as many zeros as the function $(z - z_0)^k \psi(z)$. The latter, by the condition (6-8), has in the given circle $k$ zeros; the point $z = z_0$ is its zero of order $k$. Then from (6-9) it follows that the equation

$$f(z) = a_0 + \alpha \tag{6-10}$$

has $k$ roots in the circle $|z - z_0| \leqslant \delta$, and all these roots are simple, since the point $z = z_0$ is not a root of the equation (6-10) and, by (6-8), $f'(z) \neq 0$ at the remaining points of the given circle. This means that at $k$ distinct points of the circle $|z - z_0| \leqslant \delta$ the function $f(z)$ assumes the same value $f(z) = a_0 + \alpha$. But this contradicts the condition of one-to-one mapping of the domain $\mathfrak{G}$ onto the domain $G$, which proves the theorem.

It thus follows from the theorem that we have proved that the condition of univalence of $f(z)$ in the domain $\mathfrak{G}$ is a necessary and sufficient condition for a single-valued function $f(z)$ analytic in the domain $\mathfrak{G}$ to map this domain conformally onto some domain $G$ of the $w$-plane.

(b) *The principle of correspondence of boundaries.* When solving concrete problems in the conformal mapping of a given domain $\mathfrak{G}$ onto a given domain $G$, the usual procedure is to see that the desired function $f(z)$ maps the boundary $\gamma$ of the domain $\mathfrak{G}$ onto the boundary $\Gamma$ of $G$, without specially considering the mapping of the interior points. This can be done by virtue of the so-called principle of boundary correspondence which we will prove below. First, note that if in a domain $\mathfrak{G}$ there is given a single-valued continuous function $w = f(z)$, it is obvious that the function transforms any closed curve $\gamma$ lying entirely in $\mathfrak{G}$ into a closed curve $\Gamma$ in the $w$-plane. We will say that in a mapping of the curve $\gamma$, by the function $f(z)$, the sense of the traversal is preserved if in continuous motion of the point in the positive direction along the curve $\gamma$ the corresponding point goes around the curve $\Gamma$ in the positive direction as well. Let us now examine the principle itself.

***Theorem 6.4.*** *Let there be given, in a finite domain $\mathfrak{G}$ bounded by a contour $\gamma$, a single-valued analytic function $f(z)$ which is continuous in $\bar{\mathfrak{G}}$ and maps the contour $\gamma$ one-to-one onto some contour $\Gamma$ of the complex $w$-plane. Then, if in such a mapping of contours the direction of traversal is preserved, the function $f(z)$ maps the domain $\mathfrak{G}$ conformally onto the interior domain $G$ bounded by the contour $\Gamma$.*

*Proof.* It is evidently sufficient to show that the function $f(z)$ sets up a one-to-one correspondence between the domains $\mathfrak{G}$ and $G$; that is, we have to show that the function $f(z)$ associates with every value $z \in \mathfrak{G}$ a certain point $w \in G$ and for every point $w_1 \in G$ there will be one and only one point $z_1 \in \mathfrak{G}$ such that $f(z_1) = w_1$. To do this, consider two arbitrary points $w_1 \in G$ and $w_2 \notin G$ (Fig. 6.4) and construct in $\mathfrak{G}$ the auxiliary functions

$$\begin{aligned} F_1(z) &= f(z) - w_1, \quad z \in \mathfrak{G} \\ F_2(z) &= f(z) - w_2, \quad z \in \mathfrak{G} \end{aligned} \tag{6-11}$$

Count the number of zeros of these functions in the domain $\mathfrak{G}$ using formula (5-93). Since it is given that a positive traversal of the contour $\Gamma$ corresponds to a positive traversal of the contour $\gamma$, we obtain

$$N[F_1(z)] = \frac{1}{2\pi} \text{ Var } [\arg(f - w_1)]_\gamma = 1 \qquad (6\text{-}12)$$

and

$$N[F_2(z)] = \frac{1}{2\pi} \text{ Var } [\arg(f - w_2)]_\gamma = 0 \qquad (6\text{-}13)$$

Since the choice of the point $w_2$ outside the domain $G$ is arbitrary, it follows from (6-13) that all the values of the function $f(z)$ for $z \in \mathfrak{G}$ belong to $G$. From (6-12) it follows that for any point $w_1 \in G$

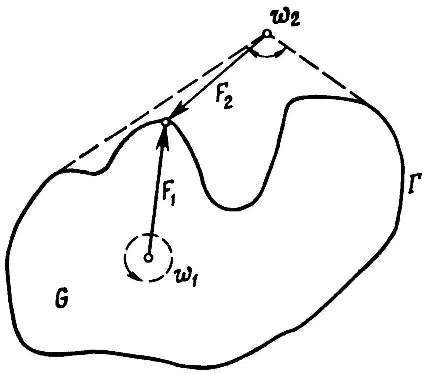


Fig. 6.4

in the domain $\mathfrak{G}$ there is one and only one point $z_1$ for which $f(z_1) = w_1$; this proves that the given mapping is one-to-one. The theorem is proved.

*Note.* If the function $f(z)$ is analytic in the domain $\mathfrak{G}$, with the exception of a unique singular point $z_0$ which is a first-order pole, and the direction of traversal is reversed in a mapping of the boundary of $\mathfrak{G}$ (the contour $\gamma$) onto the contour $\Gamma$ of the $w$-plane, then the function $f(z)$ maps the domain $\mathfrak{G}$ conformally onto the domain $G'$, which is exterior to the contour $\Gamma$, in the $w$-plane (the point $z_0$ corresponds to the point $w = \infty$).

This assertion is proved in a manner similar to that of the preceding theorem; in place of (6-12) and (6-13) we get the relations

$$N[F_1(z)] - 1 = \frac{1}{2\pi} \text{ Var } [\arg(f - w_1)]_\gamma = -1 \qquad (6\text{-}14)$$

and

$$N[F_2(z)] - 1 = \frac{1}{2\pi} \text{ Var } [\arg(f - w_2)]_\gamma = 0 \qquad (6\text{-}15)$$

from which follows the validity of the assertion.

We give the assertion without proof; in a sense it is the converse of the theorem that has just been proved.

***Theorem 6.5.*** *If a function $f(z)$ maps a domain $\mathfrak{G}$ of the complex z-plane conformally onto a bounded domain $G$ of the w-plane, the boundary of which does not contain the point $w = \infty$, then the function $f(z)$ is continuous on the boundary of $\mathfrak{G}$ and generates a continuous one-to-one correspondence between the boundaries $\gamma$ and $\Gamma$ of the domains $\mathfrak{G}$ and $G$.*

(c) *Symmetry principle.* This principle finds numerous applications in the solution of problems of conformal mapping of domains

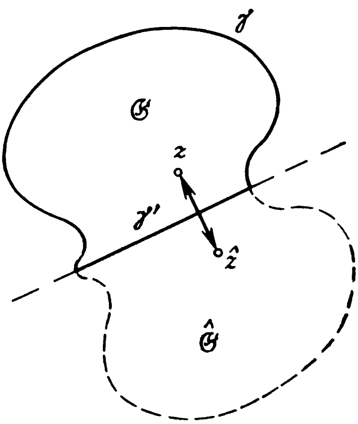


**Fig. 6.5**

whose boundaries have straight-line segments. Let the boundary $\gamma$ of the domain $\mathfrak{G}$ have a straight-line segment $\gamma'$ (Fig. 6.5). The domain $\hat{\mathfrak{G}}$ obtained by means of a mirror reflection of the domain $\mathfrak{G}$ about the straight line on which the segment $\gamma'$ lies will be called a domain symmetric to the domain $\mathfrak{G}$ with respect to $\gamma'$. The symmetry of points of the domains $\mathfrak{G}$ and $\hat{\mathfrak{G}}$ will be denoted by the symbol $z \leftrightarrow \hat{z}$. The symmetry principle may be stated in the form of a theorem.

***Theorem 6.6.*** *Let there be given, in a closed domain $\mathfrak{G}$, the boundary $\gamma$ of which has a straight-line segment $\gamma'$, a continuous function $f(z)$ which maps the domain $\mathfrak{G}$ conformally onto the domain $G$ of the complex w-plane, the segment $\gamma'$ of the boundary $\gamma$ being transformed into a straight-line segment $\Gamma'$ of the boundary $\Gamma$ of the domain $G$. Then, in the domain $\hat{\mathfrak{G}}$, which is symmetric to $\mathfrak{G}$ with respect to the segment $\gamma'$, it is possible to construct a function $\hat{f}(z)$ which is an analytic continuation of the function $f(z)$ from $\mathfrak{G}$ into $\hat{\mathfrak{G}}$ and which maps* *the domain $\check{\mathfrak{G}}$ conformally onto the domain $\check{G}$ of the complex w-plane, which ($\hat{G}$) is symmetric to the domain G with respect to the segment $\Gamma'$.*

Note that the domain $\tilde{\mathfrak{G}} = \mathfrak{G} + \hat{\mathfrak{G}}$ thus obtained can have a segment $\mathfrak{G}_{12}$ belonging simultaneously to the domains $\mathfrak{G}$ and $\hat{\mathfrak{G}}$. Then the complete analytic function $F(z)$ obtained by an analytic continuation of the function $f(z)$ into the domain $\hat{\mathfrak{G}}$ must be considered on an appropriate Riemann surface (the same refers to the domains $G$ and $\hat{G}$).

*Proof.* We associate with each point $z \in \mathfrak{G}$ a point $\hat{z} \in \hat{\mathfrak{G}}$ symmetric to it with respect to the segment $\gamma'$; and with the point $w \in G$, a point $\hat{w} \in \hat{G}$ symmetric to it with respect to the segment $\Gamma'$:

$$z \leftrightarrow \hat{z}, \quad w \leftrightarrow \hat{w} \tag{6-16}$$

In $\hat{\mathfrak{G}}$, define the function $\hat{f}(\hat{z})$ specifying its values for every $\hat{z} \in \hat{\mathfrak{G}}$ according to the scheme $\hat{z} \leftrightarrow z$; $z \rightarrow w = f(z)$; $w \leftrightarrow \hat{w}$; $\hat{f}(\hat{z}) = \hat{w}$. It is readily seen that the constructed function $\hat{f}(\hat{z})$ is analytic in the domain $\hat{\mathfrak{G}}$. Indeed, by the correspondences of (6-16), from the existence of a limit of the difference quotient $\frac{\Delta w}{\Delta z}$ there follows the existence of a limit of the difference quotient $\frac{\Delta \hat{w}}{\Delta \hat{z}}$. The analytic functions $f(z)$, $z \in \mathfrak{G}$ and $\hat{f}(\hat{z})$, $\hat{z} \in \hat{\mathfrak{G}}$, coincide and are continuous on the common segment $\gamma'$ of the boundaries of the domains $\mathfrak{G}$ and $\hat{\mathfrak{G}}$. Therefore, by the principle of analytic continuation, the function $\hat{f}(\hat{z})$ is an analytic continuation of the function $f(z)$ from the domain $\mathfrak{G}$ into the domain $\hat{\mathfrak{G}}$. The first part of the assertion of the theorem is proved. By (6-16), the mapping of the domain $\hat{\mathfrak{G}}$ onto the domain $\hat{G}$ by the function $\hat{f}(\hat{z})$ is one-to-one. Consequently, on the basis of Theorem 6.3, this mapping is conformal. The proof is complete.

*Note.* This theorem remains valid also for the case when the straight-line segment $\gamma'$ in the theorem is replaced by an arc of a circle. In that case, symmetry with respect to the arc of the circle is to be understood as a mirror reflection in the given circle carried out by an inversion transformation. It will be shown below that it is always possible to map a domain $\mathfrak{G}$ conformally onto a new domain $\mathfrak{G}_1$ so that tl e segment $\gamma'$ of the arc of a circle which is part of the boundary $\gamma$ of the domain $\mathfrak{G}$ is transformed into a straight-line segment $\gamma_1'$, which is part of the boundary $\gamma_1$ of the domain $\mathfrak{G}_1$. This proves the truth of the assertion.

### *d. Riemann's theorem*

Up to now we have reasoned on the assumption that there exists a function $f(z)$ that maps a given domain $\mathfrak{G}$ of the complex $z$-plane conformally onto a given domain $G$ of the complex $w$-plane. We will now formulate conditions which will guarantee the existence and uniqueness of such a mapping. This theorem, which is a fundamental theorem of the theory of conformal mappings, was proved by Riemann in 1851. Proof of the existence of a conformal mapping goes beyond the scope of this course and so we will confine ourselves to a statement of the theorem.*

***Theorem 6.7 (Riemann's theorem).*** *Every singly connected domain $\mathfrak{G}$ of the complex z-plane whose boundary consists of more than one point may be conformally mapped onto the interior of the unit circle $|w| < 1$ of the w-plane.*

It obviously follows from this theorem that it is possible to map a given singly connected domain $\mathfrak{G}$ of the $z$-plane conformally onto a given singly connected domain $G$ of the complex $w$-plane if

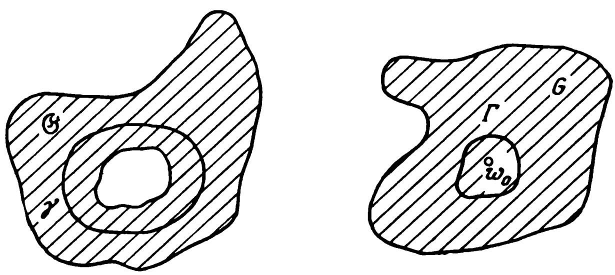


Fig. 6.6

the boundary of each of these domains consists of more than one point. Indeed, mapping the domains $\mathfrak{G}$ and $G$ onto the auxiliary circle $|\zeta| < 1$ (which is possible by Riemann's theorem), we get the desired mapping.

The condition of single connectivity of the domains $\mathfrak{G}$ and $G$ is essential, for the supposition of the possibility of a conformal mapping of a multiply connected domain $\mathfrak{G}$ onto a singly connected domain $G$ leads to a contradiction. Indeed, let us take in $\mathfrak{G}$ a closed contour $\gamma$, inside of which lie the boundary points of the domain $\mathfrak{G}$. The contour $\gamma$ is mapped onto some closed curve $\Gamma$ lying completely in the singly connected domain $G$ (Fig. 6.6). Make $\Gamma$ shrink to some interior point $w_0$ of the domain $G$; then by virtue of the continuity of the mapping, the contour $\gamma$ should also shrink to some interior

* A detailed proof is given in [1].

point $z_0$ of the domain $\mathfrak{G}$, all the while remaining inside this domain; but this is obviously impossible due to the multiple connectivity of the domain $\mathfrak{G}$ and to the indicated choice of the contour $\gamma$. Thus, a conformal mapping of a multiply connected domain onto a singly connected domain is impossible. However, as will be shown below, it is possible in a number of cases to effect a conformal mapping of domains of equal connectivity.

Let us now examine the conditions which uniquely define a function capable of carrying out a given conformal mapping. It is clear that such conditions are necessary, since, as is evident from earlier examples, the unit circle can be conformally mapped onto itself with the aid of the most elementary linear transformation, which consists in a rotation of the complex plane. Therefore, if the function $f(z)$ maps conformally a given domain $\mathfrak{G}$ onto the unit circle, then any function obtained from $f(z)$ by means of the indicated linear transformation will conformally map the domain $\mathfrak{G}$ onto the same unit circle.

**Theorem 6.8.** *A function $f(z)$ which maps conformally a given singly connected domain $\mathfrak{G}$ (the boundary of which consists of more than one point) onto the unit circle $|w| < 1$ so that $f(z_0) = 0$ and $\arg f'(z_0) = \alpha_0$ (where $z_0 \in \mathfrak{G}$ and $\alpha_0$ is a given real number) is defined uniquely.*

*Proof*. Suppose there are two different functions $w_1 = f_1(z)$ and $w_2 = f_2(z)$ in the domain $\mathfrak{G}$, effecting the given conformal mapping, i.e.,

$$f_1(z_0) = 0, \ \arg f_1'(z_0) = \alpha_0, \ |f_1(z)|_\gamma = 1$$

$$f_2(z_0) = 0, \ \arg f_2'(z_0) = \alpha_0, \ |f_2(z)|_\gamma = 1$$

We note that by Theorem 6.5, the functions $w_1 = f_1(z)$ and $w_2 = f_2(z)$ set up a one-to-one and continuous correspondence between the boundary $\gamma$ of the domain $\mathfrak{G}$ and the circles $|w_1| = 1$ and $|w_2| = 1$, respectively.

Since a one-to-one correspondence is established in a conformal mapping, this means that there is also established a one-to-one correspondence between the points of the unit circles $|w_1| \leqslant 1$ and $|w_2| \leqslant 1$. Hence, the established correspondences define the analytic function $w_2 = \varphi(w_1)$, which maps the unit circle $|w_1| < 1$ conformally onto the unit circle $|w_2| < 1$, and

$$\varphi(0) = 0, \ |\varphi(w_1)|_{|w_1|=1} = 1$$

Note that, besides, by virtue of the one-to-one correspondence of the domains $|w_1| < 1$ and $|w_2| < 1$ we have the condition

$$\varphi(w_1) \neq 0 \quad \text{for } w_1 \neq 0$$

Computing the value of the derivative $\frac{dw_2}{dw_1}$ by the rule for determining the derivative of a composite function, we get

$$\left.\frac{d\varphi}{dw_1}\right|_{w_1=0}=\left.\frac{dw_2}{dw_1}\right|_{w_1=0}=\lim_{\Delta z\to 0}\frac{\frac{\Delta w_2}{\Delta z}}{\frac{\Delta w_1}{\Delta z}}=\frac{k_2 e^{i\alpha_0}}{k_1 e^{i\alpha_0}}=\frac{k_2}{k_1}>0$$

Whence it follows that the derivative $\frac{dw_2}{dw_1}$ at the point $w_1 = 0$ is a positive real number. Let us consider the auxiliary function de fined for $|w_1| \leqslant 1$

$$\psi(w_1)=\frac{1}{w_1}\varphi(w_1) \qquad (6\text{-}17)$$

The function $\psi(w_1)$ is obviously a single-valued analytic function in the domain $0<|w_1|<1$. The point $w_1 = 0$ is a removable singular point of this function. We redefine $\psi(w_1)$ with respect to continuity for $w_1 = 0$. Expand $\varphi(w_1)$ in a Taylor series about $w_1 = 0$:

$$w_2=\varphi(w_1)=\varphi(0)+\left.\frac{d\varphi}{dw_1}\right|_{w_1=0}w_1+\ \dots\ =\left.\frac{d\varphi}{dw_1}\right|_{w_1=0}w_1+\ \dots$$

Taking the limit as $w_1 \to 0$, we have

$$\psi(0)=\lim_{w_1\to 0}\frac{\varphi(w_1)}{w_1}=\left.\frac{d\varphi}{dw_1}\right|_{w_1=0}=\frac{k_2}{k_1}>0 \qquad (6\text{-}18)$$

The function $\psi(w_1)$ is continuous in the closed domain $|w_1| \leqslant 1$, and in this domain $\psi(w_1) \neq 0$ and

$$|\psi(w_1)|_{|w_1|=1}=1 \qquad (6\text{-}19)$$

By the maximum- and minimum-modulus principle of an analytic function, there follows from (6-19) that

$$|\psi(w_1)| \equiv 1 \text{ for } |w_1| \leqslant 1$$

whence, by footnote on page 53 (Chapter 1) we find that

$$\psi(w_1) \equiv \text{constant for } |w_1| \leqslant 1 \qquad (6\text{-}20)$$

In order to find this constant, note that by (6-18) it is equal to $\frac{k_2}{k_1}$, i.e. it is a positive real number. According to (6-19), the absolute value of this number is unity, which implies that $\psi(w_1) \equiv 1$. Hence, $w_2 = \varphi(w_1) \equiv w_1$. This is proof that there do not exist two different functions accomplishing a specific conformal mapping of a given domain $\mathfrak{G}$ onto the interior of the unit circle.

*Note*. The above-stated conditions for a unique definition of a function $f(z)$ accomplishing a conformal mapping of a given singly connected domain $\mathfrak{G}$ onto the interior of the unit circle $|w| < 1$

may be replaced by the requirement that three boundary points of the boundary $\gamma$ of the domain $\mathfrak{G}$ correspond to three points of the circle $|w| = 1$.

We confine ourselves to a statement of the assertion without giving its proof.

We have examined a number of basic general properties of conformal mapping. However, these considerations do not yield general procedures for solving the basic problem of constructing a conformal mapping of a given domain $\mathfrak{G}$ of the complex $z$-plane onto a given domain $G$ of the $w$-plane. It is not possible to indicate any such procedure in the most general case. In solving concrete problems one has to resort to a variety of special methods. In this, a sufficiently full grasp of the geometric properties of a number of functions of a complex variable that are most often used in solving practical problems will be of great help.

## 6.2. Linear-Fractional Function

A linear-fractional function is the function of a complex variable of the form

$$w = f(z) = \frac{a+bz}{c+dz} \tag{6-21}$$

where $a$, $b$, $c$, $d$ are given complex constants, which must obviously satisfy the condition

$$\frac{a}{c} \neq \frac{b}{d} \tag{6-22}$$

since otherwise the function $f(z)$ would be identically constant. Without loss of generality, it may be taken that $b \neq 0$ and $d \neq 0$, for otherwise $w$ would be transformed into the already studied linear function and the function $w = \frac{1}{\zeta}$. And so we can write (6-21) in the equivalent form

$$w = f(z) = \lambda \frac{\alpha+z}{\beta+z}, \quad \lambda = \frac{b}{d}, \quad \alpha = \frac{a}{b}, \quad \beta = \frac{c}{d}, \quad \alpha \neq \beta \tag{6-23}$$

The function (6-21), (6-23) is a single-valued analytic function in the extended complex $z$-plane having one singularity—a first-order pole $z_0 = -\frac{c}{d} = -\beta$. The inverse function

$$z = \frac{\lambda\alpha - \beta w}{-\lambda + w} \tag{6-24}$$

is a linear-fractional function defined in the extended $w$-plane. Here, the point $z_0 = -\frac{c}{d} = -\beta$ is transformed into the point $w = \infty$ and the point $z = \infty$ into the point $w_0 = \lambda = \frac{b}{d}$.

Let us find the derivative of the function $w = f(z)$

$$f'(z) = \lambda \frac{\beta - \alpha}{(\beta + z)^2} \neq 0 \tag{6-25}$$

By the condition (6-22), the derivative of a linear-fractional function is nonzero at all finite points of the $z$-plane. This means that *a linear-fractional function maps the z-plane conformally onto the w-plane*. The conformality of the mapping at points at infinity is readily verified by the above-mentioned method.

The expression of the linear-fractional function includes three arbitrary parameters $\lambda$, $\alpha$, $\beta$; there are thus an infinity of linear-fractional functions that map the extended $z$-plane onto the extended $w$-plane conformally. It is natural to pose the question of the conditions that uniquely define a linear-fractional function.

***Theorem 6.9.*** *A linear-fractional function is uniquely defined by specification of a correspondence between three different points of the z-plane and three different points of the w-plane.*

*Proof.* We have to prove that the conditions

$$f(z_1) = w_1, \; f(z_2) = w_2, \; f(z_3) = w_3 \tag{6-26}$$

where $z_1$, $z_2$, $z_3$ and $w_1$, $w_2$, $w_3$—given complex numbers—uniquely define the values of the parameters $\lambda$, $\alpha$, $\beta$. Form the expressions

$$w_1 - w_3 = \lambda \frac{(z_1 - z_3)(\beta - \alpha)}{(\beta + z_1)(\beta + z_3)} \tag{6-27}$$

$$w_2 - w_3 = \lambda \frac{(z_2 - z_3)(\beta - \alpha)}{(\beta + z_2)(\beta + z_3)} \tag{6-28}$$

Dividing (6-27) by (6-28), we get

$$\frac{w_1 - w_3}{w_2 - w_3} = \frac{z_1 - z_3}{z_2 - z_3} \cdot \frac{\beta + z_2}{\beta + z_1} \tag{6-29}$$

For an arbitrary point $z$ we can write a similar relation

$$\frac{w_1 - w}{w_2 - w} = \frac{z_1 - z}{z_2 - z} \cdot \frac{\beta + z_2}{\beta + z_1} \tag{6-30}$$

Eliminating the parameter $\beta$ from the relations (6-29) and (6-30), we finally get

$$\frac{w_1 - w}{w_2 - w} : \frac{w_1 - w_3}{w_2 - w_3} = \frac{z_1 - z}{z_2 - z} : \frac{z_1 - z_3}{z_2 - z_3} \tag{6-31}$$

The relation (6-31) is an implicit expression of the desired linear-fractional function. By solving (6-31) for $w$, we evidently get an explicit expression of the coefficients $\lambda$, $\alpha$, $\beta$ of the linear-fractional function in terms of the given numbers $z_1$, $z_2$, $z_3$, $w_1$, $w_2$, $w_3$, which proves the theorem.

Note that since a linear-fractional function effects a conformal mapping of the extended $z$-plane onto the extended $w$-plane, it follows that one of the points $z_i$ and one of the points $w_i$, the specification of which defines the linear-fractional function, may be points at infinity.

Let us consider the geometric properties of a mapping by a linear-fractional function. To do this, we slightly transform the expression (6-23) to

$$f(z)=\lambda\left(\frac{\alpha-\beta}{\beta+z}+1\right) \tag{6-32}$$

and introduce the auxiliary functions

$$z_1=\beta+z,\ z_2=\frac{1}{z_1},\ z_3=\lambda(\alpha-\beta)z_2+\lambda \tag{6-33}$$

From the relations (6-33) it follows that a mapping by a linear-fractional function is a collection of elementary mappings by the linear functions $z_1$ and $z_3$ and by the function $\frac{1}{z}$ considered in Chapter 1. Thus, the mapping at hand is made up of stretchings, rotations and translations of the complex plane, and also of the inversion transformation in a circle. This mapping has a number of important properties which we will investigate.

***Theorem 6.10 (Circular property of a linear-fractional function).*** *A linear-fractional function transforms circles in the z-plane into circles in the w-plane.* We include straight lines in the family of circles, regarding them as circles of infinite radius.

*Proof.* It is obviously sufficient to show that the inversion transformation by the function $w=\frac{1}{z}$ possesses the circular property, since preservation of the circle in a linear transformation does not give rise to any doubt. Let us consider an arbitrary circle whose equation in the $z$-plane is

$$A(x^2+y^2)+Bx+Cy+D=0 \tag{6-34}$$

where $A, B, C$, and $D$ are real numbers and $A\geqslant 0$, $B^2+C^2=4AD$. Clearly, for $A=0$ we get a straight line; for $D=0$ the circle (6-34) passes through the origin (point $z=0$). In the transformation by the function $w=u+iv=\frac{1}{z}$, the coordinates $x$, $y$ are connected with the coordinates $u$, $v$ by the relations

$$x=\frac{u}{u^2+v^2},\quad y=-\frac{v}{u^2+v^2} \tag{6-35}$$

Therefore, the circle (6-34) in the new coordinates is of the form

$$D(u^2+v^2)+Bu-Cv+A=0 \tag{6-36}$$

This proves the theorem.

Note that for $D = 0$ equation (6-36) is the equation of a straight line, i.e. the circle passing through the point $z = 0$ is mapped into a straight line by the function $w = \frac{1}{z}$.

The foregoing property of a linear-fractional function is widely employed in solving many concrete problems of conformal mapping associated with the mapping of domains with circular boundaries. Indeed, suppose we have to map conformally a domain $\mathfrak{G}$ bounded by the circle $\gamma$ (in the $z$-plane) onto the domain $G$ bounded by the circle $\Gamma$ in the $w$-plane. It is known that the position of a circle in a plane is completely defined by specification of three points.

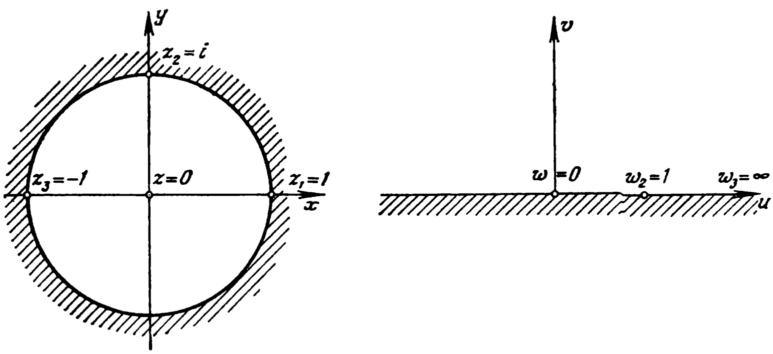


Fig. 6.7

On the other hand, by Theorem 6.9, by specifying the correspondence of three points $z_k$ in the $z$-plane lying on the circle $\gamma$ to three points $w_k$ of the $w$-plane lying on the circle $\Gamma$ we fully define a linear-fractional function which maps the $z$-plane conformally onto the $w$-plane. Then, according to Theorem 6.10, the circle $\gamma$ will be transformed into the circle $\Gamma$. If the correspondence of points $z_k$ and $w_k$ is chosen so as to preserve the direction of traversal, then by Theorem 6.4 the given function maps the domain $\mathfrak{G}$ conformally onto the domain $G$. Note that the domain exterior to the circle $\gamma$ in the $z$-plane is conformally mapped onto the domain exterior to the circle $\Gamma$ in the $w$-plane. If the correspondence of points $z_k$ and $w_k$ is established so that the traversals of the circles $\gamma$ and $\Gamma$ are in opposite senses, then the domain $\mathfrak{G}$ is conformally mapped onto the domain exterior to the circle $\Gamma$ in the $w$-plane.

*Example 1.* Find the function that maps the unit circle $|z| < 1$ conformally onto the upper half-plane $\operatorname{Im} w > 0$.

We start by establishing the following correspondence of the boundary points of the given domains (Fig. 6.7):

$$z_1 = 1 \to w_1 = 0 \tag{6-37'}$$

$$z_2 = i \to w_2 = 1 \tag{6-37''}$$

$$z_3 = -1 \to w_3 = \infty \tag{6-37'''}$$

and find the coefficients $\lambda$, $\alpha$, $\beta$ of the linear-fractional function which generates the desired mapping. From the conditions (6-37′) and (6-37‴) it is easy to see that the values of $\alpha$ and $\beta$ are determined at once, and then the desired function takes the form

$$w = \lambda \frac{z-1}{z+1}$$

The last coefficient $\lambda$ is found from the condition (6-37″):

$$1 = \lambda \frac{i-1}{i+1}$$

whence $\lambda = -i$. Thus, the function which carries out the desired mapping is of the form

$$w = i\frac{1-z}{1+z} \tag{6-38}$$

Note that the function (6-38) maps the domain $|z| > 1$ conformally onto the lower half-plane $\text{Im}\, w < 0$.

It follows from this example that construction of the desired linear-fractional function is simplest when the given points of the $w$-plane are the points $w = 0$ and $w = \infty$, for then the values of the coefficients $\alpha$ and $\beta$ are determined at once.

The next property of linear-fractional functions consists in preservation of points symmetric about a circle.

It will be recalled that the points $P$ and $P'$ are called symmetric with respect to a circle $C$ if they lie on a common ray passing through the centre $O$ of the circle $C$, and the product of their distances from the centre is equal to the square of the radius of the circle: $OP \cdot OP' = R^2$. We have

***Theorem 6.11.*** *In a mapping by a linear-fractional function, points symmetric with respect to any circle are transformed into points symmetric with respect to the image of the circle.*

*Proof*. We take advantage of the following auxiliary propositions of elementary geometry.

*Proposition 1*. Every circle $C'$ passing through the points $P$ and $P'$ is orthogonal to the circle $C$.

Indeed, drawing a ray $OP'$ and a radius $OA$ to the point of intersection of the circles $C$ and $C'$ (Fig. 6.8), we get

$$OP \cdot OP = (OA)^2 = R^2$$

by virtue of the symmetry of the points $P$ and $P'$ with respect to the circle $C$. But this, by a familiar theorem of elementary geometry,* implies that $OA$ is a tangent to the circle $C'$ drawn from point $O$, whence it follows that $C' \perp C$.

*Proposition 2.* Two mutually intersecting circles $C'$ and $C''$ orthogonal to one and the same circle $C$ intersect at the points $P$ and $P'$, which are symmetric with respect to the circle $C$.

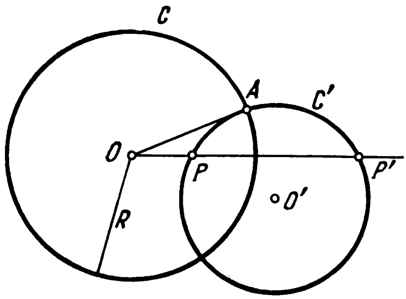


Fig. 6.8

Through the point $P$ of intersection of the circles $C'$ and $C''$, which lies inside $C$, draw a ray $OP$. Suppose that $OP$ intersects $C'$

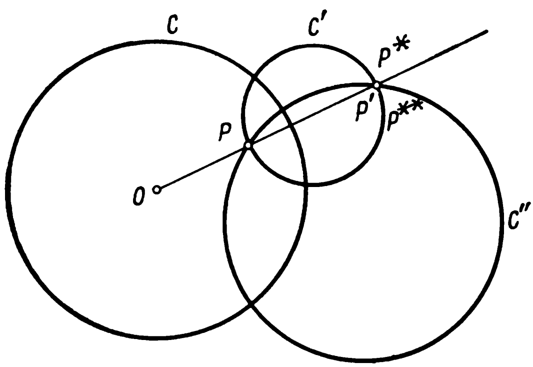


Fig. 6.9

and $C''$ at different points, $P^*$ and $P^{**}$, respectively (Fig. 6.9). Since the circles $C'$ and $C''$ are orthogonal to $C$, by the above-mentioned theorem of elementary geometry we have the relations

$$OP \cdot OP^* = R^2 \tag{6-39}$$

$$OP \cdot OP^{**} = R^2 \tag{6-40}$$

* The product of segments of a secant drawn from an exterior point of a circle is equal to the square of the segment of the tangent drawn from the same point.

But since the points $P^*$ and $P^{**}$ lie on one ray, the equalities (6-39) and (6-40) are only possible when the points $P^*$ and $P^{**}$ coincide, $P^* = P^{**} = P'$; this proves the proposition.

Let us now prove the theorem. Let the points $P$ and $P'$ be symmetric with respect to the circle $C$. Through these points draw two auxiliary circles $C'$ and $C''$. By Proposition 1, the circles $C'$ and $C''$ are orthogonal to $C$. In a conformal mapping accomplished by some linear-fractional function, the circles $C$, $C'$ and $C''$ will be transformed respectively into the circles $K$, $K'$ and $K''$ and the circles $K'$ and $K''$ will be orthogonal to $K$. The points $P$ and $P'$ of intersection of the circles $C'$ and $C''$ will be transformed into the points $Q$ and $Q'$ of intersection of their images—the circles $K'$ and $K''$. But by Proposition 2, the points $Q$ and $Q'$ must be symmetric with respect to $K$, which proves the theorem.

It is obvious that the theorem holds true in the case of circles of infinite radius (straight lines) as well.

This theorem finds numerous applications in the solution of concrete problems of conformal mapping, and in the future we will repeatedly resort to it. Here, we confine ourselves to two examples.

*Example 2.* Find a function that will map conformally the unit circle $|z| < 1$ onto itself so that a given interior point $z_0$ is transformed into the centre of the circle.

The problem can obviously be solved by using a linear-fractional function. Then the point $z_0$ and the point $z_1$, symmetric to it with respect to the circle $|z| = 1$, will be transformed into points symmetric with respect to the circle $|w| = 1$. But since a point symmetric to the centre of a circle is the point at infinity and the point $z_0$ must be transformed into the point $w = 0$, it follows that the point $z_1$ will have to transform into the point $w = \infty$. Hence, the desired linear-fractional function has the form

$$w = \lambda \frac{z - z_0}{z - z_1} \tag{6-41}$$

Since $z_1 = \frac{1}{\bar{z}_0}$, then (6-41) can be rewritten as

$$w = \lambda \bar{z}_0 \frac{z - z_0}{z\bar{z}_0 - 1} \tag{6-42}$$

So that in the mapping (6-42) the circle $|z| = 1$ is also transformed into the circle $|w| = 1$ of unit radius, the following condition must hold:

$$|\lambda \bar{z}_0| \cdot \left| \frac{e^{i\varphi} - z_0}{e^{i\varphi}\bar{z}_0 - 1} \right| = |\lambda \bar{z}_0| \left| \frac{e^{i\varphi} - z_0}{\bar{z}_0 - e^{-i\varphi}} \right| = |\lambda \bar{z}_0 = 1$$

This implies $\lambda\bar{z}_0 = e^{i\alpha}$, where $\alpha$ is an arbitrary real number, and the solution of our problem is obtained in the form

$$w = e^{i\alpha}\frac{z - z_0}{z\bar{z}_0 - 1} \tag{6-43}$$

Note that we have obtained a solution defined to within one arbitrary parameter $\alpha$, which obviously determines the rotation of the circle $|w| = 1$ about the centre. Specification of the value of the argument of the derivative of the function $w$ at the point $z = z_0$ completely defines the function $w$.

*Example 3.* Find a function that maps conformally an eccentric annulus onto a concentric annulus.

Let it be required to construct a conformal mapping of a domain bounded by two circles with noncoinciding centres (Fig. 6.10) onto

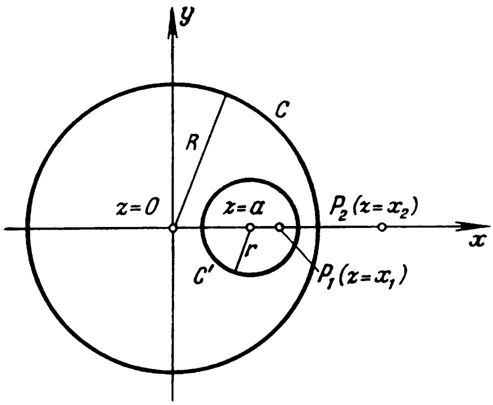


Fig. 6.10

some concentric annulus. Since we are dealing with doubly connected domains, Riemann's theorem on the existence of a conformal mapping does not hold here and, as we will see, one cannot arbitrarily specify a ratio of the radii of circles of a concentric annulus, onto which it is required to map conformally a given eccentric annulus. For future convenience, let us suppose that the centre of the larger circle $C$ lies at the point $z = 0$, its radius is $R$, and the centre of the smaller circle $C'$, of radius $r$, lies at the point $z = a$ on the real axis. We find the points $P_1$ and $P_2$, which are simultaneously symmetric with respect to both circles $C$ and $C'$. These points clearly lie on the real axis (Fig. 6.10). Then their abscissas $x_1$ and $x_2$ must satisfy the relations

$$(x_1 - a)(x_2 - a) = r^2 \tag{6-44}$$

$$x_1 \cdot x_2 = R^2 \tag{6-45}$$

From (6-44) and (6-45) it follows that $x_1$ and $x_2$ are roots of the quadratic equation

$$ax^2 - (R^2 - r^2 + a^2)\,x + aR^2 = 0 \qquad (6\text{-}46)$$

The discriminant of this equation $(R^2 - r^2 + a^2)^2 - 4a^2R^2$ is positive since the obvious relation $R - r > a$ holds. We construct the linear-fractional function

$$w = \lambda \frac{z - x_1}{z - x_2} \qquad (6\text{-}47)$$

where $x_1$ and $x_2$ are the abscissas of the points $P_1$ and $P_2$ found from equation (6-46). The function (6-47) will map the circles $C$ and $C'$ onto some circles $K$ and $K'$ of the $w$-plane and it will map the point $P_2$, which is exterior to the circles $C$ and $C'$, into the point $w = \infty$. The point $P_1$, which is symmetric to the point $P_2$ with respect to the circles $C$ and $C'$, must be transformed into a point that is symmetric to the point $w = \infty$ with respect to the circles $K$ and $K'$. But a point that is symmetric to the point at infinity is the centre of the circle. Hence, in the mapping (6-47) the point $P_1$ will be transformed into the common centre of the circles $K$ and $K'$. The desired mapping has been constructed. Note that in the expression (6-47) the definition of the parameter $\lambda$ was arbitrary; however, any variation in the parameter only results in a similarity stretching of the $w$-plane, but this cannot change the ratio of the radii of the circles of the concentric annulus thus obtained.

To conclude this section, let us examine the problem of applying a linear-fractional function in the construction of conformal mappings of lunes. *A lune (two-sided polygon) is a plane figure formed by the intersection of the arcs of two circles* of different radii, generally speaking (Fig. 6.11). It is clear that the angles at the vertices of the lune are equal. Let there be given a lune with vertices at the points $A\ (z_1)$ and $B\ (z_2)$ and angle $\alpha$ at the vertex, and let it be required to construct a conformal mapping of the interior domain of the given lune onto the upper half-plane $\operatorname{Im} w > 0$. Consider the auxiliary function

$$\zeta = \frac{z_1 - z}{z_2 - z} \qquad (\zeta = \xi + i\eta) \qquad (6\text{-}48)$$

The linear-fractional function (6-48) maps the extended $z$-plane conformally onto the extended $\zeta$-plane; the point $z = z_1$ is transformed to the point $\zeta = 0$, and the point $z = z_2$ into the point $\zeta = \infty$. By virtue of the circular property of a linear-fractional function, the circles forming the lune in the mapping (6-48) are also transformed into circles. But the circle passing through the points $\zeta = 0$ and $\zeta = \infty$ has an infinitely large radius. This means that in the mapping (6-48) the sides of the lune will be transformed into rays ($I$ and $II$) emanating from the point $\zeta = 0$, and the angle between these rays will be equal to the angle $\alpha$ at the vertex of the lune (Fig. 6.12). Thus, the function (6-48) performs a conformal mapping of the given lune in the $z$-plane onto a sector with central

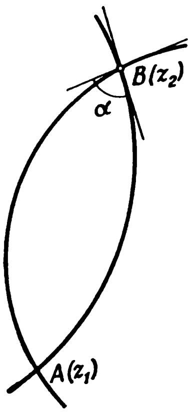


Fig. 6.11

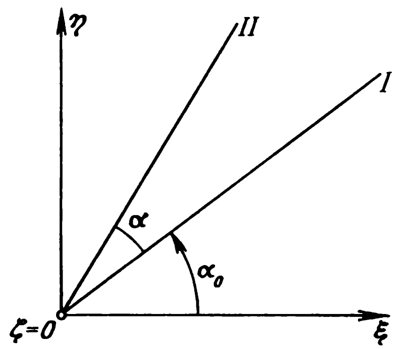


Fig. 6.12

angle $\alpha$ in the $\zeta$-plane, and the ray $I$ forms with the positive $\xi$-axis an angle $\alpha_0$ whose value is determined by the position of the vertices $A$ and $B$ of the lune. As we have seen (Chapter 6, page 160), the function

$$w = \zeta^{\frac{\pi}{\alpha}} \tag{6-49}$$

which is a direct analytic continuation of the real function $x^{\frac{\pi}{\alpha}}$, $x > 0$, maps the domain inside the sector $\alpha_0 < \arg \zeta < \alpha_0 + \alpha$ conformally onto the half-plane $\frac{\alpha_0}{\alpha}\pi < \arg w < \frac{\alpha_0}{\alpha}\pi + \pi$. It now remains to transform the half-plane thus obtained to the half-plane $\operatorname{Im} w > 0$. To do this, it is sufficient to rotate the entire plane as a whole through the angle $-\frac{\alpha_0}{\alpha}\pi$. This may be done by multiplying the function (6-49) by the complex number $e^{-i\frac{\alpha_0}{\alpha}\pi}$. Finally, then, the desired function that will map conformally the lune $AB$ onto the upper half-plane $\operatorname{Im} w > 0$ takes the form

$$w = e^{-i\frac{\alpha_0}{\alpha}\pi} \left(\frac{z_1 - z}{z_2 - z}\right)^{\frac{\pi}{\alpha}} \tag{6-50}$$

Note that the conformality of mapping is violated at the points $z_1$ and $z_2$.

*Example 4*. Construct a conformal mapping of the upper half of the circle $|z| < 1$, $\operatorname{Im} z > 0$, onto the upper half-plane $\operatorname{Im} w > 0$.

Clearly, the given domain is a lune with vertices at the points $z_1 = -1$ and $z_2 = 1$, and angle $\alpha = \frac{\pi}{2}$ at the vertex. The auxiliary function

$$\zeta = \frac{1+z}{1-z} \tag{6-51}$$

maps this lune conformally onto the first quadrant of the $\zeta$-plane and the function

$$w = \left(\frac{1+z}{1-z}\right)^2 \tag{6-52}$$

yields the desired mapping.

## 6.3. Zhukovsky's Function

The function of the complex variable

$$w = f(z) = \frac{1}{2}\left(z + \frac{1}{z}\right) \tag{6-53}$$

is called Zhukovsky's function. It was widely used by N. E. Zhukovsky in the solution of numerous problems in hydro- and aerodynamics.

The function (6-53) is clearly analytic in the entire complex plane, except at the point $z = 0$, which is a first-order pole of the given function. Computing the derivative of the function (6-53), we get

$$f'(z) = \frac{1}{2}\left(1 - \frac{1}{z^2}\right) \tag{6-54}$$

Whence it follows that the derivative of the Zhukovsky function is nonzero at all points of the $z$-plane except at the points $\pm 1$. Thus, a mapping by this function is conformal everywhere except at these two points. Let us find the domains of univalence of the Zhukovsky function. Suppose that two distinct points of the complex plane $z_1 \neq z_2$ are transformed by the function $f(z)$ into one and the same point of the $w$-plane, i.e.,

$$z_1 + \frac{1}{z_1} = z_2 + \frac{1}{z_2}$$

or

$$z_1 - z_2 = \frac{z_1 - z_2}{z_1 \cdot z_2} \tag{6-55}$$

Since $z_1 \neq z_2$, from the relation (6-55) it follows that

$$z_1 \cdot z_2 = 1 \tag{6-56}$$

The relation obtained implies that the domains of univalence of Zhukovsky's function are, in particular, the domains inside ($|z|<1$) and outside ($|z|>1$) of the unit circle. Both of these domains are mapped conformally by the function (6-53) onto one and the same domain of the $w$-plane. To determine this domain, consider the mapping of the circles $|z| = r_0$ by the function (6-53). To do this, we take the exponential form of complex numbers: $z = re^{i\varphi}$ and find the expression for the real and imaginary parts of the function (6-53):

$$u(r, \varphi) = \frac{1}{2}\left(r+\frac{1}{r}\right)\cos\varphi, \quad v(r, \varphi) = \frac{1}{2}\left(r-\frac{1}{r}\right)\sin\varphi \tag{6-57}$$

Putting $r = r_0$ and eliminating the parameter $\varphi$, we obtain

$$\frac{u^2}{\frac{1}{4}\left(r_0+\frac{1}{r_0}\right)^2} + \frac{v^2}{\frac{1}{4}\left(r_0-\frac{1}{r_0}\right)^2} = 1 \tag{6-58}$$

From the relation (6-58) it follows that the function (6-53) maps the concentric circles $|z| = r_0$ conformally onto ellipses. It will readily be seen that the foci of all the ellipses (6-58) lie at the same points of the real $u$-axis:

$$c = \pm 1 \tag{6-59}$$

Thus, the function (6-53) maps the family of concentric circles $|z| = r_0$ of the $z$-plane onto the family of confocal ellipses of the $w$-plane. Here, if $r_1 < 1$, then the positive direction of traversal around the circle $|z| = r_1$ is associated with a negative direction of traversal about the ellipse (6-58); if $r_2 = \frac{1}{r_1} > 1$, then with the positive direction about the circle $|z| = r_2$ we associate a positive direction of traversal about the ellipse (6-58). As $r_1 \to 1$ the ellipse (6-58) degenerates into the segment $[-1, 1]$ of the real $u$-axis traversed twice. As $r_1 \to 0$, the ellipse (6-58) is transformed into a circle of infinitely large radius. Thus, *the function* (6-58) *maps the domain inside the unit circle* $|z| < 1$ *in the z-plane conformally onto the w-plane cut along the segment* $[-1, 1]$ *of the real axis.* The boundary of the domain—the circle $|z| = 1$—is mapped onto this segment, the upper semicircle being mapped onto the lower lip, and the lower one onto the upper lip of the cut. Analogously, the domain $|z| > 1$ outside the unit circle in the $z$-plane is mapped onto the second sheet of the $w$-plane cut along the segment $[-1, 1]$ of the real axis, the upper semicircle $|z| = 1$, $\operatorname{Im} z > 0$ being mapped onto the upper lip, and the lower semicircle $|z| = 1$, $\operatorname{Im} z < 0$, onto the lower lip of the cut. Thus, Zhukovsky's function (6-53) maps the extended $z$-plane conformally onto the Riemann surface of the inverse function

$$z = \varphi(w) = w + \sqrt{w^2 - 1} \tag{6-60}$$

The Riemann surface of the function (6-60) is a two-sheeted surface made up of two sheets of the $w$-plane cut along the segment $[-1, 1]$ of the real axis. The lower lip of the cut of one sheet is joined to the upper lip of the cut of the other sheet and conversely. The function (6-60) is a single-valued analytic function on its Riemann surface with two branch points $w = \pm 1$, upon going around each of which we move from one sheet of this Riemann surface to the other sheet. Note that in a simultaneous traversal of both branch points $w = \pm 1$ around a closed curve that does not intersect the segment $[-1, 1]$, we are all the time on one and the same sheet.

Thus, the functions (6-53) and (6-60) establish a one-to-one correspondence between the extended $z$-plane and the given Riemann surface. The mapping defined by these functions is conformal everywhere except at the points $z = \pm 1$, at which the derivative of the function (6-53) is zero. Note that these points are associated with $w = \pm 1$, the branch points of the function (6-60), which is the inverse of the function (6-53).

In conclusion, let us find the image of the rays $\arg z = \varphi_0$ in the mapping defined by the Zhukovsky function. To do this, eliminate from the relations (6-57) the parameter $r$ and put $\varphi = \varphi_0$. Then

$$\frac{u^2}{\cos^2 \varphi_0} - \frac{v^2}{\sin^2 \varphi_0} = 1 \tag{6-61}$$

The relation (6-61) implies that in the mapping (6-53), segments of the rays $\arg z = \varphi_0$ are transformed into branches of the hyperbola (6-61). Observe that for any value of $\varphi_0$ the foci of this hyperbola lie in the points $\pm 1$. Thus the Zhukovsky function defines a transformation of the orthogonal system of polar coordinates in the $z$-plane into an orthogonal curvilinear system of coordinates, whose coordinate lines are the confocal families of ellipses (6-58) and hyperbolas (6-61).

It has already been pointed out that Zhukovsky's function finds extensive application in the solution of many concrete problems of conformal mapping, particularly those associated with the investigation of hydrodynamic problems. We will deal with these problems somewhat later; for the present we will consider one more function that finds numerous applications.

## 6.4. Schwartz-Christoffel Integral. Transformation of Polygons

In the complex $w$-plane let there be given an $n$-gon with vertices at the points $A_1, A_2, \ldots, A_n$ and with interior angles at these vertices $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_n\pi$, respectively. (Obviously, $\sum_{i=1}^{n} \alpha_i = n - 2$, $n > 2$.) Let it be required to construct a conformal mapping of the upper half of the $z$-plane onto the interior of such a polygon. This problem is solved by means of the so-called Schwartz-Christoffel integral, some properties of which will be studied in this section.

Consider a function of the complex variable $z$ defined in the upper half of the $z$-plane with the aid of the expression

$$w = f(z) = C \int_{z_0}^{z} (\zeta - a_1)^{\alpha_1 - 1} \ldots (\zeta - a_n)^{\alpha_n - 1} \, d\zeta + C_1 \qquad (6\text{-}62)$$

Here, $_0$, $C$, $C_1$ are given complex constants; $a_1, \ldots, a_n$ are real numbers arranged in increasing order; $\alpha_1, \ldots, \alpha_n$ are positive constants that satisfy the conditions

$$\sum_{i=1}^{n} \alpha_i = n - 2 \qquad (6\text{-}63)$$

$$0 < \alpha_i < 2 \qquad (6\text{-}64)$$

In the integrand we chose those branches of the functions $(\zeta - a_i)^{\alpha_i - 1}$ which are a direct analytic continuation into the upper half-plane of the real functions $(x - a_i)^{\alpha_i - 1}$ of the real variable $x > a_i$. In that case, the function (6-62) is a single-valued analytic function in the upper half-plane $\operatorname{Im} z > 0$. The points $a_i$ lying on the real axis are singularities of this function. The function (6-62) is the Schwartz-Christoffel integral. For an appropriate choice of points $a_i$, the function (6-62) defines a conformal mapping of the upper half-plane $\operatorname{Im} z > 0$ onto the domain inside some $n$-gon in the $w$-plane. To start with, we consider that all the numbers $a_i$ are bounded. We will show that the function (6-62) then remains bounded everywhere for $\operatorname{Im} z \geqslant 0$. By virtue of the condition (6-64), the integral (6-62) remains bounded in the neighbourhood of the singularities $a_i$. We will satisfy ourselves that the integral (6-62) remains bounded when $z \to \infty$ as well. Transform the integrand function by taking advantage of the condition (6-63):

$$\varphi(\zeta) = \zeta^{\alpha_1 + \ldots + \alpha_n - n} \left(1 - \frac{a_1}{\zeta}\right)^{\alpha_1 - 1} \ldots \left(1 - \frac{a_n}{\zeta}\right)^{\alpha_n - 1}$$

$$= \frac{1}{\zeta^2} \left(1 - \frac{a_1}{\zeta}\right)^{\alpha_1 - 1} \ldots \left(1 - \frac{a_n}{\zeta}\right)^{\alpha_n - 1} \qquad (6\text{-}65)$$

From this expression it follows that the integral is convergent as $z \to \infty$. Thus, the integral (6-62), which is a single-valued analytic function of $z$ in the upper half-plane $\operatorname{Im} z > 0$, defines a mapping of this half-plane onto some bounded domain $\mathfrak{G}$ of the $w$-plane.

Let us now see into what kind of curve the real axis of the $z$-plane goes. Consider the expression of the derivative of the function (6-62):

$$f'(z) = C(z-a_1)^{\alpha_1-1} \ldots (z-a_n)^{\alpha_n-1} \tag{6-66}$$

From this expression it follows that the derivative of the function $f(z)$ is nonzero everywhere in the upper half-plane $\operatorname{Im} z \geqslant 0$, with the exception of the singularities $a_i$ at which it vanishes or becomes infinite. As $z$ varies on every one of the intervals $a_k < x < a_{k+1}$

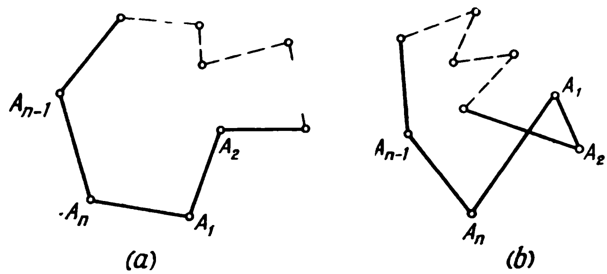


**Fig. 6.13**

($k = 1, \ldots, n-1$) of the real axis, the argument of the derivative does not change. Indeed, by virtue of the above-indicated choice of branches of the functions $(z-a_i)^{\alpha_i-1}$ the argument of these functions, on the given intervals of the real axis, takes the values

$$\arg(x-a_i)^{\alpha_i-1} = \begin{cases} \pi(\alpha_i-1), & x < a_i \\ 0 & x > a_i \end{cases} \tag{6-67}$$

which proves the proposition. In view of the geometric meaning of the argument of the derivative,* this means that the segments $a_k < x < a_{k+1}$ of the real axis are also mapped by the function $f(z)$ onto rectilinear segments of the $w$-plane. Points $a_k$ of the real axis are transformed by the function (6-62) into points $A_k$ of the $w$-plane—the ends of the corresponding straight-line segments $A_kA_{k+1}$ into which the function (6-62) maps segments $[a_k, a_{k+1}]$ of the real axis. Thus, *the function* (6-62) *is continuous and single-valued on the real axis and maps the real axis of the z-plane onto some closed polygonal line* $A_1A_2 \ldots A_n$, *the elements of which are the straight-line segments* $A_kA_{k+1}$ (Fig. 6.13). When the point $z$ traverses the entire real axis in the positive direction, the point $w$ corresponding to it

* The argument of the derivative of the function $f(z)$ at the point $z_0$ determines the size of the angle through which the tangent to a smooth curve $\gamma$ passing through $z_0$ has to be rotated in order to obtain the tangent to the image of this curve at the point $w_0 = f(z_0)$.

makes a complete circuit of the closed polygonal line $A_1A_2 \ldots A_n$. Note that, generally speaking, the polygonal line $A_1A_2 \ldots A_n$ can have points of self-intersection (Fig. 6.13*b*).

Now let us determine the size of the angles between adjacent segments of the polygonal line obtained. To do this, consider the variation of the argument of the derivative (6-66) as $z$ passes through the point $a_i$. From (6-67) it follows that as the point $z$ moves along the real axis in the positive direction, during which the singular point $a_i$ is taken round an arc of infinitesimal radius in the upper half-plane, the argument of the derivative changes its value by $-\pi(\alpha_i - 1)$. By the geometric meaning of the argument of the derivative, this means that the angle between the directions of the

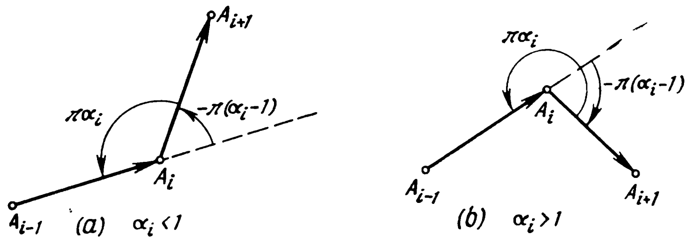


**Fig. 6.14**

vectors* $\overrightarrow{A_{i-1}A_i}$ and $\overrightarrow{A_iA_{i+1}}$ is equal to $-\pi(\alpha_i - 1)$. For $\alpha_i < 1$, the transition from the direction of the vector $\overrightarrow{A_{i-1}A_i}$ to the direction of the vector $\overrightarrow{A_iA_{i+1}}$ occurs in the positive sense (Fig. 6.14*a*) and for $\alpha_i > 1$ in the negative sense (Fig. 6.14*b*). As is easy to see, in both cases the size of the angle in the transition in the positive direction from the direction of the vector $\overrightarrow{A_iA_{i+1}}$ to the direction of the vector $\overrightarrow{A_iA_{i-1}}$ is $\pi\alpha_i$ (Fig. 6.14). If a closed polygonal line $A_1A_2 \ldots A_n$ does not have self-intersections, it bounds some $n$-gon. If, besides, the motion of a point $z$ in the positive direction of the real axis is associated with a traversal of the polygonal line $A_1A_2 \ldots$ $\ldots A_n$ in the positive direction, then the interior angle of the given $n$-gon at the vertex $A_i$, into which the point $a_i$ of the real axis of the $z$-plane is mapped, is equal to $\pi\alpha_i$. By the condition (6-63), the sum of all the interior angles of the given $n$-gon will then be equal to $(n-2)\pi$, as it should be.

On the basis of the principle of correspondence of boundaries (Theorem 6.4), it may be asserted that if the polygonal line $A_1A_2 \ldots$ $\ldots A_n$, onto which the function (6-62) maps the real axis of the

* Here, by the angle between the directions of intersecting straight lines $b_1$, $b_2$ is meant the size of the angle of the shortest rotation bringing the straight line $b_1$ to coincidence with the straight line $b_2$.

$z$-plane, does not have points of self-intersection and the direction of traversal is preserved, then the function (6-62) maps the upper half-plane $\operatorname{Im} z > 0$ conformally onto the interior of the $n$-gon bounded by the polygonal line $A_1A_2 \ldots A_n$.

A thorough investigation shows that if an arbitrary $n$-gon is given in the $w$-plane (the position of its vertices $A_1, A_2, \ldots, A_n$ and the angles at these vertices are known), then it is always possible to specify the values of the constants $C$, $C_1$ and the points $a_1, \ldots, a_n$ of the real axis so that a properly constructed function (6-62) maps the upper half-plane $\operatorname{Im} z > 0$ conformally onto the interior of the given $n$-gon. We will not go into the proof of this proposition* and will confine ourselves to a number of remarks and examples.

*Note 1.* The formula (6-62) involves a number of constants. However, when constructing a conformal mapping of the upper half-plane $\operatorname{Im} z > 0$ onto a given polygon $A_1 \ldots A_n$ of the $w$-plane, it is only possible to specify three points $a_i, a_j, a_k$ of the real $x$-axis that go into any three chosen vertices of the polygon $A_i, A_j, A_k$. The remaining constants in (6-62) are defined uniquely. Indeed, (6-62) defines $f(z)$ connected with the function $\hat{f}(z) = \int_{z_0}^{z} (\zeta - a_1)^{\alpha_1 - 1} \ldots$ $\ldots (\zeta - a_n)^{\alpha_n - 1}\, d\zeta$ by a linear transformation that is a transformation of a similarity stretching, of a rotation and of a parallel translation. Hence, if the function $f(z)$ maps the upper half-plane $\operatorname{Im} z > 0$ onto a given polygon of the $w$-plane, then the function $\hat{f}(z)$ maps this half-plane onto a polygon that is similar to the given one. For the given values of $\alpha_i$, in order that an $n$-segment closed polygonal line into which the real axis is mapped by the function $\hat{f}(z)$ be a polygon similar to the given one, it is sufficient that $n - 2$ segments of this polygonal line be proportional to the corresponding sides of the polygon. (The two extreme segments are fully defined by specifying their directions.) We thus have $n - 3$ equations in $n$ constants $a_i$. If three of these constants are arbitrarily specified, the remaining corresponding equations are defined uniquely. This is a corollary to the Riemann theorem on the unique definition of a function that carries out a conformal mapping of singly connected domains when a correspondence is specified of three points of the boundary of one domain to three points of the boundary of the other domain. Also note that the position of the given polygon (the lengths of the sides and the size of the angles at the vertices are given) in the plane is uniquely defined by the position of three vertices.

*Note 2.* We assumed that all the numbers $\alpha_i$ in formula (6-62) are positive. Then the integral (6-62) converges for all values of

* See, for example, [13].

$\text{Im}\, z \geqslant 0$. If some number $\alpha_k$ is negative, then as $z \to a_k$ the integral (6-62) diverges. This means that the corresponding vertex $A_k$ of the polygon $A_1 \ldots A_n$ lies at the point at infinity $w = \infty$. We then take the size of the angle at the vertex $A_k$ to be equal to the size of the angle (with the minus sign) between the prolongation of the segments $A_kA_{k-1}$ and $A_kA_{k+1}$ at the finite point of their intersection. It is readily seen that for such a definition, the angle at the vertex $A_k$ is equal to $\alpha_k\pi$ ($\alpha_k < 0$) and by virtue of the condition (6-63) the sum of the interior angles of the resulting $n$-gon with vertex $A_k$ at the point at infinity remains equal to $(n-2)\pi$. This remark holds true also for the case when several numbers $\alpha_k$ are negative.

*Note 3.* When we investigated formula (6-62) we assumed that all the points $a_i$ are finite. It is easy to get rid of this condition. Introduce a new complex variable $t$ connected with $z$ by the relation

$$z = a_n - \frac{1}{t} \tag{6-68}$$

Then the point $z = a_n$ will pass into the point $t = \infty$. This transformation means that in a mapping of the upper half-plane $\text{Im}\, t > 0$ onto the interior of the polygon $A_1A_2 \ldots A_n$ of the $w$-plane, the point at infinity $t = \infty$ is mapped into the vertex $A_n$. In the complex $t$-plane, the function (6-62) has the form

$$w = C \int_{t_0}^{t} \left(a_n - a_1 - \frac{1}{\tau}\right)^{\alpha_1 - 1} \ldots \left(a_n - a_{n-1} - \frac{1}{\tau}\right)^{\alpha_{n-1} - 1}$$
$$\times \left(-\frac{1}{\tau}\right)^{\alpha_n - 1} \frac{d\tau}{\tau^2} + C_1 = A \int_{t_0}^{t} (\tau - a_1')^{\alpha_1 - 1} \ldots (\tau - a_{n-1}')^{\alpha_{n-1} - 1}\, d\tau + C_1 \tag{6-69}$$

Here use is made of the relation (6-63) and the following notation has been introduced:

$$a_i' = (a_n - a_i)^{-1}, \quad t_0 = \frac{1}{a_n - z_0}$$
$$A = C(a_n - a_1)^{\alpha_1 - 1} \ldots (a_n - a_{n-1})^{\alpha_{n-1} - 1} (-1)^{\alpha_n - 1}$$

The relation (6-69) means that when in a conformal mapping of the upper half-plane onto the interior of the polygon $A_1A_2 \ldots A_n$ the point at infinity $t = \infty$ passes into one of the vertices ($A_n$), this mapping is effected by the Schwartz-Christoffel integral (6-69), in the integrand function of which the factor corresponding to the given vertex ($A_n$) has been omitted. This circumstance is frequently made use of since, as we noted above (Note 1), when solving problems on the construction of a conformal mapping of the upper half-plane

Im $z > 0$ onto a given polygon of the $w$-plane, one has to determine a large number of unknowns in the case of a large number of vertices of the polygon.

We consider a few of the simplest examples.

*Example 1.* Find a function that conformally maps the upper half-plane Im $z > 0$ onto the sector $0 < \arg w < \alpha\pi$, $0 < \alpha < 2$.

Since the given sector is a polygon with vertices $A_1$ $(w = 0)$ and $A_2$ $(w = \infty)$, the Schwartz-Christoffel integral may be employed to solve the problem. We establish the following correspondence of points of the real $z$-axis to the vertices of the given polygon:

$$\begin{aligned} a_1\,(z = 0) &\to A_1\,(w = 0) \\ a_2\,(z = \infty) &\to A_2\,(w = \infty) \end{aligned} \tag{6-70}$$

Then by (6-69) the mapping function takes the form

$$w = f(z) = C\int_{z_0}^{z} \zeta^{\alpha-1}\, d\zeta + C_1$$

Putting $z_0 = 0$ and using (6-70), we find that the constant $C_1$ is zero. Whence

$$w = C\int_{0}^{z} \zeta^{\alpha-1}\, d\zeta = \frac{C}{\alpha} z^{\alpha} \tag{6-71}$$

The function (6-71) is defined to within the constant factor defining the similarity transformation. This arbitrariness is due to the fact that the conditions (6-70) contain the demand that only two boundary points correspond, but, as we have seen (see the note on page 168), the function which effects a conformal mapping is defined uniquely by specification of a correspondence of three boundary points. Now requiring, for example, that in addition to (6-70) there occurs a supplementary correspondence of boundary points

$$z = 1 \to w = 1$$

we determine the value of the arbitrary constant $C = \alpha$ that remains in (6-71).

And so, finally, the function

$$w = z^{\alpha} \tag{6-72}$$

defines a conformal mapping of the upper half-plane Im $z > 0$ onto the given sector of the $w$-plane. And by virtue of the earlier indicated choice of branches in the integrand function of the Schwartz-Christoffel integral (6-62), that branch of the multiple-valued function (6-72) must be taken which is a direct analytic continuation of the real function $x^{\alpha}$ of the real positive variable $x$.

*Example 2.* Find a function which conformally maps the upper half-plane Im $z > 0$ onto the rectangle $A_1A_2A_3A_4$ (Fig. 6.15).

Let the vertices of the rectangle in the $w$-plane be located at the points $A_1$ ($w = a$), $A_2$ ($w = a + ib$), $A_3$ ($w = -a + ib$), $A_4$ ($w = -a$). Let us suppose that with the aid of some function $f_1(z)$ we have executed a conformal mapping of the first quadrant of the $z$-plane (Re $z > 0$, Im $z > 0$) onto the right half $OA_1A_2O'$ of the

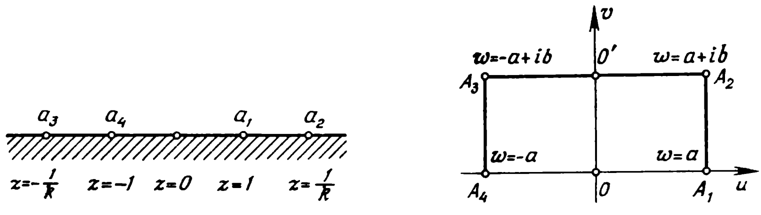


**Fig. 6.15**

rectangle (Fig. 6.15), in which mapping the right part of the imaginary axis of the $z$-plane goes into the segment $OO'$. Then on the basis of the symmetry principle (see page 164) the function which is the analytic continuation of $f_1(z)$ into the domain (Re $z < 0$, Im $z > 0$) maps conformally the given domain onto the left part of the original rectangle. Here, the symmetric points of the real $z$-axis go into the vertices $A_1$ and $A_4$. The same occurs relative to the vertices $A_2$ and $A_3$. Therefore, we can establish the following correspondence of points:

$$\begin{gathered} a_1\,(z = 1) \to A_1\,(w = a) \\ a_4\,(z = -1) \to A_4\,(w = -a) \end{gathered} \tag{6-73}$$

Besides, it is obvious that there must also be the correspondence

$$z = 0 \to w = 0 \tag{6-74}$$

The relations (6-73) and (6-74) establish a correspondence of three boundary points. It is therefore impossible to specify arbitrarily on the real $z$-axis the point $a_2$ that goes into the vertex $A_2$ of the rectangle. Let us suppose that the point $a_2$ of the real $z$-axis with coordinate $\frac{1}{k}$ (the value of which will be defined later on) goes into the vertex $A_2$. Clearly, $0 < k < 1$.

Thus, the function which defines a conformal mapping of the upper half-plane onto a given rectangle may be represented in the form

$$w = f(z) = C' \int_{z_0}^{z} (\zeta - 1)^{\frac{1}{2}-1} \left(\zeta - \frac{1}{k}\right)^{\frac{1}{2}-1} \left(\zeta + \frac{1}{k}\right)^{\frac{1}{2}-1} (\zeta + 1)^{\frac{1}{2}-1} d\zeta$$

$$+ C_1 = C \int_{z_0}^{z} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} + C_1 \tag{6-75}$$

Putting $z_0 = 0$ and using the relation (6-74), we get $C_1 = 0$. Then

$$w = C\int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \tag{6-76}$$

It remains to determine the constants $C$ and $k$ from the correspondence of the points $a_1$ and $a_2$ of the real z-axis to the vertices $A_1$ and $A_2$. Note that the integral (6-76) is not expressible in terms of elementary functions. This is a so-called *elliptic integral of the first kind*, which is ordinarily denoted as

$$F(z, k) = \int_0^z \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \tag{6-77}$$

The conditions (6-73) yield

$$a = C\int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \tag{6-78}$$

The integral on the right is the so-called *complete elliptic integral of the first kind*

$$K(k) = \int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} \tag{6-79}$$

and is a well-studied tabulated function. The correspondence of points $a_2\left(z=\frac{1}{k}\right) \leftrightarrow A_2\,(w=a+ib)$ permits us to write

$$a+ib = C\left\{\int_0^1 \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}} + \int_1^{\frac{1}{k}} \frac{d\zeta}{\sqrt{(1-\zeta^2)(1-k^2\zeta^2)}}\right\} \tag{6-80}$$

whence, taking into account (6-78), we get

$$b = C\int_1^{\frac{1}{k}} \frac{d\zeta}{\sqrt{(\zeta^2-1)(1-k^2\zeta^2)}} = C\widetilde{F}\left(\frac{1}{k}, k\right) \tag{6-81}$$

where the integral in (6-81) is denoted by $\widetilde{F}\left(\frac{1}{k}, k\right)$. From (6-78) and (6-81), given specified values of $a$ and $b$, we can solve the transcendental equation

$$a\widetilde{F}\left(\frac{1}{k}, k\right) = bK(k) \tag{6-82}$$

and determine the values of the constants $k$ and $C$. Thus, the function (6-76) which maps the upper half-plane $\operatorname{Im} z > 0$ conformally onto the given rectangle of the $w$-plane is completely defined. On the other hand, if the quantities $k$ and $C$ are given in formula (6-76), then this function defines a conformal mapping of the upper half-plane $\operatorname{Im} z > 0$ onto the rectangle of the $w$-plane, the ratio of the sides of which $\left(\frac{2a}{b}\right)$ is defined by formula (6-82), and the absolute value of the sides by the constant $C$. By varying the values of these constants at will, it is possible to obtain a conformal mapping of the upper half-plane $\operatorname{Im} z > 0$ onto any rectangle of the $w$-plane.

# CHAPTER 7

# ANALYTIC FUNCTIONS IN THE SOLUTION OF BOUNDARY-VALUE PROBLEMS

The methods of the theory of functions of a complex variable are extensively and effectively employed in the solution of a great variety of mathematical problems that arise in diverse fields of science. For example, the use of analytic functions in many cases yields sufficiently simple methods of solving boundary-value problems for the Laplace equation, to which various problems of hydro- and aerodynamics, the theory of elasticity, electrostatics and so forth reduce. This is due to the close connection between analytic functions of a complex variable and the harmonic functions of two real variables. In this chapter we will examine certain general problems of the employment of analytic functions in the solution of boundary-value problems for the Laplace equation and will give a number of examples of the solution of problems in physics and mechanics.

## 7.1. Generalities

### *a. The relationship of analytic and harmonic functions*

In a domain $\mathfrak{G}$ of the complex $z$-plane, let there be given an analytic function $f(z) = u(x, y) + iv(x, y)$. Then throughout this domain the functions $u$ and $v$ are connected by the Cauchy-Riemann conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \qquad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{7-1}$$

Since an analytic function in the domain $\mathfrak{G}$ has derivatives of all orders, the real functions $u(x, y)$ and $v(x, y)$ have partial derivatives of any order in the appropriate domain of the $x$, $y$-plane. This permits differentiation of the expressions (7-1) any number of times with respect to the variables $x$, $y$. Differentiating the first equality of (7-1) with respect to $x$ and the second with respect to $y$ and adding, we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0, \quad x, \ y \in \mathfrak{G} \tag{7-2}$$

Similarly, differentiating the first of the equalities in (7-1) with respect to $y$ and the second with respect to $x$ and subtracting one from the other, we have

$$\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}=0, \quad x, y \in \mathfrak{G} \tag{7-3}$$

whence it follows that the functions $u(x, y)$ and $v(x, y)$ are harmonic in the given domain of the $x$, $y$-plane. And so the real and imaginary parts of the function $f(z)$ analytic in the domain $\mathfrak{G}$ are harmonic functions in the corresponding domain of the $x$, $y$-plane. Also, the given harmonic functions are connected by the conditions (7-1). Conversely, if in the domain $\mathfrak{G}$ of the $x$, $y$-plane there are given two harmonic functions $u(x, y)$ and $v(x, y)$ that satisfy in this domain the conditions (7-1), then the function $f(z) = u(x, y) + iv(x, y)$ of the complex variable $z = x + iy$ is analytic in the appropriate domain of the $z$-plane. Thus, *a necessary and sufficient condition for the analyticity of the function* $f(z) = u(x, y) + iv(x, y)$ *in a domain* $\mathfrak{G}$ *is the requirement that the functions* $u(x, y)$ *and* $v(x, y)$ *be harmonic and satisfy the conditions* (7-1) *in the appropriate domain of the* $x$, $y$*-plane*. In Chapter 1 (see page 34) it was shown that when only the real (or only the imaginary) part of an analytic function of a complex variable is given, the function is defined to within an additive constant. Whence it follows that all analytic functions of a complex variable for which a given harmonic function of two real variables is the real (or imaginary) part differ solely by an additive constant.

This connection between analytic and harmonic functions permits utilizing the properties of analytic functions in the study of various properties of harmonic functions. Thus, for example, from the formula of the mean value of an analytic function (see Chapter 1, page 49) there follows in straightforward fashion the mean-value formula for a harmonic function

$$u(x_0, y_0)=\frac{1}{2\pi R_0}\int_{C_{R_0}} u(\xi, \eta)\, ds \tag{7-4}$$

where the point $x_0$, $y_0$ is the centre of the circle $C_{R_0}$ of radius $R_0$ lying wholly in the domain of harmonicity of the function $u(x, y)$.

### *b. Preservation of the Laplace operator in a conformal mapping*

Let there be given a harmonic function $u(x, y)$ in a domain $\mathfrak{G}$ of the $x$, $y$-plane, i.e.

$$\Delta u=\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}=0, \quad x, y \in \mathfrak{G} \tag{7-5}$$

With the aid of a nondegenerate transformation of independent variables

$$\xi=\xi(x\ y),\quad \eta=\eta(x,\ y) \tag{7-6}$$

$$\frac{D(\xi,\ \eta)}{D(x,\ y)}\neq 0,\quad x,\ y\in\mathfrak{G} \tag{7-7}$$

map the domain $\mathfrak{G}$ of the $x$, $y$-plane onto the new domain $\mathfrak{G}'$ of the $\xi$, $\eta$-plane. Note that specification of the two real functions (7-6) of two real variables $x$, $y$ is equivalent to the specification in the domain $\mathfrak{G}$ of the complex $z$-plane of a single function $\zeta=f(z)=\xi(x, y)+i\eta(x, y)$ of the complex variable $z=x+iy$. Here, the function $f(z)$ maps the domain $\mathfrak{G}$ of the complex $z$-plane onto the domain $\mathfrak{G}'$ of the complex $\zeta$-plane. By the condition (7-7), the equations (7-6) are uniquely solvable for the old variables, and thus the function $U(\xi, \eta)=u[x(\xi, \eta), y(\xi, \eta)]$ is defined in the domain $\mathfrak{G}'$ of the $\xi$, $\eta$-plane. Let us find out under what conditions imposed on the transformation (7-6) the function $U(\xi, \eta)$ will be a harmonic function of the variables $\xi$, $\eta$. Assuming that the functions (7-6) are twice continuously differentiable in the domain $\mathfrak{G}$, we express second partial derivatives of the function $u(x, y)$ with respect to the old variables in terms of the derivatives of the function $U(\xi, \eta)$ with respect to the new variables:

$$\begin{aligned}\frac{\partial^2 u}{\partial x^2}&=\frac{\partial^2 U}{\partial\xi^2}(\xi_x)^2+2\frac{\partial^2 U}{\partial\xi\,\partial\eta}\xi_x\eta_x+\frac{\partial^2 U}{\partial\eta^2}(\eta_x)^2+\frac{\partial U}{\partial\xi}\xi_{xx}+\frac{\partial U}{\partial\eta}\eta_{xx}\\ \frac{\partial^2 u}{\partial y^2}&=\frac{\partial^2 U}{\partial\xi^2}(\xi_y)^2+2\frac{\partial^2 U}{\partial\xi\,\partial\eta}\xi_y\eta_y+\frac{\partial^2 U}{\partial\eta^2}(\eta_y)^2+\frac{dU}{\partial\xi}\xi_{yy}+\frac{\partial U}{\partial\eta}\eta_{yy}\end{aligned} \tag{7-8}$$

Substituting these expressions into (7-5), we obtain the following equation for the function $U(\xi, \eta)$:

$$\frac{\partial^2 U}{\partial\xi^2}(\xi_x^2+\xi_y^2)+2\frac{\partial^2 U}{\partial\xi\,\partial\eta}(\xi_x\eta_x+\xi_y\eta_y)+\frac{\partial^2 U}{\partial\eta^2}(\eta_x^2+\eta_y^2)$$
$$+\frac{\partial U}{\partial\xi}(\xi_{xx}+\xi_{yy})+\frac{\partial U}{\partial\eta}(\eta_{xx}+\eta_{yy})=0 \tag{7-9}$$

For this to be Laplace's equation, the following relations must be fulfilled:

$$\xi_{xx}+\xi_{yy}=0,\quad \eta_{xx}+\eta_{yy}=0 \tag{7-10}$$

$$\xi_x\eta_x+\xi_y\eta_y=0 \tag{7-11}$$

and

$$\xi_x^2+\xi_y^2=\eta_x^2+\eta_y^2\neq 0 \tag{7-12}$$

The relations (7-10) imply that the functions $\xi(x, y)$ and $\eta(x, y)$ must be harmonic in the domain $\mathfrak{G}$. Rewrite (7-11) in the form

$$\frac{\xi_x}{\eta_y}=-\frac{\xi_y}{\eta_x}=\mu(x, y) \tag{7-13}$$

where $\mu(x, y)$ is as yet some unknown function. Then the relation (7-12) yields

$$\xi_x^2+\xi_y^2=\mu^2[\eta_y^2+\eta_x^2]=\eta_x^2+\eta_y^2\neq 0$$

Whence $\mu^2(x, y)\equiv 1$ for $x, y \in \mathfrak{G}$. Thus, the unknown function $\mu(x, y)$ is defined: $\mu=\pm 1$. For $\mu = 1$, the relations (7-13) yield

$$\xi_x=\eta_y,\ \xi_y=-\eta_x$$

That is, the functions $\xi$ and $\eta$ which are harmonic in the domain $\mathfrak{G}$ must satisfy the Cauchy-Riemann conditions in this domain. This means that the function $f(z) = \xi(x, y) + i\eta(x, y)$ must be analytic in the domain $\mathfrak{G}$ of the complex $z$-plane. Note that from (7-7) and (7-12) it follows that the mapping of the domain $\mathfrak{G}$ onto $\mathfrak{G}'$ must be reciprocal one-to-one, and the derivative of the function $f(z)$ must satisfy the condition $f'(z)\neq 0$ throughout the domain $\mathfrak{G}$. This means that the mapping of the domain $\mathfrak{G}$ of the $z$-plane onto the domain $\mathfrak{G}'$ of the $\zeta$-plane, defined by the function $f(z)$, must be conformal.

For $\mu = -1$, the relations (7-13) yield

$$\xi_x=-\eta_y,\ \xi_y=\eta_x$$

As is easily seen, in this case the function $\overline{f}(z) = \xi(x, y) - i\eta(x, y)$ must be analytic, and the mapping defined by the function $f(z) = \xi(x, y) + i\eta(x, y)$ must be conformal of the second kind.

We thus have the final answer to the question posed at the beginning of this subsection. *In a mapping of the domain $\mathfrak{G}$ of the z-plane onto the domain $\mathfrak{G}'$ of the $\zeta$-plane performed by the function $f(z) = \xi(x, y) + i\eta(x, y)$, Laplace's equation for the function $u(x, y)$ goes into Laplace's equation for the function $U(\xi, \eta) = u[x(\xi, \eta), y(\xi, \eta)]$ only if the given mapping is a conformal mapping of the first or second kind.* Note that under these mappings the Laplace operator $\Delta_{xy}$ goes into the operator $|f'(z)|^2\Delta_{\xi\eta}=\frac{1}{|\varphi'(\zeta)|^2}\Delta_{\xi\eta}$, where $z = \varphi(\zeta)$ is an inverse function defining the conformal mapping of the domain $\mathfrak{G}'$ onto the domain $\mathfrak{G}$. Thus, even the simplest equation of elliptic type with constant coefficients $\Delta u + cu = 0$, $c =$ constant $\neq 0$, will, generally speaking, in a conformal mapping go into an equation with the variable coefficient $\Delta_{\xi\eta} U + c\,|\varphi'(\zeta)|^2 U = 0$.

### *c. Dirichlet's problem*

The results obtained in the preceding subsection permit employing the method of conformal transformations in the solution of boundary-value problems for harmonic functions. Consider the basic idea of this method in an example of the solution of the Dirichlet problem.

*It is required to find a function $u(x, z)$ that satisfies the Laplace equation*

$$\Delta u = 0$$

*in the domain $\mathfrak{G}$, that is continuous in the closed domain $\overline{\mathfrak{G}} = \mathfrak{G} + \Gamma$ and that assumes specified values on the boundary $\Gamma$:*

$$u(P)|_{\Gamma} = \alpha(P) \tag{7-14}$$

where $\alpha(P)$ is a given continuous function of the point $P$ on the contour $\Gamma$. As is known (see [17]), the solution of this problem by the method of separation of variables may be obtained only for a restricted class of domains $\mathfrak{G}$ with a sufficiently simple boundary $\Gamma$.

The method of conformal transformations yields a sufficiently universal algorithm for the solution of the Dirichlet problem for two-dimensional domains. Let us begin with the solution of the Dirichlet problem for a circle of radius $a$. We introduce the polar system of coordinates $r$, $\varphi$ with origin at the centre of the circle. Then the function $\alpha(P)$ will be a function only of the variable $\varphi$. Let us try to express the value of the unknown function $u(r, \varphi)$ at an arbitrary interior point $(r_0, \varphi_0)$ of the circle in terms of its boundary values $\alpha(\varphi)$. To do this, construct a conformal mapping of the given circle onto the unit circle $|w| < 1$ of the $w$-plane in which the point $r_0$, $\varphi_0$ goes into the centre $w = 0$. The solution of this problem is readily obtained with the aid of the linear-fractional function considered in Chapter 6. The mapping function is of the form

$$w = f(z) = \lambda \frac{z - z_0}{z - \dfrac{a^2}{\bar{z}_0}} = \lambda \frac{z - r_0 e^{i\varphi_0}}{z - \dfrac{a^2}{r_0} e^{i\varphi_0}} \tag{7-15}$$

where the constant $\lambda$ is selected from the condition that the boundary points $z = ae^{i\varphi}$ of the given circle have gone into the boundary points $|w| = 1$ of the unit circle of the $w$-plane; here, $|\lambda| = \frac{a}{r_0}$, and $\arg \lambda$, which defines the rotation of the circle $|w| \leqslant 1$ about its centre $w = 0$, may be chosen at will. As a result of this transformation, the desired function $u(r, \varphi)$ goes into the function $U(\rho, \psi) = u[r(\rho, \psi), \varphi(\rho, \psi)]$, where $\rho$, $\psi$ are polar coordinates in the $w$-plane connected with the coordinates $r$, $\varphi$ by the relation (7-15). The given boundary function $\alpha(\varphi)$ will go into the function $A(\psi) = \alpha[\varphi(1, \psi)]$. Since the function $U(\rho, \psi)$ is a harmonic function of its variables, its value at the centre of the circle may be found from the mean-value formula (7-4), whence

$$u(r_0, \varphi_0) = U|_{w=0} = \frac{1}{2\pi} \int_0^{2\pi} A(\psi) \quad \psi \tag{7-16}$$

From (7-16) we get an explicit expression of the solution of the Dirichlet problem for the circle, if we express the function $A(\psi)$ in terms of the originally specified function $\alpha(\varphi)$. Note that for the correspondence of the boundary points of the circle $|z| \leqslant a$ and the circle $|w| \leqslant 1$, the formula (7-15) yields

$$e^{i\psi} = \frac{a}{r_0} \frac{ae^{i\varphi} - r_0 e^{i\varphi_0}}{ae^{i\varphi} - \frac{a^2}{r_0} e^{i\varphi_0}} \tag{7-17}$$

whence

$$d\psi = \frac{a^2 - r_0^2}{a^2 + r_0^2 - 2ar_0 \cos(\varphi - \varphi_0)} d\varphi$$

Therefore, if in the integral (7-16) we make a change of the variable of integration $\psi = \psi(\varphi)$, where the relation of the variables $\psi$ and $\varphi$ is given by formula (7-17), we obtain

$$u(r_0, \varphi_0) = \frac{1}{2\pi} \int_0^{2\pi} \frac{a^2 - r_0^2}{a^2 + r_0^2 - 2ar_0 \cos(\varphi - \varphi_0)} \alpha(\varphi) d\varphi \tag{7-18}$$

Formula (7-18) then yields an explicit analytic expression for the solution of the Dirichlet problem for a circle of radius $a$ in terms of the function of the boundary conditions $\alpha(\varphi)$. This formula, which is known as the Poisson integral, may be obtained in a number of other ways too, for example by the method of separation of variables or with the aid of a source function (see [17]).

The results obtained allow us, in principle, to solve the Dirichlet problem for any domain $\mathfrak{G}$ which may be conformally mapped onto the unit circle $|w| \leqslant 1$ of the $w$-plane. Indeed, the Laplace equation is preserved in a conformal mapping, and the solution of the Dirichlet problem for the circle is obtained. Making a change of the variable of integration in the integral (7-18) or (7-16) and proceeding from the relation of the boundary points of the domain $\mathfrak{G}$ and the unit circle $|w| = 1$ for a given conformal mapping, we obtain the expression of the solution of the Dirichlet problem at interior points of the domain in terms of the boundary function (7-14).

*Example 1.* The solution of the Dirichlet problem for a half-plane. Let it be required to determine the function $u(x, y)$ bounded at infinity, harmonic in the upper half-plane $y > 0$, continuous for $y \geqslant 0$ and assuming given values:

$$u(x, 0) = \alpha(x) \quad \text{for } y = 0 \tag{7-19}$$

Map the upper half-plane $\operatorname{Im} z > 0$ conformally onto the interior of the unit circle $|w| < 1$ so that the given point $z_0 = x_0 + iy_0$ $(y_0 > > 0)$ goes into the centre $w = 0$ of the circle. It is easy to see that the transformation is accomplished by the linear-fractional function

$$w = f(z) = \frac{z - z_0}{z - \bar{z}_0} \tag{7-20}$$

Here, the boundary points are related by

$$e^{i\psi} = \frac{x - z_0}{x - \bar{z}_0} \tag{7-21}$$

and the boundary function $\alpha(x)$ goes into the function $A(\psi) = \alpha[x(\psi)]$, where $x(\psi)$ is determined from the relation (7-21). Note that (7-21) yields

$$d\psi = \frac{2y_0}{(x - x_0)^2 + y_0^2}\, dx \tag{7-22}$$

The value of the desired function $u(x, y)$ at the point $x_0$, $y_0$ is determined by the integral (7-16). Making a change of the variable of integration in it by formulas (7-21) and (7-22), we get

$$u(x_0, y_0) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{y_0}{(x - x_0)^2 + y_0^2}\, \alpha(x)\, dx \tag{7-23}$$

which yields the solution of the problem. Formula (7-23) which gives the solution of the Dirichlet problem for a half-plane is also called Poisson's integral.

### *d. Constructing a source function*

The methods of conformal mapping permit constructing a source function of the first boundary-value problem for the Laplace equation in a two-dimensional domain $\mathfrak{G}$ which may be conformally mapped onto the unit circle $|w| < 1$ of the $w$-plane. The source function $G(M_0, M)$ of the given problem is defined by the following conditions:

(1) $$\Delta_M G(M_0, M) = 0 \quad \text{for } M \neq M_0 \tag{7-24}$$

(2) in the neighbourhood of the point $M_0$

$$G(M_0, M) = \frac{1}{2\pi} \ln \frac{1}{r_{M_0 M}} + v(M_0, M) \tag{7-25}$$

where the function $v(M_0, M)$ is a harmonic function of the point $M$ throughout the domain $\mathfrak{G}$

(3) $$G(M_0, M)_{M \in \Gamma} = 0 \tag{7-26}$$

where $\Gamma$ is the boundary of the domain $\mathfrak{G}$. The following theorem holds.

**Theorem 7.1.** *If a function $w = f(z_0, z)$ defines a conformal mapping of a given domain $\mathfrak{G}$ of the z-plane onto the interior of the unit circle $|w| < 1$ so that point $z_0 \in \mathfrak{G}$ goes into the centre $w = 0$ of the circle, then the function*

$$G(M_0, M) = \frac{1}{2\pi} \ln \frac{1}{|f(z_0, z)|} \tag{7-27}$$

*is the source function of the first boundary-value problem for the Laplace equation in the domain $\mathfrak{G}$.* Here, the coordinates of the point $M \in \mathfrak{G}$ are $x$, $y$ and $z = x + iy$.

*Proof.* To prove the theorem we see if the function defined by formula (7-27) satisfies the conditions (7-24) to (7-26). The function $f(z, z_0)$ that defines the given conformal mapping is an analytic function, and $f(z, z_0) \neq 0$ for $z \neq z_0$. Whence it follows that the function

$$\ln f(z, z_0) = \ln |f(z, z_0)| + i \arg f(z, z_0)$$

is also analytic throughout the domain $\mathfrak{G}$, with the exception of the point $z_0$. Since the real part of the analytic function is a harmonic function, the condition (7-24) is fulfilled. Since $f'(z, z_0) \neq 0$ throughout the domain $\mathfrak{G}$ including the point $z = z_0$ and $f(z, z_0) = 0$, the point $z_0$ is a first-order zero of the given function; that is, in the neighbourhood of this point we have the decomposition

$$f(z, z_0) = (z - z_0)\, \varphi(z, z_0)$$

where $\varphi(z, z_0)$ is a function analytic in the neighbourhood of the point $z_0$, and $\varphi(z, z_0) \neq 0$. Hence the condition (7-25) is fulfilled for the function (7-27). Finally, since $|f(z, z_0)|_\Gamma = 1$, the function (7-27) satisfies the condition (7-26) as well. The proof is complete.

The following is an application of this theorem.

*Example 2.* Construct the source function of the first boundary-value problem for the Laplace equation in the strip $-\infty < x < \infty$, $0 < y < \pi$.

According to the theorem that has just been proved, to solve the problem we have to construct a conformal mapping of the given strip of the z-plane onto the interior of the unit circle $|w| < 1$, in which mapping the given point $z_0$ goes into the centre of the circle $w = 0$. Clearly, the function defining the required mapping is of the form

$$f(z_0, z) = \frac{e^z - e^{z_0}}{e^z - e^{\bar{z}_0}} \tag{7-28}$$

Since we have the relation

$$|e^z - e^{z_0}| = \{(e^x \cos y - e^{x_0} \cos y_0)^2 + (e^x \sin y - e^{x_0} \sin y_0)^2\}^{1/2}$$
$$= e^{\frac{x+x_0}{2}} \sqrt{2}\, \{\cosh(x - x_0) - \cos(y - y_0)\}^{1/2}$$

it follows that after elementary transformations we get the desired function in the form

$$G(M_0, M) = \frac{1}{2\pi} \ln \frac{1}{|f(z_0, z)|} = \frac{1}{4\pi} \ln \frac{\cosh(x-x_0) - \cos(y+y_0)}{\cosh(x-x_0) - \cos(y-y_0)} \quad (7\text{-}29)$$

## 7.2. Applications to Problems in Mechanics and Physics

### *a. Two-dimensional steady-state flow of a fluid*

We will consider the two-dimensional potential steady-state flow of an incompressible ideal fluid. It is known that in the case of potential motion in a domain free from sources, the velocity vector $\mathbf{v}(x, y)$ satisfies the equations (see [17])

$$\operatorname{curl} \mathbf{v} = 0 \quad (7\text{-}30)$$

$$\operatorname{div} \mathbf{v} = 0 \quad (7\text{-}31)$$

Since the motion is potential, there exists a scalar function $u(x, y)$ called the velocity potential related to the velocity vector $\mathbf{v}$ by

$$\mathbf{v} = \operatorname{grad} u(x, y) \quad (7\text{-}32)$$

that is,

$$v_x = \frac{\partial u}{\partial x} \text{ and } v_y = \frac{\partial u}{\partial y} \quad (7\text{-}33)$$

Here, the velocity vector $\mathbf{v}$ at every point of flow is normal to the level line $u(x, y) = \text{constant}$ of the velocity potential. Putting (7-32) into equation (7-31), we get

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0 \quad (7\text{-}34)$$

The velocity potential is a harmonic function.

Construct an analytic function of the complex variable $f(z) = u(x, y) + iv(x, y)$ for which the potential $u(x, y)$ of the flow under consideration is the real part. As has been pointed out (see page 34), the function $f(z)$ is then defined to within an additive constant. Earlier (see page 34) it was shown that the level lines $u(x, y) = \text{constant}$ and $v(x, y) = \text{constant}$ of the real and imaginary parts of the analytic function are mutually orthogonal. Therefore, the velocity vector $\mathbf{v}$ at every point of the flow is tangent to the level line $v(x, y) = \text{constant}$ passing through the given point. The function $v(x, y)$, which is the imaginary part of the thus constructed analytic function $f(z)$, is called the stream function, and the function $f(z)$ is the complex potential of the flow.

The region of flow bounded by two streamlines $v(x, y) = C_1$ and $v(x, y) = C_2$ is called a stream tube. Since the velocity of the fluid at any point is tangent to the streamline the quantity of fluid flowing in unit time through any two cross sections $S_1$ and $S_2$ of the stream tube remains constant due to the incompressibility of the fluid and the stationary character of motion. Thus, the difference in the values of the constants $C_1$ and $C_2$ defines the fluid output in a given stream tube.

From the Cauchy-Riemann conditions and formulas (7-33) it follows that the components of the velocity may be expressed in terms of the partial derivatives of the stream function:

$$\mathrm{v}_x = \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}, \qquad \mathrm{v}_y = \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \tag{7-35}$$

As was noted in Chapter 1, the complex number $w = \mathrm{v}_x + i\mathrm{v}_y$ may be interpreted as a plane vector with components $\mathrm{v}_x$ and $\mathrm{v}_y$. We have the obvious relation

$$w = \mathrm{v}_x + i\mathrm{v}_y = \frac{\partial u}{\partial x} + i\frac{\partial u}{\partial y} = \frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x} = \overline{f'(z)} \tag{7-36}$$

which relates the velocity vector and the derivative of the complex potential of the flow.

In hydrodynamics an essential role is played by the concepts of circulation and flux of the velocity vector. We express these quantities in terms of the complex potential of flow.

Consider a piecewise smooth plane curve $C$ (closed or open) and introduce on it the vectors of the differentials of the arc $d\mathbf{s}$ and of the normal $d\mathbf{n}$ with the aid of the relations

$$d\mathbf{s} = \mathbf{i}\,dx + \mathbf{j}\,dy \tag{7-37}$$

$$d\mathbf{n} = \mathbf{i}\,dy - \mathbf{j}\,dx \tag{7-38}$$

We have the obvious relation $\mathbf{n}\,ds = d\mathbf{n}$, where $\mathbf{n}$ is a unit normal to the curve $C$ and $ds$ is the differential of arc length of the curve.

In a positive traversal of the closed curve $C$, formula (7-38) yields the direction of the exterior normal.

The flux of the velocity vector $\mathbf{v}$ across the curve $C$ (open or closed) is the line integral of the normal component of the velocity

$$N_C = \int_C (\mathbf{v}\ \mathbf{n})\,ds \tag{7-39}$$

This integral obviously defines the quantity of fluid flowing across the curve $C$ in unit time. Write the integral (7-39) as

$$N_C = \int_C \mathbf{v}\,d\mathbf{n} = \int_C \mathrm{v}_x\,dy - \mathrm{v}_y dx = \int_C \frac{\partial u}{\partial x}dy - \frac{\partial u}{\partial y}dx$$

$$= \int_C \frac{\partial v}{\partial x}dx + \frac{\partial u}{\partial x}dy \tag{7-40}$$

In determining the lifting force of the flow of a fluid acting on a body round which the fluid is streaming, an important role is played by the vorticity of the flow, which is characterized by the circulation. The circulation of the velocity vector along a curve $C$ is the line integral of the tangential component of the velocity vector:

$$\Gamma_C = \int_C \mathbf{v} \cdot d\mathbf{s} \tag{7-41}$$

Expressing the velocity $\mathbf{v}$ in terms of the complex potential, we obtain

$$\Gamma_C = \int_C \mathbf{v}\, d\mathbf{s} = \int_C \mathrm{v}_x\, dx + \mathrm{v}_y\, dy = \int_C \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$$

$$= \int_C \frac{\partial u}{\partial x} dx - \frac{\partial v}{\partial x} dy \tag{7-42}$$

Let us consider in the complex plane the integral of the derivative of the complex potential along the curve $C$:

$$\int_C f'(z)\, dz = \int_C \frac{\partial u}{\partial x} dx - \frac{\partial v}{\partial x} dy + i \int_C \frac{\partial v}{\partial x} dx + \frac{\partial u}{\partial x} dy \tag{7-43}$$

A comparison of (7-40), (7-42) and (7-43) leads to the formula

$$\int_C f'(z)\, dz = \Gamma_C + iN_C \tag{7-44}$$

This formula, which gives the expression of circulation and flux of the velocity vector in terms of the derivative of the complex potential, finds numerous applications in hydrodynamics. Note that if the domain $\mathfrak{G}$ in which the motion is considered is a singly connected one, then the integral (7-44) around any closed curve $C$ lying entirely in $\mathfrak{G}$ is equal to zero by the Cauchy theorem. In the case of motion in a multiply connected domain $\mathfrak{G}$, the integral around a closed curve $C$ lying entirely in $\mathfrak{G}$ may be different from zero. This will occur inside the curve $C$ when there is a domain $\mathfrak{G}'$ not belonging to $\mathfrak{G}$, in which there are sources and vortex points of the flow. Clearly, the equations (7-30) and (7-31) are violated in this domain. In a particular case, the domain $\mathfrak{G}'$ may consist of separate points which are then isolated singularities of the analytic function $f(z)$—the complex potential of flow.

Summarizing, *any two-dimensional potential flow in a domain in which there are no sources or vortex points may be described with the aid of a complex potential that is an analytic function of a complex variable.* Thus, the entire apparatus of the theory of analytic functions may be used in the study of this class of flows.

Let us consider a number of examples of elementary flows described by the elementary functions of a complex variable.

(a) Let the complex potential of flow have the form

$$f(z) = az \tag{7-45}$$

where $a = a_1 + ia_2$ is a specified complex number. Then

$$u(x, y) = a_1x - a_2y, \; v(x, y) = a_2x + a_1y$$

and the streamlines $v(x, y) = C$ are straight lines, the slope of which to the $x$-axis is defined by the expression $\tan\alpha = -\frac{a_2}{a_1}$. Formula (7-36) yields

$$w = v_x + iv_y = \overline{f'(z)} = \bar{a} = a_1 - ia_2 \tag{7-46}$$

whence it follows that the rate of flow is constant and the direction of the velocity vector coincides with the straight lines $v(x, y) = C$. And so the function (7-45) defines a plane-parallel flow.

(b) Let the complex potential of flow have the form

$$f(z) = a \ln z \tag{7-47}$$

where $a$ is a real number. In exponential notation, $z = re^{i\varphi}$, we get the expression of the potential and the flow function in polar coordinates:

$$u(r, \varphi) = a \ln r, \; v(r, \varphi) = a\varphi$$

The streamlines are thus rays emanating from the origin, and the equipotential lines are circles centred at the origin. The absolute value of velocity is then

$$|w| = |f'(z)| = \frac{|a|}{|z|} = \frac{|a|}{r} \tag{7-48}$$

and the velocity vector is directed along the ray $\varphi =$ constant. From (7-48) it follows that at the origin the velocity becomes infinite. The point $z = 0$, a singularity of the function $f(z)$, is in this case the source of flow (positive source for $a > 0$, when the velocity is directed from the origin, and negative source, or sink, for $a < 0$, when the velocity is directed to the coordinate origin). Taking an arbitrary closed contour $C$ containing the point $z = 0$, we get, by formula (7-44),

$$\int_C f'(z)\,dz = \int_C \frac{a}{z}\,dz = i2\pi a = \Gamma_C + iN_C$$

Whence $N_C = 2\pi a$. Thus, in the case at hand, the flow of fluid across any closed contour containing the source is constant and equal to $2\pi a$. This quantity is called the strength of the source.

(c) Let the complex potential have the form

$$f(z) = ia \ln z \tag{7-49}$$

where $a$ is a real number. In this case the streamlines are concentric circles centred at the coordinate origin. From formula (7-44), as in the preceding case, we get $N_C = 0$, $\Gamma_C = -2\pi a$. The point $z = 0$ is here called the vortex point of the flow.

(d) Let the complex potential of flow be of the form

$$f(z) = a \ln(z + h) - a \ln(z - h) \tag{7-50}$$

where $a$ is a positive real number and $h$ is some complex constant. According to the foregoing, this potential defines a flow with a positive source at the point $z = -h$ and a sink at the point $z = +h$, the strength of the source and sink is the same and is equal to $2\pi a$. Rewrite (7-50) as

$$f(z) = a2h \frac{\ln(z+h) - \ln(z-h)}{2h}$$

and pass to the limit as $h \to 0$ assuming that the strength of the source and sink then increases so that the quantity $m = a2h$ remains constant. We then get

$$f_0(z) = \frac{m}{z} \tag{7-51}$$

The function (7-51) is the complex potential of a dipole of strength $m$ located at the origin of coordinates. The streamlines of the dipole are obviously defined by the equation

$$-\frac{my}{x^2+y^2} = C$$

or

$$C(x^2+y^2) + my = 0 \tag{7-52}$$

that is to say, they are circles centred on the $y$-axis and tangent to the $x$-axis at the origin of coordinates. Here, the absolute value of the velocity, which is

$$|w| = \frac{m}{|z|^2} = \frac{m}{x^2+y^2} \tag{7-53}$$

tends to zero at infinity.

(e) Let us consider a flow whose complex potential has the form

$$f(z) = v_\infty z + \frac{m}{z} \tag{7-54}$$

where $v_\infty$ and $m$ are positive real numbers. It is obvious that this flow is a superposition of a plane-parallel flow with velocity parallel to the $x$-axis and equal to $v_\infty$ and a flow generated by a dipole of strength $m$ located at the origin. The streamlines of this flow are given by the equation

$$v_\infty y - \frac{my}{x^2+y^2} = C \tag{7-55}$$

The value $C = 0$ is associated with the streamline whose equation is

$$y\left(v_\infty - \frac{m}{x^2+y^2}\right) = 0$$

It breaks up into a straight line $y = 0$ and a circle $x^2 + y^2 = a^2$, where $a^2 = \frac{m}{v_\infty}$. Since

$$f'(z) = v_\infty - \frac{m}{z^2} = v_\infty\left(1 - \frac{a^2}{z^2}\right), \tag{7-56}$$

at infinity the flow velocity is $v_\infty$ and is directed along the $x$-axis. At points of the circle $x^2 + y^2 = a^2$, which is a streamline, the velocity is directed tangentially to this circle. For the absolute value of the velocity at points of the circle $z = ae^{i\varphi}$, we get, from formulas (7-36) and (7-56),

$$|w|\big|_{|z|=a} = |\overline{f'(z)}|\big|_{|z|=a} = v_\infty |1 - e^{2i\varphi}| = 2v_\infty |\sin \varphi| \tag{7-57}$$

In the examples considered above we determined the hydrodynamic characteristics of flow on the basis of a given complex potential. Let us examine the converse problem, *that of determining the complex potential of a flow from its hydrodynamic characteristics.* Note that since the physical velocity of flow is expressed in terms of the derivative of the complex potential [see formula (7-36)], the complex potential is not uniquely defined for a given flow. However, its derivative is a single-valued analytic function. This means that in the neighbourhood of any regular point of flow we have the expansion

$$f'(z) = \sum_{n=1}^{\infty} a_n (z - z_0)^n \tag{7-58}$$

and in the neighbourhood of an isolated singular point, the expansion

$$f'(z) = \sum_{n=-\infty}^{\infty} b_n (z - z_0)^n \tag{7-59}$$

From (7-59) we obtain the following expansion for the complex potential in the neighbourhood of the singular point $z_0$:

$$f(z) = b_{-1} \ln (z - z_0) + \sum_{n=-\infty}^{\infty} c_n (z - z_0)^n \tag{7-60}$$

In particular, if the point at infinity $z_\infty$ belongs to the domain of flow and the complex velocity

$$w_\infty = (v_x)_\infty + i\,(v_y)_\infty$$

of flow at this point is bounded, then the expansion of the complex potential about $z_\infty$ is of the form

$$f(z) = \bar{w}_\infty z + b_{-1} \ln z + \sum_{n=0}^{\infty} \frac{c_n}{z^n} \tag{7-61}$$

From this we have

$$\int_{C_R} f'(z)\,dz = 2\pi i b_{-1} \tag{7-62}$$

where $C_R$ is a circle $|z| = R$ of sufficiently large radius $R$, outside which there are no singularities of the function $f(z)$, except the point $z_\infty$. On the other hand, by virtue of formula (7-44), the integral (7-62) defines a flux and a circulation of the velocity vector across the curve $C_R$. Since the velocity at $z_\infty$ is bounded, this point is not a source, and so the flux of the velocity vector across the curve $C_R$ is zero, and formula (7-62) yields

$$2\pi i b_{-1} = \Gamma_\infty$$

Let us write down the final expansion of the complex potential in the neighbourhood of the point at infinity, which is a regular point of flow:

$$f(z) = \bar{w}_\infty z + \frac{\Gamma_\infty}{2\pi i} \ln z + \sum_{n=0}^{\infty} \frac{c_n}{z^n} \tag{7-63}$$

Let us now consider *the problem of a plane-parallel flux streaming around a closed contour*. Let the flux, which at infinity has a given velocity $w_\infty$ and circulation $\Gamma_\infty$, stream around a body $S$ bounded by a closed contour $C$. It is required to determine the velocity at any point of the flux from the given hydrodynamic characteristics at infinity, provided that at points of the contour $C$ the flow velocity is directed tangentially to the contour $C$. This latter condition implies that the curve $C$ is a streamline of the flow under consideration; that is, the imaginary part of the complex potential describing the given flow must preserve a constant value on the curve $C$

$$v(x, y)\,|_C = \text{constant} \tag{7-64}$$

The problem reduces to determining the analytic function $f(z)$ in the complex plane outside the contour $C$, in the expansion (7-63) of which function are given the values $\bar{w}_\infty$ and $\Gamma_\infty$, and the condition (7-64) is fulfilled on the contour $C$. Since the complex potential is defined to within an additive constant, the value of the constant in the condition (7-64) may be put equal to zero.

Let us begin with the problem of streaming around a circular cylinder of radius $R$ centred at the coordinate origin. Let the flow velocity at infinity be $v_\infty$ and directed parallel to the $x$-axis, and let the circulation be absent, $\Gamma_\infty = 0$. We have to find the complex potential whose expansion about the point at infinity is of the form

$$f(z) = v_\infty z + \sum_{n=0}^{\infty} \frac{c_n}{z^n} \tag{7-65}$$

and the imaginary part of which vanishes for $|z| = R$. We studied a complex potential of this type in Example (e) on page 203. Therefore, the solution of this problem is of the form

$$f(z) = v_\infty \left(z + \frac{R^2}{z}\right) \tag{7-66}$$

The velocity at points lying on the cylinder undergoing streaming is determined by the formula (7-57), whence it follows that it vanishes at two critical points: at $z = -R$, at which the streamline $y = 0$ branches into two streamlines coinciding with the upper and lower semicircles $|z| = R$, and at the point $z = R$, at which these streamlines converge again into the straight line $y = 0$. These points are, respectively, called the branch point and the convergence point. Note that if the velocity of flow at infinity is not parallel to the $x$-axis and has the form $w_\infty = v_\infty e^{i\varphi_0}$, then with the aid of the transformation $\zeta = ze^{-i\varphi_0}$ we arrive at a problem in the $\zeta$-plane that has already been considered. Then we get the following expression for the solution of the original problem

$$f(z) = \bar{w}_\infty z + \frac{w_\infty R^2}{z} \tag{7-67}$$

Now let the circulation $\Gamma_\infty$ be nonzero. As we have already seen [see Example (c) on page 203], the streamlines of a flow with the complex potential $ia \ln z$ ($a$ is a real number) are concentric circles centred at the origin of coordinates. Therefore, the complex potential of a flow streaming around a circular cylinder of radius $R$ with a given velocity at infinity $v_\infty$ and a given circulation $\Gamma_\infty$ has the form

$$f(z) = v_\infty \left(z + \frac{R^2}{z}\right) + \frac{\Gamma_\infty}{2\pi i} \ln z \tag{7-68}$$

Let us find the critical points of the flow at which the flow velocity vanishes. According to formula (7-36) we have

$$\bar{w} = f'(z) = v_\infty \left(1 - \frac{R^2}{z^2}\right) + \frac{\Gamma_\infty}{2\pi i z} = 0$$

Whence

$$z^2+\frac{\Gamma_\infty}{2\pi i v_\infty}z-R^2=0 \qquad (7\text{-}69)$$

and

$$z_{cr}=i\frac{\Gamma_\infty}{4\pi v_\infty}\pm\sqrt{R^2-\frac{\Gamma_\infty^2}{16\pi^2 v_\infty^2}} \qquad (7\text{-}70)$$

For $R\geqslant\left|\frac{\Gamma_\infty}{4\pi v_\infty}\right|$ the radicand in (7-70) is positive. Therefore,

$$|z_{cr}|=\sqrt{R^2-\frac{\Gamma_\infty^2}{16\pi^2 v_\infty^2}+\frac{\Gamma_\infty^2}{16\pi^2 v_\infty^2}}=R$$

that is to say, both critical points lie on the circle $|z|=R$ of the cylinder, and for $\Gamma_\infty>0$ ($v_\infty>0$) both points lie in the upper semicircle, and for $\Gamma_\infty<0$ ($v_\infty>0$) in the lower semicircle. Thus, the

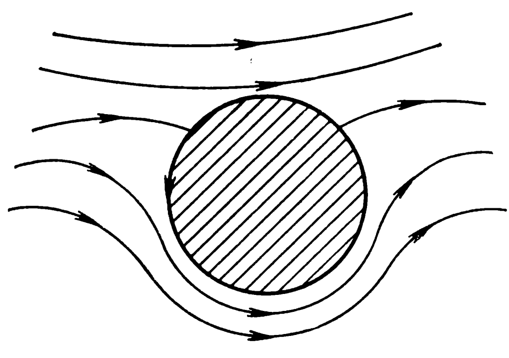

Fig. 7.1

presence of circulation brings closer together the branch point and the convergence point of the streamlines (Fig. 7.1). For $\left|\frac{\Gamma_\infty}{4\pi v_\infty}\right|=R$, both critical points coincide (with the point $z=iR$ for $\Gamma_\infty>0$ or with the point $z=-iR$ for $\Gamma_\infty<0$). Finally, for $\left|\frac{\Gamma_\infty}{4\pi v_\infty}\right|>R$, in the domain $|z|>R$ there is only one critical point lying on the imaginary $y$-axis. (As follows from equation (7-69), the product of the roots of this equation is equal to $-R^2$, and so the second critical point lies inside the circle $|z|=R$.) Through this point passes the streamline separating the closed streamlines of flow from the open streamlines (Fig. 7.2).

The results obtained permit, in principle, the problem of streaming around an arbitrary closed contour $C$ to be solved. Indeed, let the function $\zeta = \varphi(z)$ define a conformal mapping of the domain $\mathfrak{G}$ (of the complex $z$-plane) exterior to the contour $C$ onto the domain $\mathfrak{G}'$ of the $\zeta$-plane, which domain is exterior to the unit circle $|\zeta| = 1$, so that $\varphi(\infty) = \infty$. Then, obviously, the problem at hand is equivalent to the problem of streaming around a circular cylinder of unit radius. Here, the flux velocity at infinity, which generally

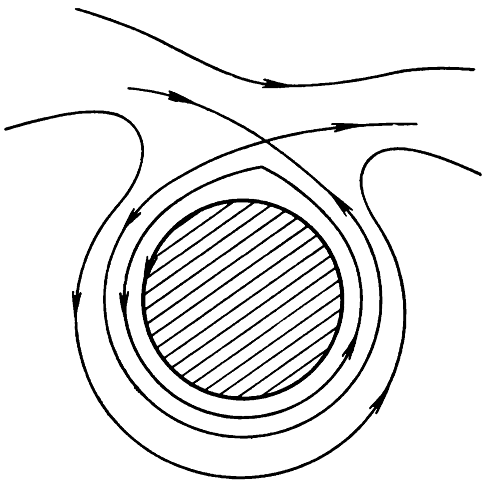

**Fig. 7.2**

speaking will vary, may be readily determined. The complex potential $f(z)$ of the initial flow goes into the function $F(\zeta) = f[z(\zeta)]$ in the given conformal mapping. And so by formula (7-36) we find

$$\overline{W}_\infty = \frac{dF}{d\zeta}\bigg|_{\zeta=\infty} = \frac{df}{dz}\bigg|_{z=\infty} \frac{dz}{d\zeta}\bigg|_{\zeta=\infty} = \overline{w}_\infty \frac{dz}{d\zeta}\bigg|_{\zeta=\infty}$$

and

$$W_\infty = w_\infty \frac{\overline{dz}}{d\zeta}\bigg|_{\zeta=\infty}$$

By formulas (7-67) and (7-68), the solution of the transformed problem is of the form

$$F(\zeta) = \overline{W}_\infty \zeta + \frac{W_\infty}{\zeta} + \frac{\Gamma_\infty}{2\pi i} \ln \zeta$$

Whence, for the solution of the original problem, we get the expression

$$f(z) = F[\zeta(z)] = \overline{w}_\infty \frac{dz}{d\zeta}\bigg|_{\zeta=\infty} \varphi(z) + \frac{w_\infty \left.\dfrac{\overline{dz}}{d\zeta}\right|_{\zeta=\infty}}{\varphi(z)} + \frac{\Gamma_\infty}{2\pi i} \ln \varphi(z) \quad (7\text{-}71)$$

By way of illustration let us consider noncirculatory flow of a two-dimensional fluid around an infinite plate. Let the $x$, $y$-plane intersect the plate along the segment $-a \leqslant x \leqslant a$, and let the velocity vector of the flow lie in the $x$, $y$-plane and at infinity have a given value $w_\infty$. As follows from a consideration of the properties of the Zhukovsky function (see Chapter 6, page 179), the function

$$z = \frac{a}{2}\left(\zeta + \frac{1}{\zeta}\right) = \psi(\zeta) \tag{7-72}$$

defines a conformal mapping of the exterior of the unit circle of the $\zeta$-plane onto the $z$-plane cut along the segment $-a \leqslant x \leqslant a$. Then, $\psi(\infty) = \infty$ and $\left.\frac{dz}{d\zeta}\right|_{\zeta=\infty} = \frac{a}{2}$. For this reason, the problem is equivalent to that of a noncirculatory flow around a circular cylinder of unit radius in the $\zeta$-plane, which at infinity has the complex velocity $W_\infty = \frac{a}{2} w_\infty$. The complex potential of the latter problem has the form

$$F(\zeta) = \frac{a}{2}\left(\overline{w}_\infty \zeta + \frac{w_\infty}{\zeta}\right)$$

Substitute in place of $\zeta$ and $\frac{1}{\zeta}$ the following quantities from (7-72):

$$\zeta = \frac{z + \sqrt{z^2 - a^2}}{a}, \quad \frac{1}{\zeta} = \frac{z - \sqrt{z^2 - a^2}}{a}$$

Here, $\sqrt{z^2 - a^2} > 0$ for $z = x > a$. Partition $w_\infty$ into a real part and an imaginary part:

$$w_\infty = (\mathrm{v}_x)_\infty + i\,(\mathrm{v}_y)_\infty$$

Then for the complex potential of the original problem we get the final expression

$$f(z) = (\mathrm{v}_x)_\infty z - i\,(\mathrm{v}_y)_\infty \sqrt{z^2 - a^2} \tag{7-73}$$

In conclusion we find the force with which the flow acts on the body. The force of pressure acting on an element $ds$ of arc of the contour $C$ is proportional to the hydrodynamic pressure $p$ at the given point of flux and is directed along the inner normal $-d\mathbf{n} = -\mathbf{i}\,dy + \mathbf{j}\,dx$. We therefore get the following expressions for the components of force acting on the contour $C$:

$$R_x = -\int_C p\,dy, \quad R_y = \int_C p\,dx$$

Determining the hydrodynamic pressure $p$ from Bernoulli's integral

$$p = A - \frac{\rho \mathrm{v}^2}{2}$$

where $A$ is constant and $\rho$ is the density of the fluid, and introducing the complex quantity $R = R_y + iR_x$, we obtain

$$R = -\frac{\rho}{2} \int_C \mathrm{v}^2 (dx - i\,dy) = -\frac{\rho}{2} \int_C \mathrm{v}^2 \overline{dz} \qquad (7\text{-}74)$$

(The integral of the constant $A$ around the closed contour $C$ is clearly zero.) We transform the integral (7-74). Since at points of the contour $C$ the velocity is directed tangentially to the contour, the complex velocity of flow $w$ is connected with the magnitude of the physical velocity v by the relation $w = \mathrm{v}e^{i\varphi}$, where $\varphi$ is the angle between the $x$-axis and the tangent to the contour. Then formula (7-36) yields $\mathrm{v}e^{-i\varphi} = f'(z)$. On the other hand, $\overline{dz} = ds\, e^{-i\varphi}$. Therefore $\mathrm{v}^2\, \overline{dz} = \mathrm{v}^2 e^{-i2\varphi}\, ds\, e^{i\varphi} = f'^2\, dz$ and formula (7-74) takes the form

$$R = -\frac{\rho}{2} \int_C f'^2 (z)\, dz \qquad (7\text{-}75)$$

This is *Chaplygin's formula*, which expresses the force exerted by a flow on the body round which it is streaming. It expresses it in terms of the derivative of the complex potential. From the expression (7-63), for the complex potential outside the body, we get

$$f'(z) = \overline{w}_\infty + \frac{\Gamma_\infty}{2\pi i} \cdot \frac{1}{z} + \sum_{n=2}^{\infty} \frac{c'_n}{z^n}$$

$$f'^2(z) = \frac{\overline{w}_\infty}{\pi i} \cdot \frac{\Gamma_\infty}{z} + \overline{w}_\infty^2 + \sum_{n=2}^{\infty} \frac{b_n}{z^n}$$

Hence,

$$\int_C f'^2 (z)\, dz = 2\overline{w}_\infty \Gamma_\infty$$

Substituting this expression into formula (7-75) and separating the real and imaginary parts, we find

$$R_x = \rho\, (\mathrm{v}_y)_\infty \Gamma_\infty, \quad R_y = -\rho\, (\mathrm{v}_x)_\infty \Gamma_\infty \qquad (7\text{-}76)$$

Whence

$$|R| = \rho\, |\mathrm{v}_\infty| \cdot |\Gamma_\infty| \qquad (7\text{-}77)$$

Formula (7-77) is Zhukovsky's theorem on a lifting force: *the force of pressure of an irrotational flow having velocity* $\mathrm{v}_\infty$ *at infinity and flowing round a contour* $C$ *with circulation* $\Gamma$ *is expressed by the formula* $|R| = \rho\, |\mathrm{v}_\infty| \cdot |\Gamma|$. The direction of this force is obtained by rotating the vector $\mathrm{v}_\infty$ through a right angle in the direction opposite that of the circulation.

The apparatus of analytic functions of a complex variable enabled Zhukovsky and Chaplygin to develop methods for solving hydroand aerodynamics problems which served as the theoretical foundation for practical aircraft construction. In this way, the methods of complex-variable theory played a great role in the development of modern aviation.

### b. A two-dimensional electrostatic field

The methods of complex-variable theory used in the preceding subsection in the study of the two-dimensional potential flow of an ideal fluid may be just as successfully employed in the study of any two-dimensional vector field of a different physical nature. Let us consider the use of these methods in solving problems of electrostatics.

Problems of electrostatics consist in determining the stationary electric field generated in a medium by a given distribution of charges. Depending on the statement of a specific physical problem, we are given either the density of distribution of charges as a function of the coordinates or the total charge distributed over the surface of an ideal conductor. In the latter case, the principal aim of the investigation is to determine the density of the distribution of charges on the surface of a conductor.

In order to obtain the basic equations for the intensity vector of an electrostatic field, we will proceed from the general system of Maxwell's equations (see [16]) in an isotropic medium:

$$\operatorname{curl} \mathbf{H} = \frac{1}{c}\frac{\partial \mathbf{D}}{\partial t} + \frac{4\pi}{c}\mathbf{j}^{st}, \quad \operatorname{curl} \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$$

$$\operatorname{div} \mathbf{D} = -4\pi\rho, \quad \operatorname{div} \mathbf{B} = 0$$

$$\mathbf{D} = \varepsilon\mathbf{E}, \quad \mathbf{B} = \mu\mathbf{H}$$

In the case of a stationary electromagnetic field, Maxwell's equations for the intensity vector $\mathbf{E}$ of an electric field in a homogeneous medium take the form

$$\operatorname{curl} \mathbf{E} = 0, \quad \operatorname{div} \mathbf{E} = \frac{4\pi}{\varepsilon}\rho \tag{7-78}$$

where $\varepsilon$ is the dielectric constant of the medium and $\rho$ is the density of static charges generating the given field. We will take $\varepsilon \equiv 1$ and will consider the two-dimensional problem, when charges generating the field are distributed in space so that their distribution density is not dependent on one of the coordinates (say the $z$-coordinate), but is a function solely of the two other coordinates, i.e. $\rho = \rho(x, y)$. Clearly, the vector $\mathbf{E}$ then has only two nonzero components, which are also functions of the coordinates $x$, $y$ alone:

$$\mathbf{E}(x, y) = \mathbf{i}E_x(x, y) + \mathbf{j}E_y(x, y) \tag{7-79}$$

By virtue of the first of the equations (7-78), the field **E** is a potential field:

$$\mathbf{E}(x, y) = -\operatorname{grad} v(x, y), \quad E_x = -\frac{\partial v}{\partial x}, \quad E_y = -\frac{\partial v}{\partial y} \qquad (7\text{-}80)$$

on the basis of the second of the equations (7-78) the function $v(x, y)$ satisfies the equation

$$\Delta v = -4\pi\rho(x, y) \qquad (7\text{-}81)$$

From (7-81) it follows that in a domain free of charges, the potential function $v(x, y)$ is harmonic. It is therefore possible in this domain to construct an analytic function of a complex variable:

$$f(z) = u(x, y) + iv(x, y) \qquad (7\text{-}82)$$

for which the potential function $v(x, y)$ of the given electrostatic field is the imaginary part.

The function (7-82) is the *complex potential of the electrostatic field.* The level lines $v(x, y) = C$ are the equipotential lines of the given field. From formulas (7-80) it follows that at every point of the equipotential line $v(x, y) = C$ the intensity vector E is normal to this line. Since the lines $v(x, y) = C$ and $u(x, y) = C$ are mutually orthogonal, the direction of the vector **E** coincides with the tangent to the line $u(x, y) = C$ at each point of the curve. The lines $u(x, y) = C$ are therefore force lines of the given field.

We associate with the vector **E** a complex number $w = E_x + iE_y$. Then by (7-80) and from the Cauchy-Riemann conditions we get

$$w = E_x + iE_y = -\frac{\partial v}{\partial x} - i\frac{\partial v}{\partial y} = -\frac{\partial v}{\partial x} - i\frac{\partial u}{\partial x}$$
$$= -i\left(\frac{\partial u}{\partial x} - i\frac{\partial v}{\partial x}\right) = -i\overline{f'(z)} \qquad (7\text{-}83)$$

Whence

$$|\mathbf{E}| = |f'(z)| \qquad (7\text{-}84)$$

Formulas (7-83) and (7-84) yield the expression of the components of the intensity vector of an electrostatic field in a domain free of charges in terms of the derivative of the complex potential.

Let the charges generating the given electrostatic field be concentrated in some domain bounded by the closed curve $C_0$.* Then the integral, around any closed contour $C$ containing $C_0$, of the normal

* This means that in space the charges are distributed inside an infinite cylinder, the contour of the cross section of which is the curve $C_0$; the distribution density of the charges does not depend on the coordinate $z$ along the generatrix of the cylinder, but is only a function of the coordinates $x$, $y$ in the cross section.

component of intensity of the electric field is, by the Gauss theorem, (see [16]) equal to the total charge (referred to unit length of the cylinder in which the charges are distributed in space):

$$\int_C E_n\, ds = 4\pi e \tag{7-85}$$

On the basis of formulas (7-80), (7-37) and (7-38), and taking into account the Cauchy-Riemann relations, we obtain

$$\int_C E_n\, ds = \int_C \frac{\partial u}{\partial x} dx - \frac{\partial v}{\partial x} dy$$

Since the electrostatic field is everywhere potential, the circulation of the field around any closed contour is zero, i.e.

$$\int_C E_s\, ds = -\int_C \frac{\partial v}{\partial x} dx + \frac{\partial u}{\partial x} dy = 0$$

Consider the integral of the derivative of the complex potential around the closed contour $C$:

$$\int_C f'(z)\, dz = \int_C \frac{\partial u}{\partial x} dx - \frac{\partial v}{\partial x} dy + i \int_C \frac{\partial v}{\partial x} dx + \frac{\partial u}{\partial x} dy \tag{7-86}$$

A comparison of the foregoing formulas yields

$$\int_C f'(z)\, dz = \int_C E_n\, ds = 4\pi e \tag{7-87}$$

that is, *a charge contained in a domain bounded by the contour $C$ is defined by the integral, along this contour, of the derivative of the complex potential of the electrostatic field generated by the given charge distribution.* If $C_0$ is the contour of the cross section of an ideally conducting cylinder, then the entire charge is concentrated on its surface with surface density $\sigma(s)$, and

$$\int_{C_0} \sigma(s)\, ds = e \tag{7-88}$$

The following relation is known to hold (see [12]):

$$\sigma(s) = \frac{1}{4\pi} E_n \Big|_{C_0} = -\frac{1}{4\pi} (\operatorname{grad} v)_n \Big|_{C_0} \tag{7-89}$$

On the other hand, from (7-83) and (7-89) we get

$$\sigma(s) = \pm \frac{1}{4\pi} | f'(z) |_{C_0} \tag{7-90}$$

The sign in (7-90) is determined by the sign of the overall charge $e$, distributed over the surface of the given ideally conducting conductor. Formula (7-90) finds extensive application in the solution of a diversity of problems in electrostatics.

Note, finally, that as in the problems of hydrodynamics, the derivative $f'(z)$ of the complex potential is, by (7-83), a single-valued analytic function of $z$. If the intensity of the given electrostatic field is bounded at infinity, then the expansion of $f'(z)$ about the point $z = \infty$ is of the form

$$f'(z) = w_\infty + \sum_{n=1}^{\infty} \frac{b_n}{z^n}$$

Whence for the complex potential itself we get the expansion

$$f(z) = w_\infty z + c_0 + b_1 \ln z + \sum_{n=1}^{\infty} \frac{c_n}{z^n} \tag{7-91}$$

Since

$$b_1 = \frac{1}{2\pi i} \int_{C_R} f'(z)\, dz$$

where the contour $C_R$ contains all the charges generating the given field, from (7-87) we get the final expansion of the complex potential about the point $z = \infty$ in the form

$$f(z) = w_\infty z - i2e \ln z + \sum_{n=0}^{\infty} \frac{c_n}{z^n} \tag{7-92}$$

We thus see that the complex potential of an electrostatic field has very much in common with the complex hydrodynamic potential.* Therefore, the investigation of a two-dimensional electrostatic field with the aid of the complex potential may be carried out by the very same methods as the solution of the corresponding hydrodynamic problems. Thus, all the examples of flows examined on pages 202-203 admit a simple electrostatic interpretation.

For example, consider the electrostatic field described by the complex potential

$$f(z) = -i2e \ln z, \quad e > 0 \tag{7-93}$$

---

* It is quite obvious that the fact that the potential function in electrostatics is the imaginary part of the complex potential and in hydrodynamics the potential of velocity is the real part of the complex potential is an unessential difference that may be eliminated by introducing an additional factor equal to $-i$. However, we hold to the established terminology in which the indicated difference exists.

Introducing the polar coordinates $r, \varphi$ and taking into account that $z = re^{i\varphi}$, we get

$$v(r, \varphi) = -2e \ln |z| = 2e \ln \frac{1}{r}, \quad u(r, \varphi) = 2e \arg z = 2e\varphi$$

This implies that the equipotential surfaces of the given field are concentric circles centred at the origin, and the force lines are the rays $\varphi$ = constant. The vector $\mathbf{E}$ is at every point $z \neq 0$ directed along a ray $\varphi$ = constant and, by formula (7-84), is in absolute value equal to

$$|\mathbf{E}| = |f'(z)| = \frac{2e}{r}$$

Since the integral of the normal component of intensity of the given field around any circle $|z| = r$ has a constant value equal to $4\pi e$, it is obvious that the field is generated by a point charge of magnitude $e$ situated at the origin (in space, the charges generating a given field are distributed with constant density $e$ along a straight line perpendicular to the $x$, $y$-plane and passing through the origin of coordinates).

Let us consider some typical problems in electrostatics that may be solved with the aid of a complex potential.

(a) *Determining the distribution density of a charge on an ideally conducting conductor.* Let the lateral surface of such a conductor be an infinite cylinder whose cross section is bounded by the contour $C$. Suppose that the distribution density of the charge is constant along the generatrices of the cylinder and there is charge $e$ per unit length of the cylinder. It is required to determine the surface density of charge $\sigma(s)$ on the contour $C$ of the cross section. The solution of the problem is obviously given by formula (7-90) for the normalization condition (7-88). The problem thus reduces to constructing the complex potential $f(z)$, which is an analytic function outside the contour $C$, provided that the imaginary part of $f(z)$ is constant on $C$ and the expansion of $f(z)$ in the neighbourhood of the point $z = \infty$ is given by (7-92), where $w_\infty = 0$ and the coefficient $e$ is equal to the charge per unit length of conductor.

Start with the simplest case when the conductor is a circular cylinder of unit radius. It was shown above (see page 214) that the equipotential lines of the complex potential (7-93) are concentric circles centred at the coordinate origin. Therefore, to satisfy the condition on the boundary of the conductor, it is natural to seek the potential of the given field in the form

$$f(z) = -iC \ln z$$

where $C$ is a constant that needs defining. From the condition at infinity (7-92) we get $C = 2e$. Then formula (7-90) yields the obvious result

$$\sigma(s)=\frac{e}{2\pi}$$

If the contour of the cross section of the conductor is an arbitrary closed curve $C$, then by mapping, by means of the function $\zeta = \varphi(z)$, the field exterior to the contour $C$ conformally onto the exterior of the unit circle $|\zeta| > 1$ in such a manner as to satisfy the condition $\varphi(\infty) = \infty$, we reduce the problem to that which has just been solved. Then the complex potential will have the form

$$f(z) = -i2e \ln \varphi(z) \tag{7-94}$$

and for the density of surface charges we get the following expression in accordance with (7-90):

$$\sigma(s)=\frac{1}{4\pi}|f'(z)|_C=\frac{e}{2\pi}\left|\frac{1}{\varphi(z)}\cdot\frac{d\zeta}{dz}\right|_C = \frac{e}{2\pi}\left|\frac{d\zeta}{dz}\right|_C=\frac{e}{2\pi}\left|\frac{dz}{d\zeta}\right|^{-1}_{|\zeta|=1} \tag{7-95}$$

By way of illustration, let us consider the problem of determining the charge density in a strip of width $2a$. Let this strip intersect the $x$, $y$-plane along the segment $-a < x < a$. The function

$$z=\frac{a}{2}\left(\zeta+\frac{1}{\zeta}\right)$$

defines a conformal mapping of the exterior of the unit circle of the $\zeta$-plane onto the $z$-plane cut along the segment $-a < x < a$ of the real axis. Therefore, formula (7-95) yields

$$\sigma(x)=\frac{e}{2\pi}\left|\frac{dz}{d\zeta}\right|^{-1}_{|\zeta|=1}=\frac{e}{a\pi}\frac{1}{|\zeta^2-1|_{|\zeta|=1}} \tag{7-96}$$

**Since**

$$\zeta=\frac{z+\sqrt{z^2-a^2}}{a}$$

**and**

$$\zeta^2-1=\frac{2}{a^2}\left(z^2-a^2+z\sqrt{z^2-a^2}\right)=\frac{2\sqrt{z^2-a^2}}{a^2}\left(z+\sqrt{z^2-a^2}\right)$$

formula (7-96) yields

$$\sigma(x)=\frac{ea}{2\pi}\cdot\frac{1}{\sqrt{a^2-x^2}}\cdot\frac{1}{\left|x+i\sqrt{a^2-x^2}\right|_{-a<x<a}}=\frac{e}{2\pi}\cdot\frac{1}{\sqrt{a^2-x^2}} \tag{7-97}$$

Note that the charge density increases without bound as the edge of the plate is approached. This fact has a simple physical interpretation. The edge of the plate has an infinite curvature, and so an infinite charge has to be placed on it in order to charge it to some potential.

(b) *Determining the field of an infinite two-dimensional capacitor.* Let it be required to find the electrostatic field between two ideally conducting nonintersecting cylindrical surfaces charged to a certain potential, the generatrices of which surfaces are parallel and the directrices pass through the point at infinity of the $z$-plane (Fig. 7.3).

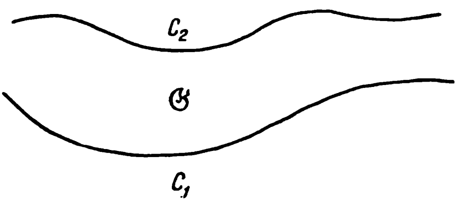


**Fig. 7.3**

Here the problem consists in determining, in a curvilinear strip $\mathfrak{G}$, the complex potential $f(z)$, which is an analytic function the imaginary part of which assumes the constant values $v_1$ and $v_2$ on the curves $C_1$ and $C_2$. Clearly, the analytic function $w = f(z)$ defines

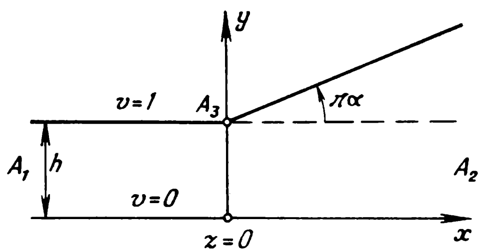


**Fig. 7.4**

a conformal mapping of the given curvilinear strip of the $z$-plane onto a strip of the $w$-plane bounded by the straight lines Im $w = v_1$, Im $w = v_2$. Thus, to solve the problem it suffices to construct the indicated conformal mapping.

To illustrate, let us find the field of the capacitor shown in Fig. 7.4 if the values of the potential on the curves $C_1$ and $C_2$ are 0 and 1, respectively. First find the function $z = \varphi(\zeta)$ that defines the conformal mapping of the upper half of the $\zeta$-plane, Im $\zeta > 0$, onto the given curvilinear strip $\mathfrak{G}$ of the $z$-plane. Since the domain is the triangle* $A_1A_2A_3$, the desired mapping may be obtained by

* Note that the vertices $A_1$ and $A_2$ lie at infinity.

means of the Schwartz-Christoffel integral (see Section 5.4). We establish the following correspondence of points of the real axis of the $\zeta$-plane and the vertices of the triangle:

$$A_1 \to \zeta = 0$$
$$A_2 \to \zeta = \infty$$
$$A_3 \to \zeta = -1$$

Since the angles at the vertices of the triangle are equal, respectively, to $\pi\alpha_1 = 0$, $\pi\alpha_2 = -\pi\alpha$ and $\pi\alpha_3 = \pi(1+\alpha)$, the desired integral must be of the form

$$z = C\int_{\zeta_0}^{\zeta} \zeta^{-1}(1+\zeta)^{\alpha}\,d\zeta + C_1 \tag{7-98}$$

From the correspondence of the points $A_3$ $(z = ih)$ and $\zeta = -1$ it follows that for $\zeta_0 = -1$, we get

$$z = C\int_{-1}^{\zeta} \frac{(1+\zeta)^{\alpha}}{\zeta}\,d\zeta + ih \tag{7-99}$$

In order to determine the constant $C$, note that to the counterclockwise traversal of the point $\zeta = 0$ in the upper half-plane around the semicircular arc of infinitely small radius $\rho$ there corresponds a transition from the side $A_2A_1$ to the side $A_1A_3$. Here, the increment of $z$ is

$$\Delta z = ih$$

On the other hand, from (7-99), putting $\zeta = \rho e^{i\varphi}$ and taking the limit as $\rho \to 0$, we get

$$\Delta z = iC\lim_{\rho\to 0}\int_0^{\pi}(1+\rho e^{i\varphi})^{\alpha}\,d\varphi = i\pi C$$

Whence $C = \frac{h}{\pi}$ and the final expression for the integral (7-99) is of the form

$$z = \frac{h}{\pi}\int_{-1}^{\zeta}\frac{(1+\zeta)^{\alpha}}{\zeta}\,d\zeta + ih$$

The function $\zeta = e^{\pi w}$ defines a conformal mapping of the strip $0 < \operatorname{Im} w < 1$ of the $w$-plane onto the upper half of the $\zeta$-plane. Therefore, the function

$$z = \frac{h}{\pi}\int_{-1}^{e^{\pi w}}\frac{(1+\zeta)^{\alpha}}{\zeta}\,d\zeta + ih \tag{7-100}$$

defines a conformal mapping of the strip $0 < \operatorname{Im} w < 1$ of the $w$-plane onto the given curvilinear strip $\mathfrak{S}$ of the $z$-plane. In the process, the straight line $\operatorname{Im} w = 0$ goes into the lower plate of the capacitor $A_1A_2$, and the straight line $\operatorname{Im} w = 1$ into the upper plate, which is the polygonal line $A_2A_3A_1$. From (7-100) for $v = \operatorname{Im} w =$ constant we get the parametric equations of the potential curves of the

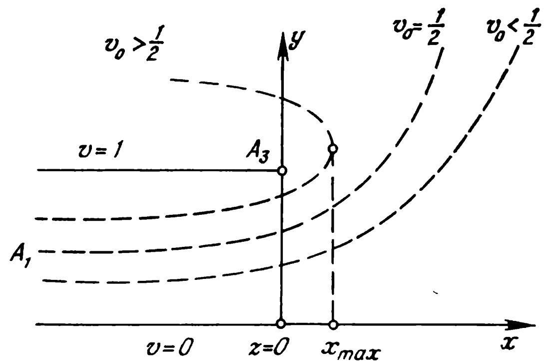


**Fig. 7.5**

given electrostatic field. For example, in the particular case of $\alpha = 1$ the integral (7-100) may be evaluated in terms of elementary functions:

$$z = \frac{h}{\pi}(1 + \pi w + e^{\pi w})$$

Then the parametric equations of the equipotential curve $v = v_0 =$ = constant $(0 \leqslant v_0 \leqslant 1)$ assume the form

$$x = \frac{h}{\pi}(1 + \pi u + \cos \pi v_0 \cdot e^{\pi u})$$

$$-\infty < u < \infty$$

$$y = \frac{h}{\pi}(\pi v_0 + \sin \pi v_0 \cdot e^{\pi u})$$

In particular, the equation of the mean equipotential line $\left(v_0 = \frac{1}{2}\right)$ has the form

$$y = \frac{h}{2} + \frac{h}{\pi} e^{\frac{\pi}{h}x - 1}$$

Equipotential lines corresponding to various values of $v$ are given in Fig. 7.5. For $v_0 > \frac{1}{2}$ it is easy to determine the value of $x_{max}$

from the formula

$$x_{\max} \doteq \frac{h}{\pi} \ln \left( -\frac{1}{\cos \pi v_0} \right)$$

The results obtained make it easy to determine the distance from the edge of the capacitor, shown in Fig. 7.5, on which the field of the capacitor may, to within a specified degree of accuracy, be considered two-dimensional.

Generally speaking, conformal-mapping methods are widely employed in designing two-dimensional electrostatic and magnetostatic lenses used for focussing electronic beams, which find extensive application in numerous physical devices.

# CHAPTER 8

# FUNDAMENTALS OF OPERATIONAL CALCULUS

The methods of operational calculus represent a peculiar approach to the solution of various mathematical problems, mainly differential equations. Underlying these methods is the idea of integral transformations. Here we associate with the solution of the original problem [a function $f(t)$ of a real variable] some function $F(p)$ of a complex variable so that the ordinary differential equation for the function $f(t)$ is transformed into an algebraic equation for $F(p)$. In a similar manner, an ordinary differential equation may be associated with a partial differential equation for a function of two real variables, and so forth. This simplifies computational techniques. In operational calculus, the fundamental entity is the Laplace transformation, to the study of which we now turn.

## 8.1. Basic Properties of the Laplace Transformation

### *a. Definition*

The Laplace transformation associates a function $F(p)$ of the complex variable $p$ with a function $f(t)$ of a real variable $t$ by means of the relation

$$F(p)=\int_0^\infty e^{-pt}f(t)\,dt$$

Naturally, this integral is not meaningful for every function $f(t)$. We therefore begin with a definition of the class of functions $f(t)$ for which the given transformation can definitely be realized. Consider the function $f(t)$ defined for all values of the real variable $-\infty < t < \infty$ and satisfying the following conditions:

1. For $t<0$ $f(t)\equiv 0$.

2. For $t \geqslant 0$, the function $f(t)$ has on every finite interval of the $t$-axis at most a finite number of discontinuities of the first kind.

3. The function $f(t)$ has a bounded order of growth at $t\to\infty$, i.e. for every function of the class under consideration there exist positive constants $M$ and $a$, such that for all $t>0$

$$|f(t)| \leqslant Me^{at} \qquad (8\text{-}1)$$

The greatest lower bound of those values of $a$ for which inequality (8-1) holds is called the *index of the order of growth* of the function $f(t)$. It is easy to see, in particular, that the index of the order of growth of the power function $f(t) = t^n$ is zero.

Note that $f(t)$ may be a complex function of the real variable $t$: $f(t) = f_1(t) + if_2(t)$, where $f_1(t)$ and $f_2(t)$ are real functions.

We introduce the basic definition:

*The Laplace transformation of a given function $f(t)$ of the real variable $t$ is a transformation which associates with a function $f(t)$ a function $F(p)$ of the complex variable $p$ defined by the integral*

$$F(p) = \int_0^\infty e^{-pt} f(t)\,dt \tag{8-2}$$

Note that the integral (8-2) is an improper integral dependent on the variable $p$ as a parameter. Generally speaking, the integral (8-2) does not converge for all values of the parameter $p$. Indeed, if the function $f(t)$ approaches a nonzero limit as $t \to \infty$, and $\operatorname{Re} p < 0$, then the integral definitely diverges. It is therefore natural to pose the problem of the domain of convergence of the integral (8-2), and hence also that of the domain of definition of the function $F(p)$.

***Theorem 8.1.*** *The integral* (8-2) *converges in the domain* $\operatorname{Re} p > a$, *where $a$ is the index of the order of growth of the function $f(t)$, and in the domain* $\operatorname{Re} p \geqslant x_0 > a$ *the integral converges uniformly.*

*Proof.* For any $p = x + iy$ for $x > a$, we can specify* an $\varepsilon > 0$, such that $x > a_1 = a + \varepsilon$, and $|f(t)| < Me^{a_1 t}$. Then, taking advantage of the comparison test for the convergence of improper integrals, we get

$$|F(p)| = \left| \int_0^\infty e^{-pt} f(t)\,dt \right| \leqslant M \int_0^\infty e^{-xt} e^{a_1 t}\,dt = \frac{M}{x - a_1}, \quad x > a_1 \tag{8-3}$$

which is grounds for concluding that the integral (8-2) converges for $x > a$. If $x \geqslant x_0 > a$, then an analogous evaluation yields

$$|F(p)| \leqslant \int_0^\infty e^{-(x_0 - a_1)t}\,dt = \frac{M}{x_0 - a_1} \tag{8-4}$$

which, by the Weierstrass test, proves the uniform convergence of the integral (8-2) with respect to the parameter $p$ in the domain $\operatorname{Re} p \geqslant x_0 > a$.

The foregoing proof rested substantially upon the conditions (2) and (3) of the definition of this class of functions $f(t)$ of the real

---

* This permits considering the unbounded functions with zero index of the order of growth.

variable $t$. However, it is possible to extend the class of functions $f(t)$ which admit the Laplace transformation. First we prove the following lemma.

***Lemma.*** *Let the function $f(t)$ of the real variable $t$ be defined for all $t \geqslant 0$ and let there exist a complex number $p_0$ such that the integral*

$$\int_0^\infty e^{-p_0 t} f(t)\, dt < M \tag{8-5}$$

*converges. Then for all $p$ satisfying the condition* $\operatorname{Re} p > \operatorname{Re} p_0$ *the integral*

$$\int_0^\infty e^{-pt} f(t)\, dt \tag{8-6}$$

*converges.*

*Proof.* Denote $\varphi(t) = e^{-p_0 t} f(t)$ and introduce an auxiliary function $F(t) = -\int_t^\infty \varphi(\tau)\, dt$. Note that $F'(t) = \varphi(t)$. Besides, by virtue of the convergence of the integral (8-5) it is obvious that for a given $\varepsilon' > 0$ it is possible to indicate a $T_0$ such that $|F(t)| < \varepsilon'$ for $t \geqslant T_0$.

Now consider the integral $\int_{T_1}^{T_2} e^{-pt} f(t)\, dt$, where $T_1$ and $T_2$ are arbitrary real numbers satisfying the condition $T_2 > T_1$, and represent it in the form

$$\int_{T_1}^{T_2} e^{-pt} f(t)\, dt = \int_{T_1}^{T_2} e^{-(p-p_0)t} \varphi(t)\, dt = \int_{T_1}^{T_2} e^{-(p-p_0)t} F'(t)\, dt$$

Evaluating the last integral by parts, we obtain

$$\int_{T_1}^{T_2} e^{-(p-p_0)t} F'(t)\, dt$$

$$= e^{-(p-p_0)T_2} F(T_2) - e^{-(p-p_0)T_1} F(T_1) + (p-p_0) \int_{T_1}^{T_2} e^{-(p-p_0)t} F(t)\, dt$$

From this, for $T_1$, $T_2 > T_0$ and $\operatorname{Re}(p-p_0) > 0$ we get

$$\left| \int_{T_1}^{T_2} e^{-pt} f(t)\, dt \right| \leqslant \left(e^{-\operatorname{Re}(p-p_0)T_2} + e^{-\operatorname{Re}(p-p_0)T_1}\right) \varepsilon'$$

$$+ \varepsilon' \frac{|p-p_0|}{\operatorname{Re}(p-p_0)} \left(e^{-\operatorname{Re}(p-p_0)T_1} - e^{-\operatorname{Re}(p-p_0)T_2}\right) < \varepsilon' \left[2 + \frac{|p-p_0|}{\operatorname{Re}(p-p_0)}\right] \times e^{-\operatorname{Re}(p-p_0)T_0}$$

Obviously, it is always possible to choose the value of $T_0$ so that the expression obtained is less than any prescribed $\varepsilon > 0$. This, on the basis of the Cauchy test, proves the convergence of the integral (8-6).

It is also possible to prove the uniform convergence, in the parameter $p$, of the integral (8-6) in the domain $\operatorname{Re} p \geqslant \operatorname{Re} p_1 > \operatorname{Re} p_0$.

On the basis of the lemma that has been proved, we can regard the functions satisfying the condition (8-5) as the basic class of functions $f(t)$ of the real variable $t$ for which the Laplace transformation (8-2) is constructed. Functions satisfying the given condition will be called functions belonging to the class $A(p_0)$.

Thus, with the aid of the transformation (8-2), the function $F(p)$ of the complex variable $p$ is defined in the half-plane of the complex $p$-plane to the right of the straight line $\operatorname{Re} p = a$, parallel to the imaginary axis.

Observe that from formula (8-3) it follows that $|F(p)| \to 0$ as $\operatorname{Re} p \to \infty$.

The function $F(p)$, defined in terms of the function $f(t)$ with the aid of the transformation (8-2) is called *the Laplace transform* of $f(t)$. The function $f(t)$ is the *original function* of $F(p)$. We will denote the relationship of the functions $f(t)$ and $F(p)$ by the symbols*

$$f(t) \risingdotseq F(p) \text{ or } F(p) \risingdotseq f(t) \tag{8-7}$$

It should be noted that in practical applications frequent use is also made of the so-called *Heaviside transformation*:

$$\widetilde{F}(p) = p \int_0^{\infty} e^{-pt} f(t)\, dt \tag{8-8}$$

which differs from the Laplace transformation by the additional factor $p$. It is clear that the domain of definition of the function $\widetilde{F}(p)$ is the same as that of the function $F(p)$. We will only consider the Laplace transformation (8-2). The properties of the Heaviside transformation (8-8) are readily obtained from the properties of the Laplace transformation that will now be examined.

As we have seen, analytic functions form the most important class of functions of a complex variable. Let us find out whether the function $F(p)$ is analytic.

***Theorem 8.2.*** *The Laplace transform* (8-2) *of the function* $f(t)$ *is an analytic function of the complex variable* $p$ *in the domain* $\operatorname{Re} p > a$, *where* $a$ *is the index of the order of growth of* $f(t)$.

---

* Other notations are: $F(p) \to f(t)$,

$F(p) \;\dot{\underset{\cdot}{\to}}\; f(t)$,

$F(p) \parallel f(t)$, etc.

*Proof.* By Theorem 8.1, the improper integral (8-2) converges in the domain $\operatorname{Re} p > a$. Let us partition the integration interval into subintervals $[t_i, t_{i+1}]$ of arbitrary finite length; also $t_0 = 0$ and $t_n \to \infty$ as $n \to \infty$. Then the function $F(p)$, for $\operatorname{Re} p > a$, is the sum of the convergent series

$$F(p) = \sum_{n=0}^{\infty} \int_{t_n}^{t_{n+1}} e^{-pt} f(t)\, dt = \sum_{n=0}^{\infty} u_n(p) \tag{8-9}$$

Note that since the $n$th remainder of the series (8-9) is equal to $\int_{t_{n+1}}^{\infty} e^{-pt} f(t)\, dt$, by Theorem 8.1 the series (8-9)converges uniformly in the domain $\operatorname{Re} p \geqslant x_0 > a$. Each of the functions

$$u_n(p) = \int_{t_n}^{t_{n+1}} e^{-pt} f(t)\, dt$$

is defined as an integral, dependent on the parameter $p$, over a subinterval of finite length in the complex $t$-plane. On the basis of the general properties of integrals of functions of two complex variables dependent on a parameter,* the functions $u_n(p)$ are entire functions of $p$. From the foregoing reasoning, it follows that the series (8-9) in the domain $\operatorname{Re} p > a$ satisfies all the conditions of the Weierstrass theorem** and, hence, the function $F(p)$ is analytic in the domain $\operatorname{Re} p > a$ and its derivatives may be computed by differentiating the integrand function in (8-2) with respect to the parameter $p$.

*b. Transforms of elementary functions*

Taking advantage of the definition (8-2), we find the transforms of a number of elementary functions of a real variable.

(a) *Heaviside unit function.* Let

$$f(t) = \sigma_0(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0 \end{cases} \tag{8-10}$$

Then

$$f(t) \risingdotseq F(p) = \int_0^{\infty} e^{-pt}\, dt = \frac{1}{p}$$

---

* See Chapter 1, page 53.
** See Chapter 2, page 63.

and the function $F(p)$ is obviously defined in the domain $\operatorname{Re} p > 0$. Hence

$$\sigma_0(t) = \begin{cases} 0, & t < 0 \\ 1, & t \geqslant 0 \end{cases} \risingdotseq \frac{1}{p}, \quad \operatorname{Re} p > 0 \tag{8-11}$$

Observe that if in place of the Laplace transformation (8-2) we use the Heaviside transformation (8-8), then the function $\widetilde{F}(p) \equiv 1$ will be the transform of the unit function $\sigma_0(t)$. This explains the relative popularity of the Heaviside transformation. However, in the case of the Heaviside transformation (8-8) a number of other formulas, including the formula of the inverse transformation and the formula of the transform of a convolution (see page 232 below), become more complicated.

Let us agree from now on (unless otherwise specified) to regard the function $f(t)$ as the product $f(t) \cdot \sigma_0(t)$, that is, as a function identically zero for $t < 0$, without specially indicating this in the appropriate formulas.

(b) *Exponential function*:

$$f(t) = e^{\alpha t} \tag{8-12}$$

Computing the integral (8-2), we get

$$F(p) = \int_0^\infty e^{-pt} e^{\alpha t}\, dt = \frac{1}{p-\alpha}, \quad \operatorname{Re} p > \operatorname{Re} \alpha$$

$$e^{\alpha t} \risingdotseq \frac{1}{p-\alpha}, \quad \operatorname{Re} p > \operatorname{Re} \alpha \tag{8-13}$$

(c) *Power function*:

$$f(t) = t^\nu, \ \nu > -1 \tag{8-14}$$

In this case, the integral (8-2) is of the form

$$F(p) = \int_0^\infty e^{-pt} f(t)\, dt = \int_0^\infty e^{-pt} t^\nu\, dt, \quad \operatorname{Re} p > 0 \tag{8-15}$$

Note that for $\nu < 0$, the function (8-14) no longer satisfies Condition 2 on page 221 (the point $t = 0$ is a discontinuity of the second kind of this function) and thus does not belong to this basic class of functions of a real variable, for which the Laplace transform exists. However, as is readily seen, for $\nu > -1$ this function belongs to an extended class introduced on page 223 [the integral (8-15) converges for $\operatorname{Re} p > 0$ and $\nu > -1$]. For this reason, in the case $-1 < \nu < 0$ as well the Laplace transform of the function (8-14) in the domain $\operatorname{Re} p > 0$ exists and is defined by formula (8-15).

Let us evaluate the integral (8-15). We begin with the case when the variable $p$ assumes a real value $p = x > 0$. Making the change of variable $xt = s$ in the integral (8-15), we get

$$F(x) = \int_0^\infty e^{-xt} t^\nu \, dt = \frac{1}{x^{\nu+1}} \int_0^\infty e^{-s} s^\nu \, ds = \frac{\Gamma(\nu+1)}{x^{\nu+1}} \tag{8-16}$$

where $\Gamma(\nu + 1)$ is Euler's gamma-function. Since the function $F(p)$ defined by formula (8-15) is analytic in the domain $\operatorname{Re} p > 0$, which on the positive real axis $x > 0$ has the value (8-16), it follows that by virtue of the uniqueness of analytic continuation for the function $F(p)$ in the domain $\operatorname{Re} p > 0$ we get the expression

$$F(p) = \int_0^\infty e^{-pt} t^\nu \, dt = \frac{\Gamma(\nu+1)}{p^{\nu+1}} \tag{8-17}$$

Here, in the case of fractional $\nu$ one should choose the branch of the multiple-valued function $\frac{1}{p^{\nu+1}}$ which is a direct analytic continuation into the domain $\operatorname{Re} p > 0$ of the real function $\frac{1}{x^{\nu+1}}$ of the real variable $x > 0$. Thus,

$$t^\nu \doteqdot \frac{\Gamma(\nu+1)}{p^{\nu+1}}, \quad \nu > -1, \quad \operatorname{Re} p > 0 \tag{8-18}$$

For integral $\nu = n$, we get from formula (8-18)

$$t^n \doteqdot \frac{\Gamma(n+1)}{p^{n+1}} = \frac{n!}{p^{n+1}}, \quad \operatorname{Re} p > 0 \tag{8-19}$$

Computing the integral (8-2), we can get the transforms of some more functions of a real variable; however, in many cases it turns out to be more convenient, when computing the transforms of a given function, to take advantage of the general properties of the Laplace transform, which we will now investigate.

### *c. Properties of a transform*

(a) *Linearity*. By virtue of familiar properties of definite integrals we have:

*Property 1*. If $F_i(p) \doteqdot f_i(t)$, $\operatorname{Re} p > a_i$ $(i = 1, \ldots, n)$, then

$$F(p) = \sum_{i=1}^{n} \alpha_i F_i(p) \doteqdot \sum_{i=1}^{n} \alpha_i f_i(t), \quad \operatorname{Re} p > \max a_i \tag{8-20}$$

where $\alpha_i$ are specified constants (real or complex) and $a_i$ are indices of the order of growth of the functions $f_i(t)$.

This property enables one, on the basis of the transforms of the functions (8-13), (8-18) and (8-19) that we have found, to find the transforms of a polynomial, and of trigonometric and hyperbolic functions. For example, with the aid of (8-13), we get

$$\cos \omega t = \frac{1}{2}\left(e^{i\omega t} + e^{-i\omega t}\right) \risingdotseq \frac{1}{2}\left(\frac{1}{p-i\omega} + \frac{1}{p+i\omega}\right) = \frac{p}{p^2+\omega^2}, \quad \operatorname{Re} p > |\operatorname{Im} \omega| \tag{8-21}$$

Similarly,

$$\sin \omega t \risingdotseq \frac{\omega}{p^2+\omega^2}, \quad \operatorname{Re} p > |\operatorname{Im} \omega| \tag{8-22}$$

(b) *Property 2.* Let $F(p) \risingdotseq f(t)$, $\operatorname{Re} p > a$, then

$$\frac{1}{\alpha} F\left(\frac{p}{\alpha}\right) \risingdotseq f(\alpha t), \quad \alpha > 0, \quad \operatorname{Re} p > a \tag{8-23}$$

Indeed,

$$\int_0^\infty e^{-pt} f(\alpha t)\, dt = \frac{1}{\alpha} \int_0^\infty e^{-\frac{p}{\alpha}\tau} f(\tau)\, d\tau = \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right)$$

(c) *Property 3 (Time-delay theorem).* Let $F(p) \risingdotseq f(t)$, $\operatorname{Re} p > a$ and let the following function be given:

$$f_\tau(t) = \begin{cases} 0, & t < \tau, \quad \tau > 0 \\ f(t-\tau), & t \geqslant \tau \end{cases} \tag{8-24}$$

Then

$$f_\tau(t) \risingdotseq F_\tau(p) = e^{-p\tau} F(p) \tag{8-25}$$

Indeed,

$$F_\tau(p) = \int_0^\infty e^{-pt} f_\tau(t)\, dt = \int_\tau^\infty e^{-pt} f(t-\tau)\, dt$$

In the last integral make a change of variable putting $t-\tau = t'$. Then,

$$F_\tau(p) = \int_0^\infty e^{-p(t'+\tau)} f(t')\, dt' = e^{-p\tau} F(p)$$

which proves Property 3.

As a first example, consider the transform of the step function

$$f(t) = \begin{cases} 0, & t < \tau \\ n f_0, & n\tau \leqslant t < (n+1)\tau, \quad n = 1, 2, \ldots \end{cases} \tag{8-26}$$

Represent $f_0(t)$ by means of Heaviside's unit function $\sigma_0$:

$$f(t) = f_0[\sigma_0(t-\tau) + \sigma_0(t-2\tau) + \ldots]$$

Using the linearity property and the time-delay theorem, we get

$$f(t) \doteqdot F(p) = f_0 e^{-p\tau}\frac{1}{p} + f_0 e^{-2p\tau}\frac{1}{p} + \ldots = \frac{f_0}{p}\frac{e^{-p\tau}}{1-e^{-p\tau}} \tag{8-27}$$

Similarly, it is easy to show that the transform of the periodic function

$$f(t) = \begin{cases} f_0, & 2n\tau \leqslant t < (2n+1)\tau \\ -f_0, & (2n+1)\tau \leqslant t < (2n+2)\tau \end{cases} \quad n = 0, 1, 2, \ldots \tag{8-28}$$

is the function

$$f(t) \doteqdot F(p) = \frac{f_0}{p} \tanh \frac{p\tau}{2} \tag{8-29}$$

The time-delay theorem permits obtaining a rather general formula for the transform of a periodic function. First consider the case when the function $f(t)$ of the real variable $t$ is of the form

$$f(t) = \begin{cases} \varphi(t), & 0 \leqslant t < \tau \\ 0 & \tau \leqslant t \end{cases} \tag{8-30}$$

Denote the transforms of the functions $\varphi(t) \doteqdot \Phi(p)$ and $\varphi(t + \tau) \doteqdot \Phi_\tau(p)$. Rewrite (8-30) in the form

$$f(t) = \varphi(t) + \begin{cases} 0, & 0 \leqslant t < \tau \\ -\varphi(t + \tau - \tau), & t \geqslant \tau \end{cases}$$

Taking advantage of the linearity of the transform and using the time-delay theorem, we obtain

$$f(t) \doteqdot F(p) = \Phi(p) - e^{-p\tau}\Phi_\tau(p) \tag{8-31}$$

Now let the function $\varphi(t)$ be a periodic function of $t$ with period $\tau$, that is,

$$\varphi(t + \tau) = \varphi(t) \tag{8-32}$$

Then $\Phi_\tau(p) = \Phi(p)$ and formula (8-31) will permit expressing the transform $\Phi(p)$ of the periodic function $\varphi(t)$ in terms of the transform $F(p)$ of the function $f(t)$, which is equal to the function $\varphi(t)$ in the first period $0 \leqslant t \leqslant \tau$ and to zero outside it for $t \geqslant \tau$:

$$\Phi(p) = \frac{F(p)}{1-e^{-p\tau}} \tag{8-33}$$

By way of illustration, let us find the transform of the function

$$\varphi(t) = |\sin \omega t|, \quad \omega > a \tag{8-34}$$

This function is periodic for $t > 0$ with a period $\frac{\pi}{\omega}$. First find the transform of the function

$$f(t)=\begin{cases}\sin \omega t, & 0 \leqslant t \leqslant \frac{\pi}{\omega} \\ 0 & \frac{\pi}{\omega} < t\end{cases} \tag{8-35}$$

By means of formulas (8.31) and (8.22) and the equality $\sin \omega \times \times \left(t+\frac{\pi}{\omega}\right) = -\sin \omega t$ we obtain

$$f(t) \doteqdot F(p)=\frac{\omega}{p^2+\omega^2}+e^{-p\frac{\pi}{\omega}}\frac{\omega}{p^2+\omega^2}=\frac{\omega}{p^2+\omega^2}\left(1+e^{-\frac{\pi}{\omega}p}\right)$$

Whence, by formula (8.33) we get

$$|\sin \omega t| \doteqdot \frac{\omega}{p^2+\omega^2}\cdot\frac{1+e^{-\frac{\pi}{\omega}p}}{1-e^{-\frac{\pi}{\omega}p}}=\frac{\omega}{p^2+\omega^2}\cdot\coth\frac{p\pi}{2\omega} \tag{8-36}$$

(d) *The transform of a derivative.* We now proceed to prove one of the basic properties of a transform which enables us to replace differentiation of the original function by multiplication of the transform by an independent variable.

*Property 4. If a function $f'(t)$ satisfies the existence conditions of the transform and $f(t) \doteqdot F(p)$,* Re $p > a$, *then*

$$f'(t) \doteqdot pF(p)-f(0), \quad \operatorname{Re} p > a \tag{8-37}$$

Indeed, integrating by parts, we obtain

$$f'(t) \doteqdot \int_0^\infty e^{-pt} f'(t)\,dt = e^{-pt} f(t)\Big|_0^\infty + p\int_0^\infty e^{-pt} f(t)\,dt = pF(p)-f(0)$$

which proves the property.

Similarly, we can prove the following property.

*Property 4'. If a function $f^{(n)}(t)$ satisfies the existence conditions of the transform and $f(t) \doteqdot F(p)$,* Re $p > a$, *then*

$$f^{(n)}(t) \doteqdot p^n\left\{F(p)-\frac{f(0)}{p}-\frac{f'(0)}{p^2}-\ldots-\frac{f^{(n-1)}(0)}{p^n}\right\}, \quad \operatorname{Re} p > a \tag{8-38}$$

Formula (8-38) is particularly simplified when $f(0) = f'(0) = = \ldots = f^{(n-1)}(0) = 0$:

$$f^{(n)}(t) \doteqdot p^n F(p) \tag{8-39}$$

The result obtained finds numerous applications.

By way of illustration let us consider the solution of the following Cauchy problem for an ordinary linear differential equation with constant coefficients:

$$a_0 y^{(n)} + a_1 y^{(n-1)} + \dots + a_n y(t) = f(t) \tag{8-40}$$

$$y(0) = y'(0) = \dots = y^{(n-1)}(0) = 0 \tag{8-41}$$

where $f(t)$ is a function of $t$ specified for $t \geqslant 0$. Putting $f(t) \equiv 0$ for $t < 0$, we can, if $f(t)$ satisfies the existence conditions of a transform, construct the transform $F(p)$ of the function $f(t)$. Suppose that a function $y(t)$, which is the solution of the problem (8-40), (8-41), and all its derivatives up to order $n$ satisfy the existence conditions of a transform. Then, multiplying both sides of (8-40) by $e^{-pt}$ and integrating with respect to $t$ from 0 to $\infty$, we get, by virtue of the linearity of the transform and the initial conditions (8-40),

$$Y(p)\{a_0 p^n + a_1 p^{n-1} + \dots + a_n\} = F(p)$$

where the transform of the desired solution of the problem (8-40), (8-41) is denoted in terms of $Y(p) = \int\limits_0^\infty e^{-pt} y(t)\, dt$. Denoting $P_n(p) = a_0 p^n + a_1 p^{n-1} + \dots + a_n$, we get

$$Y(p) = \frac{F(p)}{P_n(p)} \tag{8-42}$$

Formula (8-42) gives a sufficiently simple expression of the transform of the desired solution $y(t)$ in terms of known functions—the polynomial $P_n(p)$, the coefficients of which are defined by the equation (8-40), and the transform $F(p)$ of the given right side of the equation. Thus, if we can determine the unknown original function $y(t)$ from its known transform $Y(p)$, then the problem (8-40), (8-41) will be solved. Below we will consider various methods of determining the original function from a given transform. For the present let us continue examining a number of general properties of transforms.

(e) *The transform of an integral.*

*Property 5.* Let $f(t) \risingdotseq F(p)$, $\operatorname{Re} p > a$. Then

$$\varphi(t) = \int\limits_0^t f(\tau)\, d\tau \risingdotseq \frac{1}{p} F(p), \quad \operatorname{Re} p > a \tag{8-43}$$

Indeed, it is easy to verify that the function $\varphi(t)$ satisfies all the existence conditions of a transform, and $\varphi(t)$ has the same index of the order of growth as $f(t)$. Computing the transform of the function $\varphi(t)$ from formula (8-2), we get

$$\int_0^t f(\tau)\,d\tau \risingdotseq \int_0^\infty e^{-pt}\,dt \int_0^t f(\tau)\,d\tau$$

Changing the order of integration in the last integral, we get

$$\int_0^t f(\tau)\,d\tau \risingdotseq \int_0^\infty f(\tau)\,d\tau \int_\tau^\infty e^{-pt}\,dt = \frac{1}{p}\int_0^\infty e^{-p\tau} f(\tau)\,d\tau = \frac{1}{p}F(p).$$

and this proves formula (8-43).

In similar fashion we can prove the following property.

*Property 5'*. Let $f(t) \risingdotseq F(p)$, $\operatorname{Re} p > a$; then

$$\int_0^t dt_1 \int_0^{t_1} dt_2 \ldots \int_0^{t_{n-1}} f(t_n)\,dt_n \risingdotseq \frac{1}{p^n}F(p), \quad \operatorname{Re} p > a \tag{8-44}$$

Properties 5 and 5' find numerous applications in the computation of transforms of various functions.

For example, find the transform of the saw-tooth function $f(t)$, which is a periodically repeating isosceles triangle with base $2\tau$ and altitude $f_0\tau$. As is readily evident, this function is an integral from 0 to $t$ of the function (8-28), the transform of which is given by the formula (8-29). Therefore

$$f(t) \risingdotseq \frac{f_0}{p^2} \tanh \frac{p\tau}{2} \tag{8-45}$$

(f) *The transform of a convolution.* The convolution of the functions $f_1(t)$ and $f_2(t)$ is the function $\varphi(t)$ defined by the relation

$$\varphi(t) = \int_0^t f_1(\tau) f_2(t-\tau)\,d\tau = \int_0^t f_1(t-\tau) f_2(\tau)\,d\tau \tag{8-46}$$

The validity of this equality becomes evident if we make a change of the integration variable $t - \tau = t'$ in the first integral. The following property holds true.

*Property 6.* If $f_1(t) \risingdotseq F_1(p)$, $\operatorname{Re} p > a_1$, $f_2(t) \risingdotseq F_2(p)$, $\operatorname{Re} p > a_2$, then

$$\varphi(t) = \int_0^t f_1(\tau) f_2(t-\tau)\,d\tau \risingdotseq F_1(p) F_2(p), \ \operatorname{Re} p > \max\{a_1, a_2\} \tag{8-47}$$

The convolution of the functions $f_1(t)$ and $f_2(t)$ with bounded order of growth is also a function with bounded order of growth.

Indeed,

$$\left|\int_0^t f_1(\tau) f_2(t-\tau)\, d\tau\right| \leqslant M_1 M_2 \int_0^t e^{a_1\tau} e^{a_2(t-\tau)}\, d\tau$$

$$= \frac{M_1 M_2}{a_1 - a_2}\{e^{a_1 t} - e^{a_2 t}\} \leqslant \frac{M_1 M_2}{|a_1 - a_2|} e^{at}, \quad a = \max\{a_1, a_2\}$$

The order of growth of the convolution is obviously equal to the greatest order of growth of the functions $f_1(t)$ and $f_2(t)$. Clearly $\varphi(t)$ also satisfies the other conditions of the existence of a transform. To evaluate the transform of a convolution, use formula (8-2) and change the order of integration:

$$\int_0^\infty e^{-pt}\, dt \int_0^t f_1(\tau) f_2(t-\tau)\, d\tau = \int_0^\infty f(\tau)\, d\tau \int_\tau^\infty e^{-pt} f_2(t-\tau)\, dt$$

Making the change of variables $t - \tau = t'$ in the inner integral, we finally get

$$\int_0^t f_1(\tau) f_2(t-\tau)\, d\tau \risingdotseq \int_0^\infty e^{-p\tau} f_1(\tau) d\tau \int_0^\infty e^{-pt'} f_2(t')\, dt' = F_1(p) F_2(p)$$

This proves Property 6.

In applications, formula (8-47) is frequently used to determine the original function from a given transform, when the specified transform can be partitioned into factors for which the original functions are known.

For example, let it be required to find the original of the function

$$F(p) = \frac{p\omega}{(p^2+\omega^2)^2}$$

Earlier, we found [see formulas (8-21) and (8-22)] that

$$\frac{p}{p^2+\omega^2} \risingdotseq \cos\omega t, \quad \frac{\omega}{p^2+\omega^2} \risingdotseq \sin\omega t$$

Therefore,

$$F(p) \risingdotseq \int_0^t \sin\omega\tau \cdot \cos\omega(t-\tau)\, d\tau = \frac{t}{2}\sin\omega t$$

Let us consider some more general properties of transforms.

(g) *Differentiation of a transform.*

*Property 7.* Let $F(p) \risingdotseq f(t)$, $\operatorname{Re} p > a$, then

$$F'(p) \risingdotseq -tf(t), \quad \operatorname{Re} p > a \tag{8-48}$$

Indeed, above we pointed out that the derivative of an analytic function $F(p)$ in the domain of its definition $\operatorname{Re} p > a$ may be computed by differentiating the integrand function in the improper integral (8-2) with respect to a parameter. Doing this, we get

$$F'(p) = -\int_0^\infty e^{-pt}\,tf(t)\,dt \risingdotseq -tf(t)$$

This proves Property 7. Noting that multiplication of the function $f(t)$ by any power function $t^n$ does not change its order of growth, we get

*Property 7'.* If $F(p) \risingdotseq f(t)$, $\operatorname{Re} p > a$, then

$$F^{(n)}(p) \risingdotseq (-1)^n t^n f(t) \tag{8-49}$$

Formulas (8-48) and (8-49) may be used to evaluate the transform of the product of $t^n$ by a function $f(t)$ for which the transform is known. Later we will derive a general formula expressing the transform of a product in terms of the transforms of the factors. Let us now consider yet another property of transforms.

(h) *Integration of a transform.*

*Property 8.* If a function $\frac{f(t)}{t}$ satisfies the existence conditions of a transform and $f(t) \risingdotseq F(p)$, $\operatorname{Re} p > a$, then

$$\frac{f(t)}{t} \risingdotseq \int_0^\infty e^{-pt}\frac{f(t)}{t}\,dt = \int_p^\infty F(q)\,dq \tag{8-50}$$

Denote

$$I(p) = \int_0^\infty e^{-pt}\frac{f(t)}{t}\,dt \tag{8-51}$$

By Theorem 8.2, the function $I(p)$ is analytic in the domain $\operatorname{Re} p > a$, and by virtue of the remark on page 224, $I(\infty) = 0$. We find the derivative of the function $I(p)$ by differentiating the integral (8-51) with respect to a parameter:

$$I'(p) = -\int_0^\infty e^{-pt} f(t)\,dt = -F(p)$$

From this, taking into account the condition $I(\infty) = 0$, we obtain

$$I(p) = I(\infty) - \int_\infty^p F(q)\,dq = \int_p^\infty F(q)\,dq$$

This proves Property 8.

As an illustration, let us find the transform of the function $\frac{1}{t}\sin \omega t$. Since $\sin \omega t \risingdotseq \frac{\omega}{p^2+\omega^2}$, it follows that

$$\frac{1}{t}\sin \omega t \risingdotseq \int_p^{\infty} \frac{\omega}{p^2+\omega^2}\,dp = \frac{\pi}{2} - \arctan\frac{p}{\omega} \tag{8-52}$$

With the aid of Property 5 we get, from expression (8-52),

$$\mathrm{Si}\ t = \int_0^t \frac{\sin\tau}{\tau}\,d\tau \risingdotseq \frac{1}{p}\left(\frac{\pi}{2} - \arctan p\right) \tag{8-53}$$

The function Si $t$ is called the sine integral.

(i) The last property of transforms that we consider in this section is called the *shift theorem*.

*Property 9.* If $f(t) \risingdotseq F(p)$, $\operatorname{Re} p > a$, then for any complex number $\lambda$

$$F(p+\lambda) \risingdotseq e^{-\lambda t} f(t), \quad \operatorname{Re} p > a - \operatorname{Re}\lambda \tag{8-54}$$

Indeed, the function $\varphi(t) = e^{-\lambda t} f(t)$ obviously satisfies the conditions of existence of a transform, which, by formula (8-2), is defined in the domain $\operatorname{Re} p > a - \operatorname{Re}\lambda$, but

$$\int_0^{\infty} e^{-pt}e^{-\lambda t} f(t)\,dt = \int_0^{\infty} e^{-(p+\lambda)t} f(t)\,dt = F(p+\lambda)$$

This proves the shift theorem.

Formula (8-54) may be used to determine the transform of the product of the function $e^{-\lambda t}$ by the function $f(t)$ for which the transform is known. Thus, with the aid of this formula and the transforms already obtained we can find

$$te^{\alpha t} \risingdotseq \frac{1}{(p-\alpha)^2}, \quad \operatorname{Re} p > \operatorname{Re}\alpha \tag{8-55}$$

$$t^n e^{\alpha t} \risingdotseq \frac{n!}{(p-\alpha)^{n+1}}, \quad \operatorname{Re} p > \operatorname{Re}\alpha \tag{8-56}$$

$$e^{-\alpha t}\sin \omega t \risingdotseq \frac{\omega}{(p+\alpha)^2+\omega^2}, \quad \operatorname{Re} p > |\operatorname{Im}\omega| - \operatorname{Re}\alpha \tag{8-57}$$

and so forth.

We conclude this section with a table of the properties of transforms we have considered and a table of the transforms of a number of elementary and most frequently used special functions.

### d. Table of properties of transforms

Let $f(t) \doteqdot F(p)$. Then

(1) $\sum_{i=1}^{n} \alpha_i f_i(t) \doteqdot \sum_{i=1}^{n} \alpha_i F_i(p)$, $\alpha_i = \text{constant}$

(2) $f(\alpha t) \doteqdot \frac{1}{\alpha} F\left(\frac{p}{\alpha}\right)$, $\alpha = \text{constant}$, $\alpha > 0$

(3) $f_\tau(t) = \begin{cases} 0, & \tau > t \\ f(t-\tau), & \tau \leqslant t, \end{cases}$ $f_\tau(t) \doteqdot e^{-p\tau} F(p)$

(4) $f^{(n)}(t) \doteqdot p^n \left\{ F(p) - \frac{f(0)}{p} - \dots - \frac{f^{(n-1)}(0)}{p^n} \right\}$

(5) $\int_0^t f(\tau)\, d\tau \doteqdot \frac{1}{p} F(p)$

(6) $\int_0^t f_1(\tau) f_2(t-\tau)\, d\tau = \int_0^t f_1(t-\tau) f_2(\tau)\, d\tau \doteqdot F_1(p) F_2(p)$

(7) $F^{(n)}(p) \doteqdot (-1)^n t^n f(t)$

(8) $\int_p^\infty F(p)\, dp \doteqdot \frac{f(t)}{t}$

(9) $F(p+\lambda) \doteqdot e^{-\lambda t} f(t)$

### e. Table of transforms

(1) $1 \doteqdot \frac{1}{p}$, $\operatorname{Re} p > 0$

(2) $t^\nu \doteqdot \frac{\Gamma(\nu+1)}{p^{\nu+1}}$, $\nu > -1$, $\operatorname{Re} p > 0$

(3) $t^n \doteqdot \frac{n!}{p^{n+1}}$, $n$ an integer, $\operatorname{Re} p > 0$

(4) $e^{\alpha t} \doteqdot \frac{1}{p-\alpha}$, $\operatorname{Re} p > \operatorname{Re} \alpha$

(5) $\sin \omega t \doteqdot \frac{\omega}{p^2+\omega^2}$, $\operatorname{Re} p > |\operatorname{Im} \omega|$

(6) $\cos \omega t \doteqdot \frac{p}{p^2+\omega^2}$, $\operatorname{Re} p > |\operatorname{Im} \omega|$

(7) $\sinh \lambda t \doteqdot \frac{\lambda}{p^2-\lambda^2}$, $\operatorname{Re} p > |\operatorname{Re} \lambda|$

(8) $\cosh \lambda t \doteqdot \frac{p}{p^2-\lambda^2}$, $\operatorname{Re} p > |\operatorname{Re} \lambda|$

(9) $t^n e^{\alpha t} \doteqdot \frac{n!}{(p-\alpha)^{n+1}}$, $\quad \operatorname{Re} p > \operatorname{Re} \alpha$

(10) $t \sin \omega t \doteqdot \frac{2p\omega}{(p^2+\omega^2)^2}$, $\quad \operatorname{Re} p > |\operatorname{Im} \omega|$

(11) $t \cos \omega t \doteqdot \frac{p^2-\omega^2}{(p^2+\omega^2)^2}$, $\quad \operatorname{Re} p > |\operatorname{Im} \omega|$

(12) $e^{\lambda t} \sin \omega t \doteqdot \frac{\omega}{(p-\lambda)^2+\omega^2}$, $\quad \operatorname{Re} p > (\operatorname{Re} \lambda + |\operatorname{Im} \omega|)$

(13) $e^{\lambda t} \cos \omega t \doteqdot \frac{p-\lambda}{(p-\lambda)^2+\omega^2}$, $\quad \operatorname{Re} p > (\operatorname{Re} \lambda + |\operatorname{Im} \omega|)$

(14) $\frac{\sin \omega t}{t} \doteqdot \frac{\pi}{2} - \arctan \frac{p}{\omega}$, $\quad \operatorname{Re} p > |\operatorname{Im} \omega|$

(15) $\left\{ \begin{array}{ll} 1, & 2k\tau \leqslant t < (2k+1)\tau \\ -1, & (2k+1)\tau \leqslant t < (2k+2)\tau \end{array} \right\} \doteqdot \frac{1}{p} \tanh \frac{p\tau}{2}$, $\quad \operatorname{Re} p > 0$

$k = 0, 1, 2, \ldots$

(16) $|\sin \omega t| \doteqdot \frac{\omega}{p^2+\omega^2} \coth \frac{p\pi}{2\omega}$, $\quad \operatorname{Re} p > |\operatorname{Im} \omega|$

(17) $e^{-\alpha^2 t^2} \doteqdot \frac{\sqrt{\pi}}{2} e^{\frac{p^2}{4\alpha^2}} \left(1 - \Phi\left(\frac{p}{2\alpha}\right)\right)$

(18) $\frac{e^{-\alpha t}}{\sqrt{\pi t}} \doteqdot \frac{1}{\sqrt{p+\alpha}}$

(19) $\frac{e^{-2\alpha\sqrt{t}}}{\sqrt{\pi t}} \doteqdot \frac{1}{\sqrt{p}} e^{\frac{\alpha^2}{p}} \left(1 - \Phi\left(\frac{\alpha}{\sqrt{p}}\right)\right)$

(20) $J_0(\alpha t) \doteqdot \frac{1}{\sqrt{\alpha^2+p^2}}$

(21) $J_0(2\sqrt{t}) \doteqdot \frac{1}{p} e^{-\frac{1}{p}}$

(22) $J_n(t) \doteqdot \frac{(\sqrt{p^2+1}-p)^n}{\sqrt{p^2+1}}$

(23) $\operatorname{Si} t \doteqdot \frac{1}{p}\left(\frac{\pi}{2} - \arctan p\right)$

(24) $\Phi(\sqrt{\alpha t}) \doteqdot \frac{\sqrt{\alpha}}{p\sqrt{p+\alpha}}$

(25) $1 - \Phi\left(\frac{\alpha}{2\sqrt{t}}\right) \doteqdot \frac{1}{p} e^{-\alpha\sqrt{p}}$

For real values of the parameters in the function $f(t)$ in formulas (17) to (25), the transforms of the corresponding functions are definitely defined in the domain $\operatorname{Re} p > 0$.

## 8.2. Determining the Original Function from the Transform

In this section we consider methods for determining the original function from a given transform, and we also give certain sufficient conditions under which a given function $F(p)$ of the complex variable $p$ is a transform of the function $f(t)$ of the real variable $t$.

First, note that there are various tables of transforms of the most frequently occurring functions, so that when solving practical problems it is often possible to find, in reference works, the expression of the original function for the transform obtained.

Second, the properties of transforms (1) to (9) that were given in the preceding section enable one, in many cases, to solve the inverse problem of constructing the original function from a given transform. This applies above all to the shift theorem, integration and differentiation of transforms and to the transform of a convolution of functions. A number of examples were already considered in Section 8.1, others will be added later on.

However, all these methods are actually trial and error methods. The basic aim of this section is to give a general method for constructing the original function from the transform.

### a. Mellin's formula

We begin with the case when it is known that the given function $F(p)$ of the complex variable $p$ is the transform of a piecewise smooth function $f(t)$ with bounded order of growth $|f(t)| < Me^{at}$; the value of the constant $a$ is given. It is required, from the given function $F(p)$ to construct the desired function $f(t)$. This problem is solved with the aid of the following theorem.

***Theorem 8.3.*** *Let it be known that the given function $F(p)$ in the domain $\operatorname{Re} p > a$ is the transform of a piecewise smooth function $f(t)$ of a real variable $t$ and possesses an order of growth $a$.*

*Then*

$$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{pt} F(p)\, dp, \quad x > a \tag{8-58}$$

*Proof.* By hypothesis, the function $f(t)$ exists and we know its order of growth. Consider the auxiliary function $\varphi(t) = e^{-xt} f(t)$, $x > a$. This is a piecewise smooth function which on any bounded interval of the $t$-axis has a finite number of discontinuities of the first kind and exponentially tends to zero as $t \to \infty$. It can be represented with the aid of the Fourier integral

$$\varphi(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\xi \int_{-\infty}^{\infty} \varphi(\eta)\, e^{i\xi(t-\eta)}\, d\eta \tag{8-59}$$

Putting into (8-59) the expression of the function $\varphi(t)$ in terms of the sought-for function $f(t)$, we get

$$e^{-xt}f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}d\xi\int_{-\infty}^{\infty}e^{-x\eta}f(\eta)\,e^{i\xi(t-\eta)}\,d\eta$$

$$=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{i\xi t}\,d\xi\int_{0}^{\infty}e^{-(x+i\xi)\,\eta}f(\eta)\,d\eta \quad (8\text{-}60)$$

since $f(\eta)\equiv 0$ for $\eta<0$.

Denote $p=x+i\xi$ and note that the inner integral in (8-60) is the given transform $F(p)$ of the sought-for function $f(t)$. Then the expression (8-60) becomes

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{(x+i\xi)t}F(p)\,d\xi=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}e^{pt}F(p)\,dp$$

This proves the theorem. Observe that in formula (8-58) the integration is performed in the complex $p$-plane along a straight line parallel to the imaginary axis and passing to the right of the straight line $\operatorname{Re} p=a$. The value of the integral (8-58) is independent of $x$, provided that the straight line of integration lies to the right of the straight line $\operatorname{Re} p=a$.

Formula (8-58) is often called *Mellin's formula*. In a sense, it is the inverse of the Laplace transformation [formula (8-2)] since it expresses the original function in terms of a given transform. Note that since, in deriving the Mellin formula, we went from the unknown function $f(t)$ itself to its Fourier integral, which converges to $f(t)$ only at the points of continuity of this function, it also follows that formula (8-58) defines the function $f(t)$ only at its points of continuity.

To illustrate the application of this theorem, let us consider the question of determining the transform of a product from the known transforms of the factors.

***Theorem 8.4.*** *Let* $f_1(t)\doteqdot F_1(p)$, $\operatorname{Re} p>a_1$ *and* $f_2(t)\doteqdot F_2(p)$, $\operatorname{Re} p>a_2$. *Then*

$$f(t)=f_1(t)\,f_2(t)\doteqdot F(p)=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}F_1(q)\,F_2(p-q)\,dq$$

$$=\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty}F_1(p-q)\,F_2(q)\,dq \quad (8\text{-}61)$$

*and the function* $F(p)$ *is defined and analytic in the domain* $\operatorname{Re} p>$ $>a_1+a_2$, *and integration is performed along any straight line paral-*

*lel to the imaginary axis lying to the right of the straight lines* Re $p = a_1$ *and* Re $p = a_2$.

*Proof.* Since the function $f(t)$ satisfies all the existence conditions of a transform, the following Laplace transformation holds:

$$f(t) \risingdotseq F(p) = \int_0^\infty e^{-pt} f_1(t) f_2(t)\, dt \tag{8-62}$$

If in (8-62) we represent the function $f_1(t)$ in the form of its Mellin integral (8-58) and change the order of integration, which is possible due to the uniform convergence of the given improper integrals dependent on a parameter, we get

$$F(p) = \frac{1}{2\pi i} \int_0^\infty e^{-pt} f_2(t)\, dt \int_{x-i\infty}^{x+i\infty} e^{qt} F_1(q)\, dq$$

$$= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} F_1(q)\, dq \int_0^\infty e^{-(p-q)t} f_2(t)\, dt$$

$$= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} F_1(q) F_2(p-q)\, dq \tag{8-63}$$

Note that in (8-63) Re $q = x > a_1$, and the function $F_2(p-q)$ is defined for Re $(p-q) > a_2$, whence Re $p > a_1 + a_2$. If in (8-62) we replace the function $f_2(t)$ according to the inversion formula, we can get the second equality in (8-61). The theorem is proved. It will be seen that this theorem is, in a sense, the converse of Property 6.

*Example 1.* Let $f_1(t) = \cos \omega t$, $f_2(t) = t$. Find the transform of the function $f(t) = t \cos \omega t$.

Since $\cos \omega t \risingdotseq \frac{p}{p^2+\omega^2}$, $t \risingdotseq \frac{1}{p^2}$, it follows that

$$F(p) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} \frac{q\, dq}{(q^2+\omega^2)(p-q)^2} \tag{8-64}$$

where Re $p > |\operatorname{Im} \omega|$ and the integration is performed along any straight line parallel to the imaginary axis and lying to the right of the straight line Re $q = |\operatorname{Im} \omega|$. For such a straight line of integration we choose the straight line passing to the left of the point $q = p$, and consider (in the complex $q$-plane) the closed contour $\Gamma$, which consists of the segment $[x - iR, x + iR]$ of the given straight line and the arc of the semicircle $|q - x| = R$ completing it in the right half-plane. Within the given contour, the integrand function of (8-64) is everywhere analytic except at the point $q = p$, which is a second-order pole of the given function. The point $q = \infty$ is a third-order zero of this function. Therefore, by virtue of Lemma 1, Chapter 5, the value of the integral (8-64) is determined by the residue at a singular point of the integrand function. Noting that the contour $\Gamma$ is traversed in the negative direction, we find

$$F(p) = -\frac{d}{dq}\left[\frac{q}{(q^2+\omega^2)}\right]_{q=p} = \frac{p^2-\omega^2}{(p^2+\omega^2)^2}$$

And so

$$t\cos\omega t \doteqdot \frac{p^2-\omega^2}{(p^2+\omega^2)^2} \tag{8-65}$$

### *b. Existence conditions of the original function*

Here we will consider certain sufficient conditions under which a given function $F(p)$ of the complex variable $p$ is the transform of some function $f(t)$ of the real variable $t$, and will show how the latter may be found.

**Theorem 8.5.** *Let a function $F(p)$ of the complex variable $p = x + iy$ satisfy the following conditions:*

(a) *$F(p)$ is an analytic function in the domain* $\operatorname{Re} p > a$;

(b) *in the domain* $\operatorname{Re} p > a$, *the function $F(p)$ tends to zero as $|p| \to \infty$ uniformly in* $\arg p$;

(c) *for all* $\operatorname{Re} p = x > a$ *the following integral* converges:*

$$\int_{x-i\infty}^{x+i\infty} |F(p)|\, dy < M, \quad x > a \tag{8-66}$$

*Then $F(p)$, for* $\operatorname{Re} p > a$, *is the transform of the function $f(t)$ of the real variable $t$, which is defined by the expression*

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{pt}F(p)\, dp, \quad x > a \tag{8-67}$$

*Proof.* And so we have to prove that the integral (8-67) is the original of the function $F(p)$. First, the question arises as to the exist-

* The integral (8-66) is an improper integral of the first kind of the real function $|F(p)|$ along the straight line $\operatorname{Re} p = x$.

ence of this improper integral.* Clearly

$$\left|\frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{pt}F(p)\,dp\right| \leqslant \frac{1}{2\pi}\int_{x-i\infty}^{x+i\infty} |e^{pt}F(p)|\cdot|dp|$$

$$= \frac{e^{xt}}{2\pi}\int_{x-i\infty}^{x+i\infty} |F(p)|dy \leqslant \frac{M}{2\pi}e^{xt} \quad (8\text{-}68)$$

whence follows the convergence of the integral (8-67) for any $x > a$. We note, for the sequel, that from the evaluation (8-68) there follows the uniform convergence of the integral (8-67) with respect to the parameter $t$ on any finite interval $0 \leqslant t \leqslant T$.

In order to prove that the integral (8-67) is the original of the given function $F(p)$, we have to establish that:

(1) the integral (8-67) is independent of $x$ and defines the function $f(t)$ of the variable $t$ alone; and this function has a bounded order of growth;

(2) for $t < 0$ $f(t) \equiv 0$;

(3) the given function $F(p)$ is the Laplace transform of the function $f(t)$.

We now prove each of these propositions.

(1) In the domain Re $p > a$, consider a closed contour $\Gamma$ consisting of segments of straight lines $[x_1 - iA, x_1 + iA]$ and $[x_2 - - iA, x_2 + iA]$ parallel to the imaginary axis, and of the straight lines connecting them $[x_1 - iA, x_2 - iA]$, $[x_1 + iA, x_2 + iA]$, which are parallel to the real axis (Fig. 8.1). Here, $A > 0$, $x_1$, $x_2$ are arbitrary numbers greater than $a$. Since the function $F(p)$ is analytic in the domain Re $p > a$, then by the Cauchy theorem the integral of the function $e^{pt}F(p)$ around the contour $\Gamma$ is zero. Let $A$ tend to infinity and let $x_1$ and $x_2$ remain fixed. Then by condition (b) of the theorem, the integrals over the horizontal segments of the integration path will in the limit yield zero whereas the integrals over the vertical straight lines will pass into the integral (8-67). This yields

$$\int_{x_1-i\infty}^{x_1+i\infty} e^{pt}F(p)\,dp = \int_{x_2-i\infty}^{x_2+i\infty} e^{pt}F(p)\,dp$$

---

* The improper integral (8-67) is calculated along the straight line Re $p = x$ and is to be taken in the sense of the principal value, i.e.

$$\int_{x-i\infty}^{x+i\infty} e^{pt}F(p)\,dp = \lim_{A\to\infty}\int_{x-iA}^{x+iA} e^{pt}F(p)\,dp$$

which, since $x_1$ and $x_2$ are arbitrary, proves proposition (1). Thus, the integral (8-67) is a function of only the single variable $t$. It will

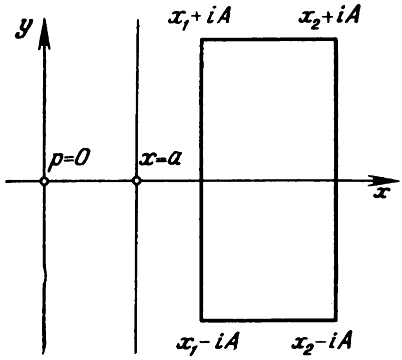


Fig. 8.1

be noted that from the evaluation (8-68) it immediately follows that the integral (8-67) is a function of bounded order of growth with respect to $t$, and the index of the order of growth of this function is $a$.

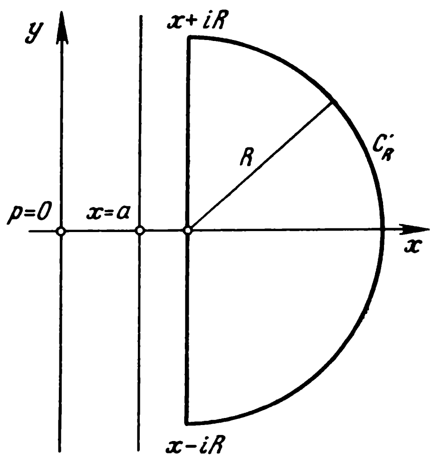


Fig. 8.2

(2) Consider the value of the integral (8-67) for $t < 0$. To do this, consider, in the domain $\operatorname{Re} p > a$, a closed contour $C$ consisting of the straight-line segment $[x - iR, x + iR]$, $x > a$, and of the arc $C'_R$ of the semicircle $|p - x| = R$ completing it (Fig. 8.2). By the

Cauchy theorem, the integral of the function $e^{pt}F(p)$ over the given contour is zero. By the remark concerning the Jordan lemma (see Chapter 5, page 135) the integral over the arc $C'_R$ tends to zero for $t < 0$ as $R \to \infty$. Therefore

$$f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{pt} F(p)\, dp \equiv 0, \quad t < 0, \quad \operatorname{Re} p > a \tag{8-69}$$

and the assertion (2) is proved.

(3) Construct the Laplace transform of the function (8-67) and consider its value for some arbitrary $p_0$, where $\operatorname{Re} p_0 > a$:

$$\int_0^\infty e^{-p_0 t} f(t)\, dt = \frac{1}{2\pi i} \int_0^\infty e^{-p_0 t}\, dt \int_{x-i\infty}^{x+i\infty} e^{pt} F(p)\, dp \tag{8-70}$$

The inner integral in (8-70) is independent of $x$. Choose a value of $x$ that satisfies the condition $a < x < \operatorname{Re} p_0$ and change the order of integration. This is possible by virtue of the uniform convergence of the corresponding integrals. We get

$$\int_0^\infty e^{-p_0 t} f(t)\, dt = \frac{2}{2\pi i} \int_{x-i\infty}^{x+i\infty} F(p)\, dp \int_0^\infty e^{-(p_0-p)t}\, dt$$

$$= \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} F(p) \frac{dp}{p_0 - p} \tag{8-71}$$

The integral (8-71) can be computed with the aid of residues, since by condition (b) of the theorem, the integrand function, as $|p| \to \infty$, tends to zero faster than the function $\frac{1}{p}$. Therefore, taking into account that the only singularity of the integrand function—a first order pole—is the point $p = p_0$ and, upon completion of (8-71) in the right half-plane, the integration is performed in the negative direction, we obtain

$$f(t) \doteqdot \int_0^\infty e^{-p_0 t} f(t) dt = F(p_0) \tag{8-72}$$

Since $p_0$ is an arbitrary point in the domain $\operatorname{Re} p > a$, the theorem is proved. The integral (8-67) naturally coincides with Mellin's formula (8-58) which was derived on the assumption of the existence of an original function. We have thus established certain sufficient conditions under which a given function $F(p)$ of the complex variable $p$ is a transform.

### c. Computing the Mellin integral

In many cases of practical importance, the integral (8-58), (8-67), which yields an expression of the original function in terms of the given function $F(p)$ of a complex variable, may be computed with the aid of earlier considered (see Chapter 5) methods of computing contour integrals of functions of a complex variable. Let $F(p)$, originally specified in the domain $\operatorname{Re} p > a$, be analytically extended to the entire $p$-plane. Let its analytic continuation satisfy, for $\operatorname{Re} p < a$, the conditions of Jordan's lemma. Then for $t > 0$

$$\int_{C''_R} e^{pt} F(p)\, dp \to 0, \quad R \to \infty \tag{8-73}$$

where $C''_R$ is the arc of the semicircle $|p - x| = R$ in the left-hand half-plane. In this case, the integral (8-67) may be computed with the aid of the calculus of residues. Let us consider some examples.

*Example 2.* Find the original function of $F(p) = \frac{\omega}{p^2+\omega^2}$, $\operatorname{Re} p > 0$, $\omega^2 > 0$. Since the conditions of Theorem 8.5 are fulfilled,

$$F(p) \risingdotseq f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{pt} \frac{\omega}{p^2+\omega^2}\, dp, \quad x > 0$$

The analytic continuation of the function $F(p)$ into the left half-plane $\operatorname{Re} p < 0$, the function $\frac{\omega}{p^2+\omega^2}$, satisfies the conditions of the Jordan lemma and has two singularities—poles of the first order for $p_{1,2} = \pm i\omega$. Therefore, for $t \geqslant 0$,

$$f(t) = \sum_{k=1}^{2} \operatorname{Res}\left[e^{pt} \frac{\omega}{p^2+\omega^2},\ p_k\right] = \frac{\omega e^{i\omega t}}{2i\omega} - \frac{\omega e^{-i\omega t}}{2i\omega} = \sin \omega t, \quad t \geqslant 0$$

The conditions of Theorem 8.5, in particular (c), are sufficient conditions for the existence of the original function of the function $F(p)$, analytic in the domain $\operatorname{Re} p > a$. It is easy to indicate examples which show that if this condition does not hold, then the function $F(p)$ can still be the transform of some function of a real variable.

*Example 3.* Find the original function of the function $F(p) = \frac{1}{p^{\alpha+1}}$, $-1 < \alpha < 0$, $\operatorname{Re} p > 0$. This function is multiple-valued in the domain under consideration. For the function $F(p)$ we will take that branch of the given multiple-valued function which is a direct analytic continuation into the domain $\operatorname{Re} p > 0$ of the real function $\frac{1}{x^{\alpha+1}}$ of the real variable $x > 0$. Then we will obviously have to take $\arg p = 0$ for $p = x$, $x > 0$. The function $F(p)$ does not satisfy the condition (c) of Theorem 8.5. However, we will show that the function

$$f(t) = \frac{1}{2\pi i} \int\limits_{x-i\infty}^{x+i\infty} e^{pt} \frac{1}{p^{\alpha+1}} \, dp, \quad x > 0 \tag{8-74}$$

is the original function of the given function $F(p)$.

The analytic continuation of $F(p)$ into the left half-plane $\operatorname{Re} p < 0$ is a multiple-valued function having as branch points the

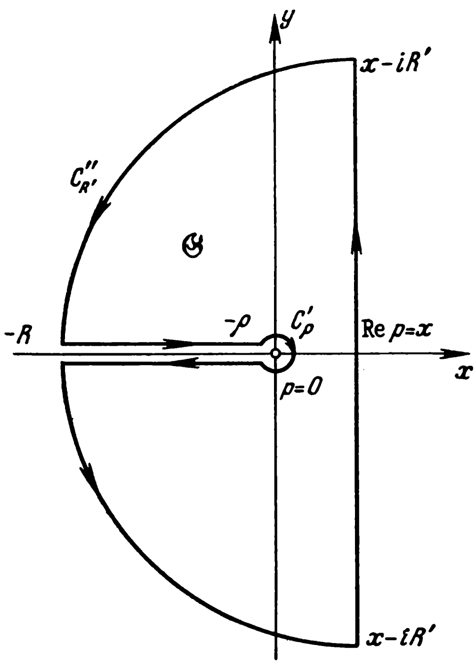


**Fig. 8.3**

points $p = 0$ and $p = \infty$. In the domain $\mathfrak{G}$, which is the complex $p$-plane cut along the negative real axis, we will consider that branch of the multiple-valued function $\frac{1}{p^{\alpha+1}}$, which is the immediate analytic continuation of the function $F(p)$ originally specified in the right half-plane $\operatorname{Re} p > 0$. In the domain $\mathfrak{G}$ we consider a closed contour $\Gamma$ consisting of a straight-line segment $[x - iR', x + iR']$, $x > 0$, of the segments $-R < x < -\rho$ on the lips of the cut and of the arcs of the circle $C'_\rho$, $|p| = \rho$ which complete them, and of the arcs of the circle $C''_R$, $|p - x| = R'$, connecting the lips of the cut with the vertical segment $[x - iR', x + iR']$ (Fig. 8.3). Since the function $e^{pt}\frac{1}{p^{\alpha+1}}$ does not have any singularities in the domain $\mathfrak{G}$, by the Cauchy theorem the integral of this function around the contour $\Gamma$ is zero. Let $R'$ go to infinity, and $\rho$ to zero. By virtue of the Jordan lemma, the integrals along the curves $C''_{R'}$ will yield zero in the limit. Evaluate the integral around the circle $C'_\rho$, putting $p = \rho e^{i\varphi}$:

$$\left|\frac{1}{2\pi i}\int_{C'_\rho} e^{pt}\frac{dp}{p^{\alpha+1}}\right| \leqslant \frac{1}{2\pi\rho^\alpha}\int_{-\pi}^{\pi} e^{t\rho\cos\varphi}\,d\varphi$$

Since $-1 < \alpha < 0$, the integral around $C'_\rho$ will also tend to zero as $\rho \to 0$. This leaves only the integrals along the straight-line segments of the contour of integration. Note that on the lower lip of the cut $\arg p = -\pi$, on the upper lip, $\arg p = \pi$. We thus get

$$f(t) = \frac{1}{2\pi i}\int_{x-i\infty}^{x+i\infty} e^{pt}\frac{dp}{p^{\alpha+1}}$$

$$= \frac{1}{2\pi i}\left\{\int_{-\infty}^{0} e^{xt}\frac{dx}{(-x)^{\alpha+1}e^{i\pi\alpha}} + \int_{0}^{-\infty} e^{xt}\frac{dx}{(-x)^{\alpha+1}e^{-i\pi\alpha}}\right\}$$

$$= \frac{1}{2\pi i}\left\{e^{-i\pi\alpha}\int_0^\infty e^{-xt}x^{-\alpha-1}\,dx - e^{i\pi\alpha}\int_0^\infty e^{-xt}x^{-\alpha-1}\,dx\right\}$$

$$= \frac{\sin(-\pi\alpha)}{\pi}\int_0^\infty e^{-xt}x^{-\alpha-1}\,dx \quad (8\text{-}75)$$

Making the change of variable of integration $xt = s$ in the integral (8-75), we get

$$f(t) = t^\alpha\frac{\sin(-\pi\alpha)}{\pi}\Gamma(-\alpha)$$

Taking advantage of the equality

$$\Gamma(-\alpha)\Gamma(1+\alpha) = \frac{\pi}{\sin(-\pi\alpha)}$$

we finally obtain the formula

$$\frac{1}{p^{\alpha+1}} \doteqdot f(t) = \frac{t^{\alpha}}{\Gamma(1+\alpha)} \tag{8-76}$$

which is the inverse of the formula (8-18). This proves our assertion.

*Example 4.* Find the original function of $F(p) = \frac{1}{p} e^{-\alpha\sqrt{p}}$, $\alpha > 0$, Re $p > 0$. As in the preceding example, we consider that branch of the multiple-valued function $\sqrt{p}$ which is a direct analytic continuation into the domain Re $p > 0$ of the real function $\sqrt{x}$ of the real variable $x > 0$. It will be recalled that in this case we have to put $\arg p = 0$ for $p = x > 0$. The analytic continuation of the function $F(p)$ into the left half-plane Re $p < 0$ again has for branch points the points $p = 0$ and $p = \infty$. We consider the domain $\mathfrak{G}$ — the $p$-plane cut along the negative real axis. In this domain is defined the single-valued analytic function $\frac{1}{p} e^{-\alpha\sqrt{p}}$ which is a direct analytic continuation of the function $F(p)$. We note that the function $F(p)$, for Re $p > 0$, satisfies the conditions of Theorem 8.5, and its analytic continuation in the domain $\mathfrak{G}$ in the left half-plane Re $p < 0$ satisfies for $t > 0$ the conditions of the Jordan lemma. For this reason, if we choose the same contour of integration $\Gamma$ as in the previous example and note that on the upper lip of the cut, $\arg p = \pi$, which yields $p = \xi e^{i\pi} = -\xi$, $\sqrt{p} = \sqrt{\xi}\, e^{i\frac{\pi}{2}} = i\sqrt{\xi}$ and on the lower lip of the cut, $\arg p = -\pi$, which yields $p = \xi e^{-i\pi} = -\xi$, $\sqrt{p} = \sqrt{\xi}\, e^{-i\frac{\pi}{2}} = -i\sqrt{\xi}$ $(\xi > 0)$, we get

$$F(p) \doteqdot f(t) = \frac{1}{2\pi i} \int_{x-i\infty}^{x+i\infty} e^{pt} \frac{e^{-\alpha\sqrt{p}}}{p}\, dp$$

$$= \frac{1}{2\pi i} \left\{ \int_0^{\infty} e^{-\xi t} \frac{e^{-i\alpha\sqrt{\xi}}}{\xi}\, d\xi - \int_0^{\infty} e^{-\xi t} \frac{e^{i\alpha\sqrt{\xi}}}{\xi}\, d\xi \right\} + \lim_{\rho\to 0} \frac{1}{2\pi i} \int_{C'_\rho} e^{pt} \frac{e^{-\alpha\sqrt{p}}}{p}\, dp$$

Since

$$\lim_{\rho\to 0} \frac{1}{2\pi i} \int_{-\pi}^{\pi} e^{\rho t e^{i\varphi}} \frac{e^{-\alpha\sqrt{\rho}\, e^{i\frac{\varphi}{2}}}}{\rho e^{i\varphi}}\, i\rho e^{i\varphi}\, d\varphi = 1$$

it follows that

$$f(t) = -\frac{1}{\pi} \int_0^{\infty} e^{-\xi t} \frac{\sin \alpha\sqrt{\xi}}{\xi}\, d\xi + 1$$

Change the variable in this integral, putting $\sqrt{\xi}=x$, taking into account that

$$\frac{\sin \alpha x}{x}=\int_0^\infty \cos \beta x \, d\beta$$

and change the order of integration. We get

$$\int_0^\infty e^{-\xi t} \frac{\sin \alpha \sqrt{\xi}}{\xi} \, d\xi = 2 \int_0^\alpha d\beta \int_0^\infty e^{-tx^2} \cos \beta x \, dx \tag{8-77}$$

It is easy to compute the inner integral in (8-77)*. It is

$$\int_0^\infty e^{-tx^2} \cos \beta x \, dx = \frac{1}{2} \sqrt{\frac{\pi}{t}} \, e^{-\frac{\beta^2}{4t}}$$

Whence

$$f(t) = 1 - \frac{2}{\sqrt{\pi}} \sqrt{\frac{1}{4t}} \int_0^\alpha e^{-\frac{\beta^2}{4t}} \, d\beta$$

Putting $\frac{\beta}{\sqrt{4t}}=\eta$, we finally get

$$F(p) = \frac{1}{p} e^{-\alpha\sqrt{p}} \doteqdot 1 - \Phi\left(\frac{\alpha}{2\sqrt{t}}\right), \quad \alpha>0, \ \operatorname{Re} p>0 \tag{8-78}$$

where the function

$$\Phi(z) = \frac{2}{\sqrt{\pi}} \int_0^z e^{-\eta^2} \, d\eta \tag{8-79}$$

is the so-called error function. **

### *d. The case of a function regular at infinity*

Let us examine one more special case when it is particularly easy to determine the original function for a given function $F(p)$ of a complex variable. Let the analytic continuation of a function $F(p)$, originally specified in the domain $\operatorname{Re} p > a$, be a single-valued function in the extended plane of the complex variable $p$, and let the point $p=\infty$ be a regular point of $F(p)$. This means that a Laurent-series expansion of the function $F(p)$ about the point $p=\infty$

---

* For example by differentiating with respect to a parameter.

** For a definition and the properties of the function $\Phi(z)$ see [17].

is of the form

$$F(p)=\sum_{n=0}^{\infty}\frac{c_n}{p^n} \tag{8-80}$$

When considering the properties of a transform, we noted that $|F(p)|\to 0$ as $\operatorname{Re} p\to+\infty$. Therefore, in the expansion (8-80) the coefficient $c_0$ is zero and

$$F(p)=\sum_{n=1}^{\infty}\frac{c_n}{p^n} \tag{8-81}$$

It is easy to find the function $f(t)$ of the real variable $t$ for which the function (8-81) is the transform.

***Theorem 8.6.*** *If the point $p=\infty$ is a regular point of the function $F(p)$ and $F(\infty)=0$, then $F(p)$ is the Laplace transform of the function of the real variable*

$$f(t)=\begin{cases} 0, & t<0 \\ \sum\limits_{n=0}^{\infty} c_{n+1}\dfrac{t^n}{n!}, & t>0 \end{cases} \tag{8-82}$$

*where $c_n$ are the coefficients of the Laurent-series expansion* (8-81) *of the function $F(p)$ about the point $p=\infty$.*

*Proof.* Earlier (see page 116) it was shown that the coefficients of the expansion (8-81) are given by the formula

$$c_n=\frac{1}{2\pi i}\int_{C_R} F(p)\,p^{n-1}\,dp$$

where $C_R$ is the circle $|p|=R$, exterior to which there are no singularities of the function $F(p)$. Since the point $p=\infty$ is a zero of $F(p)$, it follows that $|F(p)|<\frac{M}{R}$ or $|z|>R$. Therefore, the formula for $c_n$ yields

$$|c_n|<MR^{n-1}$$

From this evaluation there follows the convergence of the series (8-82). Indeed,

$$\left|\sum_{n=0}^{\infty} c_{n+1}\frac{t^n}{n!}\right|\leqslant\sum_{n=0}^{\infty}|c_{n+1}|\frac{|t|^n}{n!}<M\sum_{n=0}^{\infty}\frac{R^n|t|^n}{n!}=Me^{R|t|}$$

From this it follows that in a circle of any finite radius, the series (8-82) converges uniformly, thus defining some entire function of the complex variable $t$:

$$\tilde{f}(t)=\sum_{n=0}^{\infty} c_{n+1}\frac{t^n}{n!}$$

Note that we can regard the function $f(t)$ defined by the formula (8-82) as the product of the function $\tilde{f}(t)$ by the Heaviside unit function $\sigma_0(t)$.

Multiplying $f(t)$ by $e^{-pt}$ and integrating term-by-term with respect to $t$ the uniformly convergent series (8-82), we get, on the basis of $t^n \risingdotseq \frac{n!}{p^{n+1}}$ (see 8-19), the relation

$$\sum_{n=0}^{\infty} c_{n+1}\frac{t^n}{n!} \risingdotseq \sum_{n=0}^{\infty} c_{n+1}\frac{1}{p^{n+1}} = \sum_{n'=1}^{\infty} c_{n'}p^{-n} = F(p) \tag{8-83}$$

This proves the theorem

*Example 5.* Let

$$F(p)=\frac{1}{\sqrt{p^2+1}} \tag{8-84}$$

This function has two singularities $p_{1,2} = \pm i$ and is a single-valued analytic function about the point $p=\infty$; as was shown above (see example on page 123), in the neighbourhood of this point the function $F(p)$ may be expanded in a Laurent series:

$$F(p)=\sum_{k=0}^{\infty} (-1)^k \frac{(2k)}{2^{2k}(k!)^2}\cdot\frac{1}{p^{2k+1}}$$

And so formula (8-83) yields

$$\frac{1}{\sqrt{p^2+1}} \risingdotseq \sum_{k=0}^{\infty} (-1)^k \frac{t^{2k}}{2^{2k}(k!)^2} = \sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{t}{2}\right)^{2k}}{(k!)^2} \tag{8-85}$$

The series on the right of (8-85) is the expansion of an extremely important special function called the Bessel function* of zero order:

$$J_0(t)=\sum_{k=0}^{\infty} (-1)^k \frac{\left(\frac{t}{2}\right)^{2k}}{(k!)^2}$$

And so

$$\frac{1}{\sqrt{p^2+1}} \risingdotseq J_0(t) \tag{8-86}$$

* For definition and properties of the Bessel function see [17].

Observe that by representing

$$\frac{1}{p^2+1}=\frac{1}{\sqrt{p^2+1}}\cdot\frac{1}{\sqrt{p^2+1}}$$

and taking advantage of the transform of the function $\sin t$ [see formula (8-22)], we get

$$\int_0^t J_0(\tau)\,J_0(t-\tau)\,d\tau=\sin t$$

on the basis of the convolution theorem.

*Example 6.* Let

$$F(p)=\frac{1}{p}e^{-\frac{1}{p}}$$

This function evidently satisfies the conditions of Theorem 8.6, and

$$F(p)=\sum_{n=1}^{\infty}(-1)^{n-1}\frac{1}{(n-1)!\,p^n}$$

Then

$$\frac{1}{p}e^{-\frac{1}{p}}\doteqdot\sum_{n=0}^{\infty}(-1)^n\frac{t^n}{(n!)^2}=\sum_{k=0}^{\infty}(-1)^k\frac{\left(\frac{2\sqrt{t}}{2}\right)^{2k}}{(k!)^2}=J_0(2\sqrt{t}) \qquad (8\text{-}87)$$

## 8.3. Solving Problems for Linear Differential Equations by the Operational Method

In this section we consider the application of the methods of operational calculus to the solution of a number of problems for linear differential equations.

### *a. Ordinary differential equations*

In Section 8.1 we saw how operational methods could be used to reduce the Cauchy problem with zero initial conditions for a linear differential equation to a simple algebraic problem involving the transform. Let us consider the more general Cauchy problem:

$$a_0y^{(n)}(t)+a_1y^{(n-1)}(t)+\ldots+a_ny(t)=f(t) \qquad (8\text{-}88)$$

$$y(0)=y_0,\ y'(0)=y_1,\ \ldots,\ y^{(n-1)}(0)=y_{n-1} \qquad (8\text{-}89)$$

where $a_0$, $a_1$, ..., $a_n$, $y_0$, $y_1$, ..., $y_{n-1}$ are specified constants, $f(t)$ is the given function of an independent variable $t$, which is assumed to satisfy all the existence conditions of a transform. (See page 221 on the existence conditions of a transform.) Since the problem (8-88), (8-89) is linear, we can consider separately the solution of the homogeneous equation with initial conditions (8-89) and the solution of the nonhomogeneous equation (8-88) with zero initial conditions.

We begin with the first problem. It suffices to construct a fundamental system of solutions of the homogeneous equation (8-88). For this system we choose the solutions of the homogeneous equation

$$a_0\psi_k^{(n)}(t) + a_1\psi_k^{(n-1)}(t) + \ldots + a_n\psi_k(t) = 0, \quad k = 0,\ 1,\ \ldots,\ n-1 \tag{8-90}$$

which satisfy the initial conditions

$$\psi_k^{(j)}(0) = \delta_{kj} \quad \begin{matrix} k = 0,\ 1,\ \ldots,\ n-1 \\ j = 0,\ 1,\ \ldots,\ n-1 \end{matrix} \tag{8-91}$$

where

$$\delta_{kj} = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}$$

The functions $\psi_k(t)$ clearly form a fundamental system, since their Wronskian determinant, for $t = 0$, is definitely nonzero. The solution of the problem (8-88), (8-89) for $f(t) \equiv 0$ is expressed in the most elementary manner in terms of these functions:

$$\tilde{y}(t) = \sum_{k=0}^{n-1} y_k\psi_k(t)$$

We apply the operational method to determine the functions $\psi_k(t)$. Bearing in mind that the function $\psi_k(t)$ and all its derivatives up to order $n$ satisfy the existence conditions of a transform,* we, by (8-91) and (8-38), get

$$\psi_k(t) \risingdotseq \Psi_k(p), \quad \psi_k^{(j)}(t) \risingdotseq p^j\left[\Psi_k(p) - \frac{\varepsilon_{kj}}{p^{k+1}}\right], \quad j = 1, 2, \ldots, n$$

where

$$\varepsilon_{kj} = \begin{cases} 0, & j \leqslant k \\ 1, & j > k \end{cases}$$

Multiplying both sides of the equation (8-90) by $e^{-pt}$ and integrating with respect to $t$, we obtain

$$\Psi_k(p)\cdot P_n(p) = P_k(p) \tag{8-92}$$

where the polynomials $P_n(p)$ and $P_k(p)$ are equal, respectively, to

$$P_n(p) = a_0p^n + a_1p^{n-1} + \ldots + a_n$$

---

* Indeed, the functions $\psi_k(t)$, as solutions of the equation (8-90), are smooth functions which at infinity do not grow faster than an exponential function with linear exponent.

and

$$P_k(p) = a_0 p^{n-(k+1)} + a_1 p^{n-(k+2)} + \ldots + a_{n-(k+1)} \tag{8-93}$$

From (8-92)

$$\Psi_k(p) = \frac{P_k(p)}{P_n(p)}, \quad k = 0, 1, \ldots, n-1 \tag{8-94}$$

and, in particular,

$$\Psi_{n-1}(p) = \frac{P_{n-1}(p)}{P_n(p)} = \frac{a_0}{P_n(p)} \tag{8-95}$$

Formula (8-95) will be used in the sequel. The originals of the functions $\Psi_k(p)$ may be found by Mellin's formula:

$$\Psi_k(p) \risingdotseq \psi_k(t) = \frac{1}{2\pi i} \int\limits_{x-i\infty}^{x+i\infty} e^{pt} \frac{P_k(p)}{P_n(p)} dp, \quad x > a \tag{8-96}$$

where the straight line $x = a$ passes to the right of all singularities of the integrand function of (8-96). Since the function (8-94) is a ratio of two polynomials, only the zeros of the denominator $P_n(p)$ (all of them are poles) can be its singular points. Besides, for $t > 0$, the integrand of (8-96) obviously satisfies the conditions of the Jordan lemma in the left half-plane Re $p < a$. Therefore,

$$\psi_k(t) = \sum_{i=1}^{n} \text{Res}\left[e^{pt} \frac{P_k(p)}{P_n(p)}, \ p_i\right] \tag{8-97}$$

where the $p_i$ are zeros of the polynomial $P_n(p)$.

If all the zeros $p_i$ of the polynomial $P_n(p)$ are simple, then by representing it in the form of a product $P_n(p) = a_0 \prod\limits_{j=1}^{n} (p - p_j)$ we get, from formula (8-97),

$$\psi_k(t) = \sum_{i=1}^{n} a_{ki} e^{p_i t} \tag{8-98}$$

where

$$a_{ki} = \frac{P_k(p_i)}{a_0 \prod\limits_{j \neq i}^{1-n} (p_i - p_j)} \tag{8-99}$$

If the zeros $p_i$ of the polynomial $P_n(p)$ are multiple, then the expansion of the polynomial is of the form $P_n(p) = a_0 \prod\limits_{i=1}^{m} (p - p_i)^{\alpha_i}$, where $\alpha_i$ is the multiplicity of the appropriate zero and $\sum_{i=1}^{m} \alpha_i = n$. In this case, utilizing the rule for evaluating the residue at a pole of order $k > 1$ and computing the derivative of the product by the Leibniz formula, we get

$$\psi_k(t) = \sum_{i=1}^{m} q_{ki}(t)\, e^{p_i t} \tag{8-100}$$

where the polynomials $q_{ki}(t)$ have the form

$$q_{ki}(t) = b_{0,\,ki} t^{\alpha_i - 1} + b_{1,\,ki} t^{\alpha_i - 2} + \ldots + b_{\alpha_i - 1,\,ki} \tag{8-101}$$

and the coefficients $b_{m,\,k,\,i}$ are computed from the formula

$$b_{m,\,k,\,i} = \frac{1}{m!\,(\alpha_i - m - 1)} \frac{d^m}{dp^m} \left[ \frac{P_k(p)}{a_0 \prod\limits_{j \neq i}^{1-m} (p - p_j)^{\alpha_j}} \right]_{p = p_i} \tag{8-102}$$

Note that the zeros $p_i$ of the polynomial $P_n(p)$ coincide with the zeros of the characteristic polynomial for the equation (8-90). Therefore, formulas (8-98) and (8-100) yield a representation of each of the particular solutions of equation (8-90), satisfying the initial conditions (8-91), in terms of the particular solutions of equation (8-90) which are obtained with the aid of the characteristic equation.

*Example 1.* Solve the Cauchy problem

$$y^{(IV)} + 2y'' + y = 0, \; y(0) = y'(0) = y''(0) = 0, \; y'''(0) = 1$$

The obvious solution is the function $\psi_3(t)$, which can be found from formula (8-96):

$$y(t) = \psi_3(t) = \frac{1}{2\pi i} \int\limits_{x - i\infty}^{x + i\infty} e^{pt} \frac{dp}{p^4 + 2p^2 + 1} \tag{8-103}$$

The integrand function in (8-103) has two singularities $p_{1,2} = \pm i$, which are second-order poles. Therefore

$$y(t) = \frac{d}{dp}\left[ e^{pt} \frac{1}{(p+i)^2} \right]_{p=i} + \frac{d}{dp}\left[ e^{pt} \frac{1}{(p-i)^2} \right]_{p=-i} = \frac{1}{2}(\sin t - t\cos t) \tag{8-104}$$

Let us now tackle the Cauchy problem with zero initial conditions for the nonhomogeneous equation (8-88):

$$L[y(t)] = f(t)$$

By virtue of the zero initial conditions, when we pass to the transforms* $Y(p) \doteqdot y(t)$, $F(p) \doteqdot f(t)$ we obtain

$$Y(p)\,P_n(p) = F(p)$$

whence

$$Y(p) = \frac{F(p)}{P_n(p)} \tag{8-105}$$

Since the function $Y(p)$ is a transform, its original, by Theorem 8.3, may be found with the aid of the Mellin integral. However, in the given case we can dispense with computing this integral. Indeed, by (8-95) the function $\frac{a_0}{P_n(p)}$ is the transform of the function $\psi_{n-1}(t)$—the solution of the Cauchy problem for the homogeneous equation (8-90) with initial conditions of a special kind:

$$\psi_{n-1}^{(j)}(0) = \delta_{n-1,\,j}, \quad j = 0,\ 1,\ \ldots,\ n-1$$

And so by the convolution theorem from (8-105) we get

$$Y(p) \doteqdot y(t) = \frac{1}{a_0}\int_0^t \psi_{n-1}(t-\tau)\,f(\tau)\,d\tau$$

The function $\psi_{n-1}(t)$ is often called the unit point-source function for the equation (8-90) and is denoted by $g(t)$. Using this notation, rewrite the solution of the Cauchy problem with zero initial conditions for the equation (8-88) as

$$y(t) = \frac{1}{a_0}\int_0^t g(t-\tau)\,f(\tau)\,d\tau \tag{8-106}$$

Formula (8-106) is called Duhamel's integral.**

*Example 2*. Solve the Cauchy problem

$$y'' + y = \sin t, \quad y(0) = y'(0) = 0$$

---

* Note that for the existence of the transform $F(p)$ of the right side of equation (8-88)—of the function $f(t)$—the behaviour of this function as $t \to \infty$ is in many cases inessential. Indeed, we are often interested in the solution of (8-88) only over a bounded interval of time $0 \leqslant t \leqslant T$, which solution is fully determined by specifying the function $f(t)$ on this interval and is independent of the behaviour of the function $f(t)$ for $t > T$. We can therefore vary the values of the function $f(t)$ any way we please for $t > T$, provided the conditions for the existence of the transform $F(p)$ of the function $f(t)$ are fulfilled. For example, we can put $f(t) \equiv 0$ for $t > T$. (We stress the fact that for the determination of the transform $F(p)$, the function $f(t)$ must be specified over the entire infinite interval $0 \leqslant t < \infty$.) We then, of course, obtain different transforms, but their original functions naturally coincide for $t \leqslant T$. One should bear in mind that this situation refers not only to the case of equation (8-88), but also to many other physical problems in which the solution is sought in a restricted interval of time variation.

** On the use of Duhamel's integral in problems of mathematical physics, see [17].

Find the function $g(t)$:

$$g''+g=0, \quad g(0)=0, \quad g'(0)=1$$

By formula (8-95), for its transform $G(p)$ we get

$$G(p)=\frac{1}{p^2+1}$$

Whence by the table of transforms we find $G(p) \doteqdot \sin t$ and thus

$$y(t)=\int_0^t \sin(t-\tau)\sin\tau\, d\tau=\frac{1}{2}(\sin t - t\cos t)$$

### b. Heat-conduction equation

Let us consider the operational method in the solution of boundary-value problems for the heat-conduction equation in the case of the propagation of the boundary conditions over a semi-infinite rod.

Let it be required to find the temperature distribution in a semi-infinite rod $0 < x < \infty$ if from an instant $t = 0$ onwards a specified temperature regime is maintained on the left end $x = 0$. The rod is at an initial temperature of zero. Mathematically, the problem consists in determining the solution $u(x, t)$, bounded for $0 \leqslant x < \infty$, $t \geqslant 0$, of the equation

$$u_t = a^2 u_{xx}, \quad x > 0, \; t > 0 \tag{8-107}$$

with the supplementary conditions

$$u(x, 0) = 0, \quad u(0, t) = q(t) \tag{8-108}$$

where $q(t)$ is a given function of time, which we will assume satisfies the existence conditions of the Laplace transformation. Suppose that the desired solution $u(x, t)$ and also its derivatives that enter into equation (8-107) satisfy the existence conditions for the Laplace transformation with respect to $t$; and the conditions of bounded order of growth, with respect to $t$, of the function $u(x, t)$ and its derivatives are independent of $x$. We then have

$$\begin{aligned} u(x, t) &\doteqdot U(x, p) \\ u_t(x, t) &\doteqdot pU(x, p) \\ u_{xx}(x, t) &\doteqdot U_{xx}(x, p) \end{aligned} \tag{8-109}$$

The second of the formulas (8-109) is obtained with account taken of the zero initial condition (8-108). The last of the formulas (8-109) is valid due to the fact that the assumptions are sufficient for computing the derivatives of the improper integrals dependent on a parameter by differentiation of the integrand functions with respect to the parameter.

Now passing to transforms, for the function $u(x, t)$ we get the boundary-value problem for the transform $U(x, p)$ in place of the problem (8-107), (8-108):

$$U_{xx}(x, p) - \frac{p}{a^2} U(x, p) = 0 \tag{8-110}$$

$$U(0, p) = Q(p), \quad |U(x, p)| < M \tag{8-111}$$

This is a boundary-value problem for an ordinary differential equation; in this problem the variable $p$ plays the role of a parameter. It is easy to see that the solution of the problem (8-110), (8-111) has the form

$$U(x, p) = Q(p) e^{-\frac{\sqrt{p}}{a} x} \tag{8-112}$$

The solution $u(x, t)$ of the original problem may be found from its transform (8-112) with the aid of the Mellin formula; however, in the case of the arbitrary function $Q(p)$, computation of the appropriate integral can involve considerable difficulties. It is therefore natural to attempt to avoid direct evaluation of the Mellin integral in determining the original of the function (8-112). Note that above we found the original function for the function (see Example 4, p. 248)

$$\frac{1}{p} e^{-\alpha\sqrt{p}} \doteqdot 1 - \Phi\left(\frac{\alpha}{2\sqrt{t}}\right) = \frac{2}{\sqrt{\pi}} \int\limits_{\frac{\alpha}{2\sqrt{t}}}^{\infty} e^{-\eta^2} d\eta \tag{8-113}$$

Therefore, representing $U(x, p) = Q(p) \cdot p \cdot e^{\left\{-\frac{\sqrt{p}}{a} x\right\}} \frac{1}{p}$ and taking into account that by (8-113)

$$\frac{1}{p} e^{-\frac{\sqrt{p}}{a} x} \doteqdot 1 - \Phi\left(\frac{x}{2a\sqrt{t}}\right) = G(x, t) \tag{8-114}$$

on the basis of the theorems of the transform of a derivative and a convolution, we get

$$U(x, p) \doteqdot u(x, t) = \int\limits_0^t \frac{\partial}{\partial t} G(x, t-\tau)\, q(\tau)\, d\tau$$

Substituting the explicit expression (8-114) of the function $G(x, t)$ and differentiating, we get an expression of the solution of the problem* (8-107), (8-108) in the form

$$u(x, t) = \frac{x}{2a\sqrt{\pi}} \int_0^t e^{-\frac{x^2}{4a^2(t-\tau)}} \frac{q(\tau)}{(t-\tau)^{3/2}} d\tau \qquad (8\text{-}115)$$

### c. The boundary-value problem for a partial differential equation

The method described in Subsection 8.3.b may be formally extended to the solution of the boundary-value problem for a partial differential equation of a more general type.

Consider the equation

$$P_n[u(x, t)] - L_2[u(x, t)] = f(x, t) \qquad (8\text{-}116)$$

where $P_n[u]$ is a linear differential operator with constant coefficients of the type

$$P_n[u] = a_0 \frac{\partial^n u}{\partial t^n} + a_1 \frac{\partial^{n-1} u}{\partial t^{n-1}} + \ldots + a_{n-1} \frac{\partial u}{\partial t}$$

$L_2(u)$ is a linear differential operator of the second order** of the type

$$L_2[u] = b_0(x) \frac{\partial^2 u}{\partial x^2} + b_1(x) \frac{\partial u}{\partial x} + b_2(x) u(x, t)$$

the coefficients $b_i(x)$ of which are functions of only one independent variable $x$; $f(x, t)$ is a given function of the variables $x$ and $t$, which is sufficiently smooth in the domain of solution of the problem. We will seek the solution $u(x, t)$ of the equation (8-116) in the domain*** $t > 0$, $a < x < b$, which solution satisfies the initial conditions

$$u(x, 0) = \varphi_0(x), \quad \frac{\partial u}{\partial t}(x, 0) = \varphi_1(x), \quad \ldots, \quad \frac{\partial^{n-1} u}{\partial t^{n-1}}(x, 0) = \varphi_{n-1}(x)$$

and the boundary conditions

$$\alpha_1 \frac{\partial u}{\partial x}(a, t) + \beta_1 u(a, t) = \psi_1(t), \quad \alpha_2 \frac{\partial u}{\partial x}(b, t) + \beta_2 u(b, t) = \psi_2(t)$$

---

* Observe that this expression is obtained on the assumption that a solution exists; thus, the foregoing reasoning amounts to a uniqueness proof of the solution of the given problem in the class of functions under consideration. If the existence of a solution of the problem is not known beforehand, then it is necessary to demonstrate that the formally obtained expression (8-115) is indeed a solution of the problem at hand.

** The method at hand does not depend on the order of the differential operator $L$ (like $P$); however, because of its special importance and for vividness of exposition we confine ourselves to the case of the second-order operator $L$.

*** This method can also be applied when $a = -\infty$ or $b = +\infty$ or, simultaneously, $a = -\infty$, $b = +\infty$.

We assume that the initial and boundary conditions of the problem and also the function $f(x, t)$ are such that there exist Laplace transforms, with respect to $t$, of the function $u(x, t)$ and of all its derivatives that enter into the equation (8-116):

$$\begin{gathered} u(x, t) \risingdotseq U(x, p) = \int_0^\infty e^{-pt} u(x, t)\, dt \\ \frac{\partial u}{\partial x} \risingdotseq \int_0^\infty e^{-pt} \frac{\partial u}{\partial x}(x, t)\, dt \end{gathered} \tag{8-117}$$

and so forth; and let us suppose that the conditions involving bounded order of growth, with respect to $t$, of the function $u(x, t)$ and its derivatives are independent of $x$. Then, since the integral (8-117) is uniformly convergent in the parameter $x$, we get

$$\frac{\partial u}{\partial x}(x, t) \risingdotseq \frac{\partial U}{\partial x}(x, p), \quad \frac{\partial^2 u}{\partial x^2}(x, t) \risingdotseq \frac{\partial^2 U}{\partial x^2}(x, p)$$

and

$$\frac{\partial^k u}{\partial t^k}(x, t) \risingdotseq p^k U(x, p) - p^{k-1}\varphi_0(x) - p^{k-2}\varphi_1(x) - \ldots - \varphi_{k-1}(x)$$

Besides, we assume that there exist transforms, with respect to $t$, of the functions $f(x, t)$, $\psi_1(t)$ and $\psi_2(t)$:

$$f(x, t) \risingdotseq F(x, p), \quad \psi_1(t) \risingdotseq \Psi_1(p), \quad \psi_2(t) \risingdotseq \Psi_2(p)$$

Then, taking transforms in equation (8-116), we get for the function $U(x, p)$ an ordinary differential equation with respect to the independent variable $x$:

$$-P_n(p)\, U(p) + L_2[U(x, p)] = -F(x, p) - F_0(x, p) \tag{8-118}$$

where

$$F_0(x, p) = \sum_{k=0}^{n-1} P_k(p)\, \varphi_{n-k-1}(x)$$

and the polynomials $P_k(p)$ are defined by formula (8-93).

Equation (8-118) has to be solved with the boundary conditions

$$\begin{aligned} \alpha_1 U_x(a, p) + \beta_1 U(a, p) &= \Psi_1(p) \\ \alpha_2 U_x(b, p) + \beta_2 U(b, p) &= \Psi_2(p) \end{aligned} \tag{8-119}$$

The boundary-value problem (8-118), (8-119), in which $p$ plays the role of a parameter, is solved by the usual methods of solving boundary-value problems for ordinary differential equations. The inverse transformation from transform $U(x, p)$ to the solution of the original problem may be performed with the aid of the inversion formula (8-67).

# APPENDIX I

## SADDLE-POINT METHOD

The saddle-point method is widely used for constructing asymptotic expansions* of certain contour** integrals of functions of a complex variable. We consider integrals of the type

$$F(\lambda) = \int_C \varphi(z)\, e^{\lambda f(z)}\, dz \qquad \text{(I-1)}$$

where $\varphi(z)$ and $f(z)$ are functions of the complex variable $z$ analytic in some domain $\mathfrak{G}$ containing the curve $C$, which may be unbounded; $\lambda$ is a large positive number. We assume that integral (I-1) exists and our aim will be to obtain an asymptotic expansion of the function $F(\lambda)$ in inverse powers of the parameter $\lambda$. Integrals of type (I-1) are frequently encountered in studies of the integral representations of a number of special functions and also in the solution of many problems of mathematical physics and other divisions of mathematics.

### I.1. Introductory Remarks

Let us begin with suggestive remarks. We consider the integral defining Euler's gamma function

$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}\, dx \qquad \text{(I-2)}$$

and we shall try to find an approximate expression for it for large positive values of $p$. Note that by putting $x^p = e^{p \ln x}$ we reduce integral (I-2) to integral (I-1). The integrand in (I-2) tends to zero as $x \to 0$ and $x \to \infty$. Therefore the magnitude of the integral is mainly determined by the value of the integrand function in the

---

* Recall that an asymptotic expansion of the function $f(x)$ in the neighbourhood of a point $x_0$ is a representation of the form $f(x) = \sum_{k=1}^{N} a_k \varphi_k(x) + o(\varphi_N(x))$, where $a_k$ are constant coefficients and the functions $\varphi_k(x)$ as $x \to x_0$ satisfy the condition $\varphi_{k+1}(x) = o(\varphi_k(x))$.

** Following established practice, the contour of integration need not here be understood only as a closed curve.

neighbourhood of its maximum. Let us transform the integrand to

$$x^p e^{-x} = e^{p \ln x - x} = e^{f(x)} \qquad \text{(I-3)}$$

The function $f(x)$ attains a maximum value at $x = p$, and

$$f(p) = p \ln p - p, \quad f'(x)|_{x=p} = 0, \quad f''(x)|_{x=p} = -\frac{1}{p} \qquad \text{(I-4)}$$

Expanding the function $f(x)$ in a Taylor series in the $\delta$-neighbourhood of the point $x = p$ and taking only the first terms of the expansion, we get

$$\Gamma(p+1) \simeq \int_{p-\delta}^{p+\delta} e^{p \ln p - p - \frac{1}{2p}(x-p)^2}\, dx = p^p e^{-p} \int_{p-\delta}^{p+\delta} e^{-\frac{(x-p)^2}{2p}}\, dx$$

$$\simeq p^p e^{-p} \int_{-\infty}^{\infty} e^{-\frac{(x-p)^2}{2p}}\, dx \qquad \text{(I-5)}$$

The approximate equalities occur because the integrand is small for $|x - p| > \delta$ and rapidly tends to zero. In (I-5) make a change of the integration variable, putting $\sqrt{\frac{1}{2p}}\,(x - p) = y$. Then

$$\Gamma(p+1) \simeq \sqrt{2p}\, p^p e^{-p} \int_{-\infty}^{\infty} e^{-y^2}\, dy = \sqrt{2\pi p} \left(\frac{p}{e}\right)^p \qquad \text{(I-6)}$$

Formula (I-6) yields an approximate expression of integral (I-2) for large values of $p$. As will be shown later on, it is the first term of the asymptotic expansion of the integral (I-2). It is often called Stirling's formula.

In deriving this formula we did not evaluate the accuracy of the approximations and so our considerations are only illustrative. Later on, we will estimate the accuracy of formula (I-6). For the present we offer some more remarks aimed at facilitating a grasp of the basic idea of the saddle-point method. Formula (I-6) expresses an approximate value of integral (I-2) in terms of the value of the integrand at the point of its maximum ($p^p e^{-p}$) and a certain additional factor corresponding to the length of the interval of integration on which the value of the integrand is sufficiently close to maximum.

Let us examine integral (I-1), in which the integrand is analytic in the domain $\mathfrak{G}$ of the complex $z$-plane. This integral can also be approximately evaluated in terms of the maximum value of the modulus of the integrand function with account taken of the speed of its decrease on the contour of integration. If the path of integration connecting the points $z_1$ and $z_2$ is such that on a small section of it the absolute value of the integrand reaches its maximum and then rapidly decreases, it is natural to suppose that the quantity thus found yields a good approximation. Since the function $f(z)$ is analytic in the domain $\mathfrak{G}$, then by virtue of the Cauchy theorem the value of integral (I-1) is determined solely by specification of the initial point $z_1$ and the terminal point $z_2$ of the path of integration and not by the type of curve $C$. It then follows that for a given integral (I-1) the possibility of its approximate evaluation with the aid of the methods under study is associated with the possibility of choosing a contour of integration that will satisfy the requirements indicated above. We are interested in the values of integral (I-1) for large positive values of the parameter $\lambda$ in the exponent of the exponential factor. It is therefore natural to expect that the major contribution to the value of the integral will come from those portions of the path of integration on which the function $u(x, y)$—the real part of the function $f(z) = u(x, y) + iv(x, y)$—attains the greatest values. We must also bear in mind here that the function $u(x, y)$, which is harmonic in the domain $\mathfrak{G}$, cannot attain an absolute maximum at interior points of the domain; i.e. inside the domain $\mathfrak{G}$ there are no points at which the function $u(x, y)$ can increase or decrease in all directions. The surface of the function $u(x, y)$ *can only have saddle points.*

Let the point $z_0 = x_0 + iy_0$ be the sole saddle point of the surface $u(x, y)$ in the domain $\mathfrak{G}$. Let us consider lines of constant value $u(x, y) = u(x_0, y_0) =$ constant of the function $u(x, y)$ which pass through this point. By virtue of the maximum principle for harmonic functions (see [17]), these lines cannot form closed curves (we do not consider the trivial case $f \equiv$ constant in $\mathfrak{G}$); that is, they either end at the boundary of the domain $\mathfrak{G}$ or recede to infinity in the case of an unbounded domain. The curves $u(x, y) = u(x_0, y_0)$ partition the domain $\mathfrak{G}$ into sectors within which the values of the function $u(x, y)$ are, respectively, either less or greater than $u(x_0, y_0)$. We call the former sectors *negative*, the latter, *positive*.

If the end points $z_1$ and $z_2$ of the integration curve lie in one sector and the function $u(x, y)$ at these points takes on different values, then it is obviously possible to deform the contour so that the function $u(x, y)$ will vary monotonically on it. Here, the dominant contribution to the value of the integral is made by the neighbourhood of that end point at which the value of the function $u(x, y)$ is greatest. The same occurs when $z_1$ lies in the positive sector and $z_2$ lies in the negative sector or vice versa. The saddle-point method is employed when the points $z_1$ and $z_2$ lie in different negative sectors, which fact enables one to choose a contour of integration passing through the saddle point $x_0, y_0$ on which the function $u(x, y)$ is maximum at the point $x_0$, $y_0$ and rapidly decreases in the direction of the end points. In this case, clearly, the dominant contribution to the value of integral (I-1) will be made by a small section in the neighbourhood of the saddle point, which section may be chosen the smaller the faster the values of the function $u(x, y)$ decrease along the integration curve. The saddle-point method is also sometimes called the *method of steepest descent*. This "mountaineering" terminology has to do, most likely, with the topography of the surface of the function $u(x, y)$ in the neighbourhood of the saddle point. Let us now estimate the accuracy of the method by means of which the asymptotic formula (I-6) was obtained. We will also establish a number of propositions underlying the saddle-point method.

## I.2. Laplace's Method

We prove a number of auxiliary propositions underlying the so-called Laplace method of asymptotic representation of integrals of functions of a real variable.

*Lemma 1. For $p>0$ and as $A \to \infty$ we have the asymptotic formula**

$$\int_0^A x^{p-1}e^{-x}\,dx = \Gamma(p) + O\left(e^{-\frac{A}{2}}\right) \tag{I-7}$$

*Proof*. We evaluate, for $p>1$, the integral**

$$\int_A^\infty e^{-x}x^{p-1}\,dx = e^{-A}\int_0^\infty e^{-y}(y+A)^{p-1}\,dy < e^{-A}\left\{\int_0^A (2A)^{p-1}e^{-y}\,dy + \int_0^\infty (2y)^{p-1}e^{-y}\,dy\right\} = e^{-A}\{(2A)^{p-1}(1-e^{-A}) + 2^{p-1}\Gamma(p)\} \tag{I-8}$$

Whence (I-7) follows for $A^p < e^{A/2}$.

---

* The symbol $O(t^2)$, or more generally, $O(t^n)$ in an expansion of the form $\varphi(t) = \sum_{k=0}^{n-1} c_k t^k + O(t^n)$ implies that for $|t| \leqslant \delta$ we have the uniform evaluation $\left|\varphi(t) - \sum_{k=0}^{n-1} c_k t^k\right| < C|t|^n$, where $C$ is a constant.

** For $0 < p \leqslant 1$ $\int_A^\infty e^{-x}x^{p-1}\,dx \leqslant \int_A^\infty e^{-x}\,dx = e^{-A}$.

In what follows a prominent role will be played by integrals of the form

$$\Phi(\lambda)=\int_{-a}^{a}\varphi(t)\,e^{-\lambda t^2}\,dt,\quad 0<a<\infty$$

The following lemma is valid.

*Lemma 2. Let the function $\varphi(t)$ for $|t|\leqslant\delta$ be representable in the form*

$$\varphi(t)=c_0+c_1t+O(t^2) \tag{I-9}$$

*and for some $\lambda_0>0$ let the integral*

$$\int_{-a}^{a}|\varphi(t)|\,e^{-\lambda_0t^2}\,dt<M \tag{I-10}$$

*converge. Then for $\lambda>\lambda_0$ the asymptotic formula*

$$\Phi(\lambda)=\int_{-a}^{a}\varphi(t)\,e^{-\lambda t^2}\,dt=c_0\sqrt{\frac{\pi}{\lambda}}+O(\lambda^{-3/2}) \tag{I-11}$$

*holds true.*

*Proof.* The principal term of formula (I-11) is readily obtainable from the following suggestive reasoning. If the function $\varphi(t)$ is bounded for $|t|>a$, then it is natural to expect that the value of integral (I-11) will change but slightly if the limits of integration are changed: $-a$ to $-\infty$ and $a$ to $\infty$. Then the first term in the expansion (I-9) yields the principal term of formula (I-11), the integral of the second term is zero due to the oddness of the integrand, and it remains to evaluate the remainder term. It is this evaluation and the possibility of the indicated change of limits of integration that comprise the basic content of the lemma. We now begin a rigorous proof.

Split up the integral $\Phi(\lambda)$ into three terms:

$$\Phi(\lambda)=\int_{-a}^{-\delta}\varphi(t)\,e^{-\lambda t^2}\,dt+\int_{-\delta}^{\delta}\varphi(t)\,e^{-\lambda t^2}\,dt+\int_{\delta}^{a}\varphi(t)\,e^{-\lambda t^2}\,dt \tag{I-12}$$

where $\delta>0$ is some fixed number. Evaluate the last term:

$$\left|\int_{\delta}^{a}\varphi(t)\,e^{-\lambda t^2}\,dt\right|\leqslant e^{-(\lambda-\lambda_0)\delta^2}\int_{\delta}^{a}|\varphi(t)|\,e^{-\lambda_0t^2}\,dt$$
$$\leqslant Me^{\lambda_0\delta^2}\cdot e^{-\lambda\delta^2}=O(e^{-\lambda\delta^2}) \tag{I-13}$$

Here we took advantage of the condition (I-10) and the obvious inequality

$$\lambda t^2=\lambda\delta^2+\lambda(t^2-\delta^2)>\lambda\delta^2+\lambda_0(t^2-\delta^2)=(\lambda-\lambda_0)\delta^2+\lambda_0t^2$$

that occurs for $\lambda > \lambda_0$, $t > \delta$. Analogously, we evaluate the first term in (I-12). It then follows that for sufficiently large $\lambda$ the principal contribution to the value of the integral $\Phi(\lambda)$ is made by the second term, whereas the extreme terms in (I-12) tend to zero exponentially as $\lambda \to \infty$.

Let us examine the principal term in (I-12). Putting in place of $\varphi(t)$ its expansion (I-9), we get

$$\Phi_2(\lambda) = \int_{-\delta}^{\delta} \varphi(t)\, e^{-\lambda t^2}\, dt$$

$$= c_0 \int_{-\delta}^{\delta} e^{-\lambda t^2}\, dt + c_1 \int_{-\delta}^{\delta} e^{-\lambda t^2} t\, dt + \int_{-\delta}^{\delta} O(t^2)\, e^{-\lambda t^2}\, dt \qquad \text{(I-14)}$$

Due to the oddness of the integrand, the second integral in (I-14) is zero. To evaluate the first integral, make a change of variable, putting $\lambda t^2 = \tau$. We get

$$\int_{-\delta}^{\delta} e^{-\lambda t^2}\, dt = 2\int_{0}^{\delta} e^{-\lambda t^2}\, dt = \frac{1}{\sqrt{\lambda}} \int_{0}^{\lambda\delta^2} \tau^{-\frac{1}{2}} e^{-\tau}\, d\tau \qquad \text{(I-15)}$$

But by virtue of Lemma 1, as $\lambda \to \infty$, we have the asymptotic formula

$$\int_{0}^{\lambda\delta^2} \tau^{-\frac{1}{2}} e^{-\tau}\, d\tau = \Gamma\left(\frac{1}{2}\right) + O\left(e^{-\frac{\lambda\delta^2}{2}}\right) = \sqrt{\pi} + O\left(e^{-\frac{\lambda\delta^2}{2}}\right) \qquad \text{(I-16)}$$

Since for any fixed $\delta$ the function $e^{-\frac{\lambda\delta^2}{2}}$ tends to zero faster than $\lambda^{-3/2}$ as $\lambda \to \infty$, we can write

$$c_0 \int_{-\delta}^{\delta} e^{-\lambda t^2}\, dt = c_0 \sqrt{\frac{\pi}{\lambda}} + O(\lambda^{-3/2}) \qquad \text{(I-17)}$$

It remains to evaluate the last term in (I-14):

$$\int_{-\delta}^{\delta} O(t^2)\, e^{-\lambda t^2}\, dt < C \int_{-\delta}^{\delta} t^2 e^{-\lambda t^2}\, dt = 2C \int_{0}^{\delta} t^2 e^{-\lambda t^2}\, dt \qquad \text{(I-18)}$$

In integral (I-18) again make the change of variable $\lambda t^2 = \tau$. Then we have

$$2\int_{0}^{\delta} e^{-\lambda t^2}\, dt = \frac{1}{\lambda^{3/2}} \int_{0}^{\lambda\delta^2} \tau^{1/2} e^{-\tau}\, d\tau \qquad \text{(I-19)}$$

Integral (I-19) also satisfies the conditions of Lemma 1. Therefore we finally obtain

$$\int_{-\delta}^{\delta} O(t^2)\, e^{-\lambda t^2}\, dt = C\frac{\Gamma\left(\frac{3}{2}\right)}{\lambda^{3/2}} + O\left(e^{-\frac{\lambda\delta^2}{2}}\right) = O\left(\lambda^{-\frac{3}{2}}\right) \quad \text{(I-20)}$$

Formulas (I-13), (I-17) and (I-20), after their substitution into (I-12), prove the lemma.

A number of remarks are in order regarding this lemma.

*Note 1.* Repeating the earlier arguments, it is possible to prove that if the function $\varphi(t)$ for $|t| \leqslant \delta$ is expandable in the Taylor series

$$\varphi(t) = \sum_{k=0}^{n-1} c_k t^k + O(t^n), \quad c_k = \frac{\varphi^{(k)}(0)}{k!} \quad \text{(I-21)}$$

then we have the asymptotic expansion

$$\Phi(\lambda) = \int_{-a}^{a} \varphi(t)\, e^{-\lambda t^2}\, dt = \sum_{m=0}^{\left[\frac{n-1}{2}\right]} c_{2m} \frac{\Gamma\left(m+\frac{1}{2}\right)}{\lambda^{m+\frac{1}{2}}} + O\left(\lambda^{-\frac{n+1}{2}}\right) \quad \text{(I-22)}$$

where the symbol $\left[\frac{n-1}{2}\right]$ denotes the greatest integer less than or equal to $\frac{n-1}{2}$.

In particular, for $n = 1$, when the expansion of the function $\varphi(t)$ is of the form $\varphi(t) = c_0 + O(t)$, the remainder term in formula (I-22) is of the order $\lambda^{-1}$, since in the evaluation of the remainder the main role is played by the integral

$$\int_{-\delta}^{\delta} O(t)\, e^{-\lambda t^2}\, dt < C \int_{-\delta}^{\delta} |t|\, e^{-\lambda t^2}\, dt = 2C \int_{0}^{\delta} t e^{-\lambda t^2}\, dt$$

*Note 2.* The lemma holds true also for the case when the integration is performed over the interval $[a_1, a_2]$, where $a_1 < 0$, $a_2 > 0$ and $-a_1 \neq a_2$. The next remark is so essential for what follows that we state it in the form of a separate lemma.

***Lemma 3.*** *On the interval* $|t| \leqslant \delta_0$ *let the functions* $\varphi(t)$ *and* $\mu(t)$ *be representable in the form*

$$\varphi(t) = c_0 + c_1 t + O(t^2) \quad \text{(I-9)}$$

$$\mu(t) = c_3 t^3 + O(t^4) \quad \text{(I-23)}$$

*and as* $\lambda \to \infty$ *let the function* $\delta(\lambda) \leqslant \delta_0$ *satisfy the conditions**

$$\lambda\delta^2(\lambda) \to \infty, \quad \lambda\delta^3(\lambda) \to 0 \qquad \text{(I-24)}$$

*Then as* $\lambda \to \infty$ *we have the asymptotic formula*

$$I(\lambda) = \int_{-\delta(\lambda)}^{\delta(\lambda)} \varphi(t)\, e^{\lambda[-t^2+\mu(t)]}\, dt = c_0 \sqrt{\frac{\pi}{\lambda}} + O(\lambda^{-3/2}) \qquad \text{(I-25)}$$

*Proof.* It is easy to see that if the conditions (I-9), (I-23) and (I-24) are fulfilled on the interval $|t| \leqslant \delta(\lambda)$, the following equality holds:

$$\varphi(t)\, e^{\lambda\mu(t)} = c_0 + c_1 t + c_0 c_3 \lambda t^3 + O(t^2) + O(\lambda^2 t^6) + O(\lambda t^4) \qquad \text{(I-26)}$$

Then, repeating the argument given in the proof of Lemma 2, we find that upon substitution of the expansion (I-26) into formula (I-25) the first term, by condition (I-24), yields the principal term of the right side of (I-25); the second and third terms of the resulting expression vanish due to the oddness of the integrands; the last three terms are infinitesimals of the same order $O(\lambda^{-3/2})$. The lemma is proved.

These lemmas enable us to prove the following theorem which underlies the Laplace method of the asymptotic expansion of integrals of functions of a real variable.

***Theorem I.1.*** *Let a function* $f(t)$ *given on the interval* $[a, b]$ *attain its absolute maximum at some interior point* $t_0$, $f''(t_0) < 0$, *and let there be a* $\delta_0 > 0$ *such that for* $|t - t_0| < \delta_0$ *the following representation holds*:

$$f(t) = f(t_0) + \frac{f''(t_0)}{2}(t - t_0)^2 + \mu(t) \qquad \text{(I-27)}$$

*Then, if the functions* $\varphi(t)$ *and* $\mu(t)$ *for* $|t - t_0| \leqslant \delta_0$ *satisfy the conditions of Lemma 3, i.e.*

$$\varphi(t) = c_0 + c_1(t - t_0) + O[(t - t_0)^2] \qquad \text{(I-9)}$$

$$\mu(t) = c_3(t - t_0)^3 + O[(t - t_0)^4] \qquad \text{(I-23)}$$

*the asymptotic formula*

$$\Psi(\lambda) = \int_a^b \varphi(t)\, e^{\lambda f(t)}\, dt = e^{\lambda f(t_0)} \left\{ \sqrt{-\frac{2\pi}{\lambda f''(t_0)}}\, \varphi(t_0) + O(\lambda^{-3/2}) \right\} \qquad \text{(I-28)}$$

*holds if the following supplementary conditions are fulfilled*:

---

* As is readily seen, the function $\delta(\lambda) = \lambda^{-2/5}$ for example satisfies the conditions (I-24).

(a) *for a given* $\delta_0$ *the following relations are fulfilled simultaneously*:

$$\begin{aligned} &\textit{for } |t-t_0|\leqslant\delta_0 \quad |\mu(t)|<-\frac{f''(t_0)}{4}(t-t_0)^2 \\ &\textit{for } |t-t_0|>\delta_0 \quad f(t_0)-f(t)\geqslant h>0 \end{aligned} \tag{I-29}$$

(b) *for some* $\lambda_0>0$ *the integral*

$$\int_a^b |\varphi(t)|\, e^{\lambda_0 f(t)}\, dt\leqslant M \tag{I-30}$$

*converges*.

*Proof*. Let us split the integral in (I-28) into a sum of the following terms:

$$\begin{aligned} \Psi(\lambda) &= \int_a^b \varphi(t)\, e^{\lambda f(t)}\, dt = \int_a^{t_0-\delta_0} \varphi(t)\, e^{\lambda f(t)}\, dt + \int_{t_0-\delta_0}^{t_0-\delta(\lambda)} \varphi(t)\, e^{\lambda f(t)}\, dt \\ &+ \int_{t_0-\delta(\lambda)}^{t_0+\delta(\lambda)} \varphi(t)\, e^{\lambda f(t)}\, dt + \int_{t_0+\delta(\lambda)}^{t_0+\delta_0} \varphi(t)\, e^{\lambda f(t)}\, dt + \int_{t_0+\delta_0}^{b} \varphi(t)\, e^{\lambda f(t)}\, dt \end{aligned} \tag{I-31}$$

where the function $\delta(\lambda)$ satisfies the conditions (I-24) of Lemma 3. The extreme integrals in (I-31) are evaluated as in Lemma 2. Indeed, using the obvious inequality

$$\begin{aligned} \lambda[f(t_0)-f(t)] &= (\lambda-\lambda_0)[f(t_0)-f(t)]+\lambda_0[f(t_0)-f(t)] \\ &\geqslant h(\lambda-\lambda_0)+\lambda_0 f(t_0)-\lambda_0 f(t) \end{aligned} \tag{I-32}$$

which holds for $a\leqslant t\leqslant t_0-\delta_0$ and $\lambda>\lambda_0$, we obtain

$$\begin{aligned} \left|\int_a^{t_0-\delta_0} \varphi(t)\, e^{\lambda f(t)}\, dt\right| &\leqslant e^{\lambda f(t_0)} \int_a^{t_0-\delta_0} |\varphi(t)|\, e^{-\lambda[f(t_0)-f(t)]}\, dt \\ &< e^{(\lambda-\lambda_0)f(t_0)-h(\lambda-\lambda_0)} \int_a^{t_0-\delta_0} |\varphi(t)|\, e^{\lambda_0 f(t)}\, dt \\ &< M e^{\lambda f(t_0)+\lambda_0[h-f(t_0)]} e^{-\lambda h} = e^{\lambda f(t_0)} O(e^{-\lambda h}) \end{aligned} \tag{I-33}$$

In the same way we evaluate the integral over the interval $[t_0+\delta_0, b]$. To evaluate the second integral, take advantage of the conditions (I-27), (I-29), by virtue of which, for $t_0-\delta_0\leqslant t\leqslant t_0-\delta(\lambda)$, we have the inequality

$$f(t_0)-f(t)>-\frac{f''(t_0)}{4}(t-t_0)^2\geqslant-\frac{f''(t_0)}{4}\delta^2(\lambda) \tag{I-34}$$

Therefore, repeating the computations carried out in deriving formula (I-33), we obtain

$$\left| \int_{t_0-\delta_0}^{t_0-\delta(\lambda)} \varphi(t)\, e^{\lambda f(t)}\, dt \right| = e^{\lambda f(t_0)} O\left(e^{-C\lambda\delta^2(\lambda)}\right), \quad C > 0 \qquad \text{(I-35)}$$

But by condition (I-24), the quantity on the right side of (I-35) is also of an exponential order of smallness.* The fourth integral is evaluated in similar fashion.

We now examine the principal integral of the formula (I-31):

$$\Psi_3(\lambda) = \int_{t_0-\delta(\lambda)}^{t_0+\delta(\lambda)} \varphi(t)\, e^{\lambda f(t)}\, dt \qquad \text{(I-36)}$$

By virtue of condition (I-27), this integral may be rewritten as

$$\Psi_3(\lambda) = e^{\lambda f(t_0)} \int_{t_0-\delta(\lambda)}^{t_0+\delta(\lambda)} \varphi(t)\, e^{\lambda\left[\frac{f''(t_0)}{2}(t-t_0)^2+\mu(t)\right]} dt \qquad \text{(I-37)}$$

Reduce (I-37) to the form (I-25) by the change of variable $-\frac{f''(t_0)}{2} \times (t-t_0)^2 = \tau^2$. It is clearly seen that the resulting integral satisfies all the conditions of Lemma 3. And so we finally get

$$\Psi_3(\lambda) = e^{\lambda f(t_0)} \left\{ \sqrt{-\frac{2\pi}{\lambda f''(t_0)}}\, \varphi(t_0) + O\left(\lambda^{-3/2}\right) \right\} \qquad \text{(I-38)}$$

Since $\Psi_3(\lambda)$ differs from the integral being evaluated by an exponentially small term, formula (I-38) proves the theorem.

*Note 1.* The theorem holds true for the case when one or both of the limits of integration are infinite, since the evaluation of integral (I-33) holds true for $a = -\infty$ as well.

*Note 2.* We obtained only the first term in the asymptotic expansion of integral (I-28). It is possible, in similar fashion, to obtain an expression for the succeeding terms of the asymptotic expansion, but we will not dwell on this point.

*Note 3.* The proof given here may be extended also to the case when the maximum value of the function $f(t)$ is attained at one of the boundary points of the interval $[a, b]$. Then a supplementary factor $\frac{1}{2}$ appears in formula (I-28).

*Note 4.* When the function $f(t)$ inside the interval $[a, b]$ has several maxima of equal magnitude, the asymptotic expansion of integral (I-28) in terms of inverse powers of the large parameter $\lambda$ may

* For $\delta(\lambda) = \lambda^{-2/5}$ we get $O\left(e^{-C\lambda^{1/5}}\right)$.

be obtained by evaluating integrals of the type (I-36) about the δ-neighbourhood of each of the maximum points and summing the results.

We give an example of the application of this theorem.

*Example 1.* Obtain the asymptotic expansion of Euler's gamma function

$$\Gamma(p+1)=\int_0^\infty x^p e^{-x}\,dx \tag{I-2}$$

Represent the integrand in the form $x^p e^{-x} = e^{p\ln x - x}$ and make the change of variable $x = pt$. Then integral (I-2) is transformed to

$$\Gamma(p+1)=p^{p+1}\int_0^\infty e^{p(\ln t-t)}\,dt \tag{I-39}$$

This is an integral of type (I-28) with $\varphi(t)\equiv 1$ and $f(t) = \ln t - t$. The function $f(t)$ attains its maximum for $t_0 = 1$, and

$$f(1) = -1,\ f'(t)\,|_{t=1} = 0,\ f''(t)\,|_{t=1} = -1 \tag{I-40}$$

Therefore, by formula (I-28) we get

$$\Gamma(p+1)=e^{-p}\left\{\sqrt{\frac{2\pi}{p}}+O(p^{-3/2})\right\}p^{p+1} = \sqrt{2\pi p}\left(\frac{p}{e}\right)^p\left\{1+O\left(\frac{1}{p}\right)\right\} \tag{I-41}$$

We have thus obtained an asymptotic evaluation of the accuracy of formula (I-6) that was earlier obtained from suggestive reasoning. As has been pointed out, these methods permit obtaining the subsequent terms of the asymptotic expansion as well. We give without derivation the first few terms of Stirling's formula:

$$\Gamma(p+1)=\sqrt{2\pi p}\left(\frac{p}{e}\right)^p\left\{1+\frac{1}{12p}+\frac{1}{288p^2}-\frac{139}{51\,840p^3}+\dots\right\} \tag{I-42}$$

## I.3. The Saddle-Point Method

We now examine the saddle-point method proper for obtaining asymptotic expansions of integrals of type (I-1):

$$F(\lambda)=\int_C \varphi(z)\,e^{\lambda f(z)}\,dz$$

By the suggestive arguments of Section I.1, it is natural to suppose that if the contour $C$ is such that on any small section of it the values of the real part $u(x, y)$ of the function $f(z) = u(x, y) + iv(x, y)$ attain a maximum and then decay rapidly, while the imaginary part $v(x, y)$ remains constant (in order to guarantee the absence of undesirable rapid oscillations of the integrand function), then the major contribution to the magnitude of integral (I-1) is made by integration over the given section of the contour $C$. It is therefore advisable in an approximate calculation of integral (I-1) to deform the contour $C$ so that the integrand possesses the indicated properties. In this process, as was established by our earlier reasoning, the necessary deformation of the contour $C$ is determined primarily by the topography of the level surface of the function $u(x, y)$. In particular, the contour of integration has to pass through the saddle point of the surface of the function $u(x, y)$ in the direction of fastest variation of this function.

Let us examine in more detail the topography of the surface of the harmonic function $u(x, y)$ in the neighbourhood of the saddle point $M_0(x_0, y_0)$. We determine the directions of fastest variation of this function passing through the point $M_0$. These directions are known to be defined by the direction of the vector grad $u$. Let grad $u \neq 0$. Since for the analytic function $\nabla u \cdot \nabla v = 0$ (see page 34), the direction of the vector grad $u$ defines the curve $v(x, y)$ = constant. Thus, if on the curve $v(x, y)$ = constant, grad $u \neq 0$, then the function $u(x, y)$ varies along this curve with greatest rapidity. However, at the saddle point itself $M_0(x_0, y_0)$ of the surface of the function $u(x, y)$ the vector grad $u(M_0) = 0$. Let us examine in more detail the behaviour of the functions $u(x, y)$ and $v(x, y)$ in the neighbourhood of this point. Obviously, at the point $M_0$ the derivatives of the functions $u(x, y)$ and $v(x, y)$ are zero with respect to the direction $l$ of the tangent to the curve $v(x, y)$ = constant passing through $M_0$:

$$\frac{\partial u}{\partial l}(x_0, y_0) = 0, \quad \frac{\partial v}{\partial l}(x_0, y_0) = 0 \tag{I-43}$$

Since the derivative of an analytic function is independent of direction, it follows that

$$f'(z_0) = 0 \tag{I-44}$$

Consequently, the expansion of the function $f(z)$ in the neighbourhood of the point $z_0$ is of the form

$$f(z) = f(z_0) + (z - z_0)^p \{c_0 + c_1 (z - z_0) + \dots\} \tag{I-45}$$

where $p \geqslant 2$ and $c_0 \neq 0$. Putting $c_n = r_n e^{i\theta_n}$, $n = 0, 1, \dots$, $z - z_0 = \rho e^{i\varphi}$, we get

$$f(z) - f(z_0) = \rho^p \{r_0 e^{i(p\varphi + \theta_0)} + \rho r_1 e^{i[(p+1)\varphi + \theta_1]} + \dots\} \tag{I-46}$$

With the aid of the introduced notation let us write down the equations of the curves $u(x, y)$ = constant and $v(x, y)$ = constant passing through the point $z_0$. We have

$$U(\rho, \varphi) = r_0 \cos(p\varphi + \theta_0) + \rho r_1 \cos[(p+1)\varphi + \theta_1)] + \ldots$$
$$\ldots = 0 \quad \text{(I-47)}$$

$$V(\rho, \varphi) = r_0 \sin(p\varphi + \theta_0) + \rho r_1 \sin[(p+1)\varphi + \theta_1] + \ldots$$
$$\ldots = 0 \quad \text{(I-48)}$$

Here,

$$u(x, y) - u(x_0, y_0) = \rho^p U(\rho, \varphi)$$
$$v(x, y) - v(x_0, y_0) = \rho^p V(\rho, \varphi)$$

Since, when $\varphi$ varies from 0 to $2\pi$ the function $\cos(p\varphi + \theta_0)$ changes sign $2p$ times, it follows from formula (I-47) that the neighbourhood of the point $z_0$ is broken up into $2p$ curvilinear sectors, inside which the function $U(\rho, \varphi)$ does not change sign. The boundaries of these sectors are defined from the solution of equation (I-47). The sectors in which $U(\rho, \varphi) < 0$ will as before be called negative and the sectors in which $U(\rho, \varphi) > 0$, positive. The directions of steepest descent of the function $u(x, y)$ obviously lie in the negative sectors and are determined by those values of the angle $\varphi$ for which, in the neighbourhood of the point $(x_0, y_0)$, $V(\rho, \varphi) = 0$ and $U(\rho, \varphi) < 0$, i.e. $\cos(p\varphi + \theta_0) = -1$. These values are equal to

$$\varphi_m = -\frac{\theta_0}{p} + \frac{2m+1}{p}\pi, \quad m = 0, 1, \ldots, p-1 \qquad \text{(I-49)}$$

Note that the directions of steepest descent coincide with the bisectors of the negative sectors.

In future we will only consider the case $p = 2$ when $f''(z_0) \neq 0$. Here, $c_0 = \frac{1}{2} f''(z_0)$ and $\theta_0 = \arg f''(z_0)$. In this case, there are only two negative sectors inside which passes the line of steepest descent of the function $u(x, y)$. The direction of the tangent to this line at the point $z_0$ is, according to formula (I-49), determined by the angles

$$\varphi_0 = \frac{-\theta_0 + \pi}{2} \text{ and } \varphi_1 = \frac{-\theta_0 + 3\pi}{2} = \varphi_0 + \pi \qquad \text{(I-50)}$$

Evidently, the choice of angle $\varphi_0$ or $\varphi_1$ is determined by specification of the direction of integration along the line of steepest descent.

Now let us take up the proof of the basic theorem of the saddle-point method.

***Theorem I.2.*** *Let the functions $\varphi(z)$ and $f(z) = u(x, y) + iv(x, y)$ be analytic in the domain $\mathfrak{G}$ and satisfy the following conditions:*

(1) *The surface of the function $u(x, y)$ has inside $\mathfrak{G}$ a unique saddle point $z_0 = x_0 + iy_0$; $f''(z_0) \neq 0$.*

(2) *There is a $\delta > 0$ such that on the line $L$ of constant value of the function $v(x, y) = v(x_0, y_0)$ passing through the point $z_0$, in both*

*negative sectors of this point the function $u(x, y)$ outside the $\delta$-neighbourhood of $z_0$ satisfies the condition*

$$u(x_0, y_0) - u(x, y) \geqslant h > 0 \tag{I-51}$$

(3) *For some value $\lambda_0 > 0$ the following line integral converges:*

$$\int_C |\varphi(z)| e^{\lambda_0 u(x, y)} ds < M \tag{I-52}$$

*where the curve $C$ lies entirely in the domain $\mathfrak{G}$, and its initial point $(z_1)$ and terminal point $(z_2)$ lie in different negative sectors of the point $z_0$ so that they may be joined with the curve $L$ by curves $\gamma_1$ and $\gamma_2$ of finite length, on which the function $u(x, y)$ satisfies the condition* (I-51).

*Then for all $\lambda \geqslant \lambda_0$ the following asymptotic formula holds:*

$$F(\lambda) = \int_C \varphi(z) e^{\lambda f(z)} dz = e^{\lambda f(z_0)} \left\{ \sqrt{\frac{2\pi}{\lambda |f''(z_0)|}} \varphi(z_0) e^{i\varphi_m} + O(\lambda^{-3/2}) \right\} \tag{I-53}$$

*where* $\varphi_m = \frac{\pi - \theta_0}{2} + m\pi$ $(m = 0, 1)$ *and* $\theta_0 = \arg f''(z_0)$. The choice of value of $\varphi_m$ determines the sign in formula (I-53) and, naturally, depends on the direction of integration along the contour $C$.

*Proof*. Integral (I-53) does not change value if the integration curve $C$ is deformed into the curve $\Gamma = L + \gamma_1 + \gamma_2$. By condition (I-51), for integrals along the curves $\gamma_1$ and $\gamma_2$ we have the evaluation

$$\int_{\gamma_{1,2}} \varphi(z) e^{\lambda f(z)} dz = e^{\lambda f(z_0)} O(e^{-\lambda h}) \tag{I-54}$$

Consider the integral

$$F_1(\lambda) = \int_L \varphi(z) e^{\lambda f(z)} dz \tag{I-55}$$

On the curve $L$ we introduce the natural parameter $s$ and consider that the value $s = 0$ corresponds to the point $z_0$. Write the equation of the curve $L$ in the form $z = z(s)$. Making the change of variable $z = z(s)$ in (I-55), we obtain

$$F_1(\lambda) = e^{i\lambda v(x_0, y_0)} \int_{-a}^{b} \Phi(s) e^{\lambda U(s)} \frac{dz}{ds} ds \tag{I-56}$$

where

$$\Phi(s) = \varphi[z(s)], \quad U(s) = u[x(s), y(s)]$$
$$0 < a < \infty, \quad 0 < b < \infty$$

The integral (I-56) satisfies all the conditions of Theorem 1; the function $U(s)$ attains its maximum at the point $s=0$ and $\left.\frac{d^2U}{ds^2}\right|_{s=0} < 0$.

Then, according to (I-28)

$$F_1(\lambda)=e^{\lambda f(z_0)}\left\{\sqrt{-\frac{2\pi}{\lambda U''(0)}}\,\Phi(0)\,z'(0)+O(\lambda^{-3/2})\right\} \qquad \text{(I-57)}$$

and it remains to express the quantities entering into (I-57) in terms of the values of the functions $\varphi(z)$ and $f(z)$ at the point $z_0$. Clearly, $\Phi(0)=\varphi(z_0)$. Since $\left.\frac{d^2V}{ds^2}\right|_L=0$, it follows that

$$\left.\frac{d^2U}{ds^2}\right|_L=\left.\frac{d^2f[z(s)]}{ds^2}\right|_L=f''(z)\left(\frac{dz}{ds}\right)^2+f'(z)\frac{d^2z}{ds^2} \qquad \text{(I-58)}$$

Whence, by (I-44), we obtain

$$\left.\frac{d^2U}{ds^2}\right|_{s=0}=f''(z_0)\left[\left(\frac{dz}{ds}\right)_{s=0}\right]^2 \qquad \text{(I-59)}$$

Since in the neighbourhood of the point $z_0$ we have the relation $z-z_0=se^{i\varphi}$ to within higher-order infinitesimals, it follows that $\left.\frac{dz}{ds}\right|_{s=0}=e^{i\varphi}$ and it remains to determine the direction of the tangent to the curve $L$ at the point $z_0$. However, by the very mode of construction of the curve $L$, the tangent to this curve at the point $z_0$ coincides with the direction of the fastest variation of the function $u(x, y)$. Then, from (I-50), for the angle $\varphi_m$ we get the formula

$$\varphi_m=\frac{\pi-\theta_0}{2}+m\pi, \quad m=0,\ 1 \qquad \text{(I-60)}$$

where $\theta_0=\arg f''(z_0)$ and the value of $m$ is determined by the direction of integration. Note that $\left.\frac{d^2U}{ds^2}\right|_{s=0}<0$ and $\left|\frac{dz}{ds}\right|_{s=0}=1$. Then formula (I-59) may be written as

$$\left.\frac{d^2U}{ds^2}\right|_{s=0}=-|f''(z_0)| \qquad \text{(I-61)}$$

We then finally get

$$F(\lambda)=e^{\lambda f(z_0)}\left\{\sqrt{\frac{2\pi}{\lambda|f''(z_0)|}}\,\varphi(z_0)\,e^{i\varphi_m}+O(\lambda^{-3/2})\right\} \qquad \text{(I-62)}$$

where the value of the angle $\varphi_m$ is given by formula (I-60). The sign of the principal term on the right side of (I-62) is determined by the choice of value of $m$ and is dependent on the direction of integration along the curve $C$.

Some remarks are in order concerning this theorem.

*Note 1.* From this theorem it follows that if both end points $z_1$ and $z_2$ of the integration curve $C$ lie in the same negative sector of the saddle point $z_0$, then the evaluation (I-54) is valid for integral (I-1).

*Note 2.* In applications, one is particularly often involved with type (I-1) integrals in an unbounded domain with integration curve $C$ receding to infinity. From the foregoing arguments it is clear that in this case it is necessary for the convergence of integral (I-1) that the integration curve recede to infinity in the negative sectors of the saddle point $z_0$. Here, Theorem 2 and formula (I-53) remain valid.

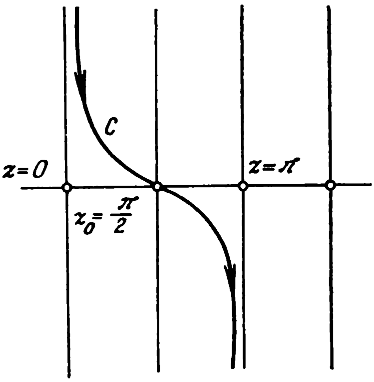


**Fig. I.1**

*Note 3.* Theorem 2 was proved on the assumption that the point $z_0$ is a unique saddle point of the surface of the function $u(x, y)$ in the domain $\mathfrak{G}$ and $f''(z_0) \neq 0$. If these assumptions are not fulfilled, similar reasoning may be carried out leading to asymptotic expansions of integral (I-1) similar to formula (I-53). However, when there are several saddle points in the domain $\mathfrak{G}$, the choice of the integration contour requires a special investigation. If the contour of integration passes through several saddle points, the asymptotic expansion of integral (I-1) may contain several terms, having the same order, like the first term in (I-53), and this is capable of altering substantially the final result.

We consider a number of examples involving the results obtained.

*Example 2. An asymptotic formula for the Hankel function.*

A Hankel function (see [17]) of the first kind $H_\nu^{(1)}(x)$ may be represented with the aid of the integral

$$H_\nu^{(1)}(x) = \frac{1}{\pi}\int_C e^{ix\sin z - i\nu z}\,dz \qquad \text{(I-63)}$$

where the contour of integration $C$ in the complex $z$-plane goes from the half-strip $-\frac{\pi}{2} < \operatorname{Re} z < \frac{\pi}{2}$, $\operatorname{Im} z > 0$ to the half-strip $\frac{\pi}{2} < \operatorname{Re} z < \frac{3}{2}\pi$, $\operatorname{Im} z < 0$ through the point $z_0 = \frac{\pi}{2}$ (Fig. I.1). This is a saddle point of the function $f(z) = i\sin z$ in the strip $0 < \operatorname{Re} z < \pi$ since $f'\left(\frac{\pi}{2}\right) = 0$, $f''\left(\frac{\pi}{2}\right) = -i \neq 0$. The above-indicated half-strips are negative sectors of this saddle point, which, for one thing, ensures convergence of the given improper integral. Let us find the asymptotic value of this integral for large positive values of $x \gg |\nu|$. The given integral, where $f(z) = i \sin z$, $\varphi(z) = e^{-i\nu z}$, obviously satisfies the conditions of Theorem 2. Therefore, the saddle-point method can be used to evaluate it. Since $f'(z) = i \cos z$, there is only one saddle point $z_0 = \frac{\pi}{2}$ in the strip $-\frac{\pi}{2} < \text{Re}\, z < \frac{3}{2}\pi$. Here, $f(z_0) = i$, $f'(z_0) = 0$, $|f''(z_0)| = 1$, $\theta_0 = \frac{3\pi}{2}$. Taking into account the direction of integration, we get, from (I-50), $\varphi_0 = -\frac{\pi}{4}$. Note that this direction coincides with the bisector of the negative sector of the saddle point $z_0 = \frac{\pi}{2}$ Finally, on the basis of formula (I-53) we get

$$H_\nu^{(1)}(x) = \frac{1}{\pi} e^{ix} \left\{ \sqrt{\frac{2\pi}{x}} e^{-i\nu\frac{\pi}{2} - i\frac{\pi}{4}} + O(x^{-3/2}) \right\}$$

$$= \sqrt{\frac{2}{\pi x}} \left\{ e^{i\left(x - \frac{\nu\pi}{2} - \frac{\pi}{2}\right)} + O\left(\frac{1}{x}\right) \right\} \quad \text{(I-64)}$$

Formula (I-64) finds extensive application in the solution of various problems that involve asymptotic representation of cylindrical functions

*Example 3. An asymptotic formula for Legendre polynomials.** We will proceed from the integral representation (see [9]) of the Legendre polynomials

$$P_n(\cos\theta) = \frac{1}{\pi\sqrt{2}} \int_{-\theta}^{\theta} \frac{e^{i\left(n+\frac{1}{2}\right)\varphi}}{\sqrt{\cos\varphi - \cos\theta}} d\varphi, \quad 0 < \theta < \pi \quad \text{(I-65)}$$

It is readily seen that the integrand function has an integrable singularity for $\varphi = \pm\theta$. Our aim is to obtain an asymptotic expression for the function $P_n(\cos\theta)$ for large values of the index $n$. We consider the analytic continuation of the integrand into the complex plane $z = x + iy$:

$$w(z) = \frac{e^{i\left(n+\frac{1}{2}\right)z}}{\sqrt{\cos z - \cos\theta}} \quad \text{(I-66)}$$

The function $w$ is analytic in the upper half-plane $\text{Im}\, z > 0$. Therefore, the integral of this function along any closed contour lying entirely in the upper half-plane is zero. We choose the closed contour** $\Gamma$ consisting of the segment ($y=0$, $-\theta \leqslant x \leqslant \theta$) of the real

* For a definition of Legendre polynomials and their basic properties, see [17].

** Here we traverse the singularities $z = \pm\theta$ along arcs of circles of infinitely small radius; the radius is then allowed to tend to zero.

axis, the vertical segments ($x = -\theta$, $0 \leqslant y \leqslant H$), ($x = \theta$, $0 \leqslant y \leqslant H$) parallel to the imaginary axis, and a closing horizontal

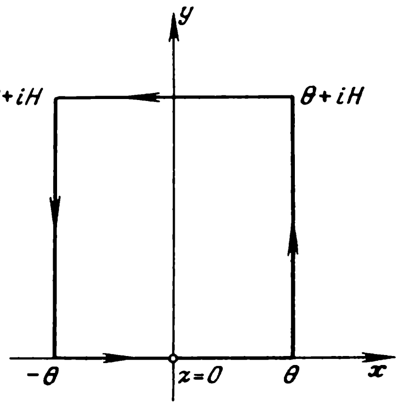


**Fig. I.2**

segment ($y = H$, $-\theta \leqslant x \leqslant \theta$) (Fig. I.2). It is easy to see that on the last segment the modulus

$$|w| = \frac{e^{-\left(n+\frac{1}{2}\right)H}}{\left|\sqrt{\cos(x+iH)-\cos\theta}\right|} \tag{I-67}$$

tends to zero exponentially as $H \to \infty$. For this reason, passing to the limit as $H \to \infty$, we obtain

$$I = \int_{-\theta}^{\theta} \frac{e^{i\left(n+\frac{1}{2}\right)\varphi}}{\sqrt{\cos\varphi - \cos\theta}}\, d\varphi = I_1 + I_2 \tag{I-68}$$

where

$$I_1 = ie^{-i\left(n+\frac{1}{2}\right)\theta} \int_0^{\infty} \frac{e^{-\left(n+\frac{1}{2}\right)y}}{\sqrt{\cos(\theta - iy) - \cos\theta}}\, dy \tag{I-69}$$

and

$$I_2 = -ie^{i\left(n+\frac{1}{2}\right)\theta} \int_0^{\infty} \frac{e^{-\left(n+\frac{1}{2}\right)y}}{\sqrt{\cos(\theta + iy) - \cos\theta}}\, dy \tag{I-70}$$

The saddle-point method is applicable for an approximate computation of integrals $I_1$ and $I_2$ for large values of $n$. Consider integral

$I_1$ ($I_2$ is computed in similar fashion). Put $y = t^2$ and denote $n + \frac{1}{2} = \lambda$. Then from (I-69) we have

$$\Psi(\lambda) = -ie^{i\lambda\theta}I_1 = 2\int_0^\infty \frac{e^{-\lambda t^2}\, t\, dt}{\sqrt{\cos(\theta - it^2) - \cos\theta}} \quad \text{(I-71)}$$

Clearly, integral (I-71) satisfies all the conditions of Theorem 1, and $f(t) = -t^2$ and the point $t_0 = 0$, at which the function $f(t)$ attains its maximum $f(0) = 0$, coincides with the end point of the interval of integration. Then $f''(0) = -2$ and

$$\varphi(t_0) = \lim_{t\to 0} \frac{t}{\sqrt{\cos(\theta - it^2) - \cos\theta}} = \frac{e^{-i\frac{\pi}{4}}}{\sqrt{\sin\theta}} \quad \text{(I-72)}$$

Therefore, by formula (I-28), into which we have to introduce a supplementary factor $\frac{1}{2}$, since the point $t_0$ coincides with the end point of the interval of integration, we get

$$\Psi(\lambda) = \sqrt{\frac{\pi}{\lambda}} \cdot \frac{1}{\sqrt{\sin\theta}}\, e^{-i\frac{\pi}{4}} + O(\lambda^{-3/2}) \quad \text{(I-73)}$$

whence

$$I_1 = ie^{-i\left(n+\frac{1}{2}\right)\theta}\left\{\sqrt{\frac{\pi}{\left(n+\frac{1}{2}\right)\sin\theta}}\, e^{-i\frac{\pi}{4}} + O(n^{-3/2})\right\} \quad \text{(I-74)}$$

Similarly

$$I_2 = -ie^{i\left(n+\frac{1}{2}\right)\theta}\left\{\sqrt{\frac{\pi}{\left(n+\frac{1}{2}\right)\sin\theta}}\, e^{i\frac{\pi}{4}} + O(n^{-3/2})\right\} \quad \text{(I-75)}$$

Then, after simplifications, taking into account that $\frac{1}{\sqrt{n+\frac{1}{2}}}$ differs from $\sqrt{\frac{1}{n}}$ by a quantity of the order of $O(n^{-3/2})$, we finally get the asymptotic formula for Legendre polynomials which holds for $n \gg 1$ and $0 < \theta < \pi$:

$$P_n(\cos\theta) = \sqrt{\frac{2}{\pi n \sin\theta}}\left\{\cos\left[\left(n+\frac{1}{2}\right)\theta - \frac{\pi}{4}\right] + O(n^{-1})\right\} \quad \text{(I-76)}$$

# APPENDIX II

## THE WIENER-HOPF METHOD

The Wiener-Hopf method finds extensive application when solving certain integral equations and various boundary-value problems of mathematical physics by means of the integral transformations of Laplace, Fourier, and others. This method was first employed, in a joint study by N. Wiener and E. Hopf (1931), in the solution of integral equations with a kernel depending on the difference of arguments in the case of a semi-infinite interval:

$$u(x) = \lambda \int_0^\infty v(x-s)\, u(s)\, ds + f(x)$$

Subsequently, equations of this kind were considered by V. A. Fock in [5] who made a substantial contribution to the development of general methods of their solution.

The general method of solving functional equations which became known as the Wiener-Hopf method or the factorization method, has been successfully employed in the solution of many problems of diffraction and the theory of elasticity, of boundary-value problems involving the heat-conduction equation, of integral equations in the theory of radiative transport (known as Milne's problem), and many other problems of mathematical physics.* Our aim is not to give a rigorous mathematical substantiation of the Wiener-Hopf method, but only to illustrate the basic idea in a series of examples involving the solution of a number of important problems.

### II.1. Introductory Remarks

Let us begin with suggestive arguments illustrating the application of methods of integral transformations in the solution of integral equations. Let us consider an integral equation of the type

$$u(x) = \lambda \int_{-\infty}^{\infty} v(x-s)\, u(s)\, ds + f(x) \tag{II-1}$$

* Numerous examples involving the Wiener-Hopf technique are given in [12] which contains an extensive bibliography.

with kernel $v(x-s)$ depending on the difference of the arguments. We will not investigate the solvability conditions of this equation or substantiate the methods of its solution, but only point out that for real values of $\lambda$, upon fulfilment of the conditions

$$\int_{-\infty}^{\infty} |f(x)|^2\,dx < A, \quad \lambda \int_{-\infty}^{\infty} |v(t)|\,dt < 1 \tag{II-2}$$

where $A$ is an arbitrary fixed number, equation (II-1) has a unique solution* $u(x)$ that is quadratically integrable in the infinite interval

$$\int_{-\infty}^{\infty} |u(x)|^2\,dx < \infty \tag{II-3}$$

We take it that there exist Fourier transforms of all functions involved in equation (II-1):

$$U(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(x)\,e^{ikx}\,dx \tag{II-4}$$

$$V(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(t)\,e^{ikt}\,dt \tag{II-5}$$

$$F(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x)\,e^{ikx}\,dx \tag{II-6}$$

Then, multiplying (II-1) by $\frac{1}{\sqrt{2\pi}}e^{ikx}$ and integrating over the infinite interval, we obtain

$$U(k) = F(k) + \frac{\lambda}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{ikx}\,dx \int_{-\infty}^{\infty} v(x-s)\,u(s)\,ds = F(k) + I(k) \tag{II-7}$$

Inverting the order of integration in the last term, we represent this integral in the form

$$I(k) = \frac{\lambda}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(s)\,ds \int_{-\infty}^{\infty} e^{ikx} v(x-s)\,dx \tag{II-8}$$

Making the change of variable $x-s=t$, we have, by (II-4) and (II-5),

$$I(k) = \frac{\lambda}{\sqrt{2\pi}} \int_{-\infty}^{\infty} u(s)\,e^{iks}\,ds \int_{-\infty}^{\infty} v(t)\,e^{ikt}\,dt = \lambda\sqrt{2\pi}\,U(k)\,V(k) \tag{II-9}$$

---

* This question is discussed in detail by E. C. Titchmarsh in [18].

Actually, formula (II-9) implies that the formula for the transformation of a convolution which we obtained for the one-sided Laplace transformation (see page 232) is also valid in the case of the Fourier transformation.

Formula (II-7) may now be rewritten in the form

$$U(k) = F(k) + \lambda\sqrt{2\pi}\,U(k)\,V(k) \tag{II-10}$$

Thus, with the aid of the Fourier transformation we have succeeded in reducing the solution of the original integral equation (II-1) to the solution of the algebraic equation (II-10) for the Fourier transformation of the desired solution. It is easy to solve the last equation:

$$U(k) = \frac{F(k)}{1-\lambda\sqrt{2\pi}\,V(k)} \tag{II-11}$$

Thus, the Fourier transformation (II-11) of the solution of the original integral equation proved to be expressed in terms of the Fourier transformation of the given functions—the kernel and the right-hand side of the equation. The solution itself can readily be expressed in terms of its Fourier transformation with the aid of the familiar formula of inverse transformation:

$$u(x) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} U(k)\,e^{-ikx}\,dk = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \frac{F(k)\,e^{-ikx}}{1-\lambda\sqrt{2\pi}\,V(k)}\,dk \tag{II-12}$$

Formula (II-12) actually solves the problem, but it is not always convenient since it requires the computation of the Fourier transformation $F(k)$ for every right-hand side of $f(x)$. In many cases it is more convenient to represent the solution of the nonhomogeneous integral equation in terms of the *resolvent kernel* for the original equation:

$$u(x) = f(x) + \lambda\int_{-\infty}^{\infty} g(x-s)\,f(s)\,ds \tag{II-13}$$

In order to obtain the required representation, note that formula (II-10) may be transformed to the form

$$U(k) - F(k) = \lambda\sqrt{2\pi}\,F(k)\,G(k) \tag{II-14}$$

where

$$G(k) = \frac{V(k)}{1-\lambda\sqrt{2\pi}\,V(k)} \tag{II-15}$$

From relation (II-14), with the aid of the formula of inverse transformation (II-12) and noting that by virtue of formula (II-9) the original of the function $\sqrt{2\pi}F(k)\,G(k)$ is the function

$$\int_{-\infty}^{\infty} g(x-s)\,f(s)\,ds$$

where

$$g(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(k)\,e^{-ikt}\,dk \tag{II-16}$$

we get

$$u(x)=f(x)+\lambda\int_{-\infty}^{\infty} g(x-s)\,f(s)\,ds \tag{II-13}$$

Thus, to determine the solution of the original integral equation (II-1) it is sufficient to find the function $g(t)$ defined by formula (II-16).

The function $g(t)$ is a solution of equation (II-1) for the special type of function $f(x)$. Indeed, from formulas (II-11) and (II-15) it follows that for $U(k)=G(k)$ the function $F(k)$ is equal to $V(k)$. This means that the solution of equation (II-1) for $f(x)\equiv v(x)$ is the function $u(x)\equiv g(x)$, that is, the resolvent kernel for equation (II-1) satisfies the integral equation

$$g(x)=\int_{-\infty}^{\infty} v(x-s)\,g(s)\,ds+v(x) \tag{II-17}$$

*Example 1.* Solve the integral equation

$$u(x)=\lambda\int_{-\infty}^{\infty} v(x-s)\,u(s)\,ds+f(x) \tag{II-18}$$

where

$$v(t)=e^{-\alpha|t|},\quad \alpha>0 \tag{II-19}$$

Let us find the function $g(t)$. To do this, calculate

$$V(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\alpha|t|}e^{ikt}\,dt=\frac{1}{\sqrt{2\pi}}\,\frac{2\alpha}{\alpha^2+k^2} \tag{II-20}$$

Then by formula (II-15)

$$G(k)=\frac{V(k)}{1-\lambda\sqrt{2\pi}\,V(k)}=\frac{1}{\sqrt{2\pi}}\,\frac{2\alpha}{k^2+\alpha^2-2\alpha\lambda} \tag{II-21}$$

whence

$$g(t)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} G(k)\,e^{-ikt}\,dk=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\alpha e^{-ikt}}{k^2+\alpha^2-2\alpha\lambda}\,dk \qquad \text{(II-22)}$$

Assume $\lambda<\frac{\alpha}{2}$. Then the integral (II-22) is meaningful and may readily be computed with the aid of the calculus of residues by applying Jordan's lemma. Simple manipulations yield

$$g(t)=\alpha\frac{e^{-|t|\sqrt{\alpha^2-2\alpha\lambda}}}{\sqrt{\alpha^2-2\alpha\lambda}} \qquad \text{(II-23)}$$

and, finally

$$u(x)=f(x)+\frac{\alpha\lambda}{\sqrt{\alpha^2-2\alpha\lambda}}\int_{-\infty}^{\infty} e^{-|x-s|\sqrt{\alpha^2-2\alpha\lambda}}f(s)\,ds \qquad \text{(II-24)}$$

Thus, the use of this method, which reduces the solution of the original integral equation (II-1) to the solution of an algebraic equation, was associated with the possibility of applying the Fourier transformation to the functions in this equation and of using the convolution formula. Our immediate aim is to transfer these methods to the solution of integral equations with a difference kernel in the case of a semi-infinite interval

$$u(x)=\lambda\int_{0}^{\infty} v(x-s)\,u(s)\,ds+f(x) \qquad \text{(II-25)}$$

But for this we need some analytic properties of the Fourier transformation, in particular, the definition of the domains of analyticity of the Fourier transformation of functions of a real variable, functions which both decrease and increase at infinity.

## II.2. Analytic Properties of the Fourier Transformation

Let the function $f(x)$ be defined for all values $-\infty<x<\infty$. We consider the Fourier transformation of this function

$$F(k)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)\,e^{ikx}\,dx \qquad \text{(II-26)}$$

We assume here that the parameter $k$ of the transformation (II-26) can, generally speaking, take on complex values as well. We pose the question of the properties of the function $F(k)$, which is regarded as a function of the complex variable $k$. To do this, represent $f(x)$ as

$$f(x)=f_+(x)+f_-(x) \qquad \text{(II-27)}$$

where the functions $f_-(x)$ and $f_+(x)$ are, respectively,

$$f_-(x) = \begin{cases} f(x), & x < 0, \\ 0, & x > 0, \end{cases} \qquad f_+(x) = \begin{cases} 0, & x < 0 \\ f(x), & x > 0 \end{cases}$$

The Fourier transform $F(k)$ of the function $f(x)$ is then obviously equal to the sum of the Fourier transforms $F_+(k), F_-(k)$ of the functions $f_+(x)$ and $f_-(x)$. We find the analytic properties of $F(k)$ by establishing the analytic properties of the functions $F_+(k)$ and $F_-(k)$. Thus, consider the function

$$f_+(x) = \begin{cases} 0, & x < 0 \\ f(x), & x > 0 \end{cases} \tag{II-28}$$

Its Fourier transform is the function

$$F_+(k) = \frac{1}{\sqrt{2\pi}} \int_0^\infty f_+(x)\, e^{ikx}\, dx \tag{II-29}$$

Repeating the arguments of Theorems 8.1 and 8.2, it is easy to show that if the function $f_+(x)$ satisfies the condition

$$|f_+(x)| < Me^{\tau_- x} \text{ as } x \to \infty \tag{II-30}$$

then the function $F_+(k)$ defined by formula (II-29) is an analytic function of the complex variable $k = \sigma + i\tau$ in the domain $\operatorname{Im} k > \tau_-$, and in this domain $F_+(k) \to 0$ as $|k| \to \infty$. With the aid of reasoning similar to that in Theorem 8.5, it may be shown that the functions $f_+(x)$ and $F_+(k)$ are connected by the inverse relation

$$f_+(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} F_+(k)\, e^{-ikx}\, dk \tag{II-31}$$

where the integration is performed over any straight line $\operatorname{Im} k = \tau > \tau_-$ parallel to the real axis in the complex $k$-plane.

For $\tau_- < 0$ [i.e. for the functions $f(x)$ decreasing at infinity] the domain of analyticity of the function $F_+(k)$ contains the real axis and in formula (II-31) the integration may be performed along the real axis. If $\tau_- > 0$ [i.e. the function $f_+(x)$ increases at infinity, but not faster than an exponential function with linear exponent] then the domain of analyticity of the function $F_+(k)$ lies above the real axis of the complex $k$-plane [here the integral (II-29) may diverge on the real $k$-axis]. Similarly, if the function

$$f_-(x) = \begin{cases} f(x), & x < 0 \\ 0, & x > 0 \end{cases} \tag{II-32}$$

satisfies the condition

$$f_-(x) < Me^{\tau_+ x} \quad \text{as} \quad x \to -\infty \tag{II-33}$$

then its Fourier transform, the function

$$F_-(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} f_-(x)\, e^{ikx}\, dx \tag{II-34}$$

is an analytic function of the complex variable $k$ in the domain $\operatorname{Im} k < \tau_+$. The function $f_-(x)$ is expressed in terms of the function $F_-(x)$ with the aid of the relation

$$f_-(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} F_-(k)\, e^{-ikx}\, dk \tag{II-35}$$

where $\operatorname{Im} k = \tau < \tau_+$.

If $\tau_+ > 0$, then the domain of analyticity of the function $F_-(k)$ contains the real axis.

Clearly, for $\tau_- < \tau_+$ the function $F(k)$ defined by formula (II-26) is an analytic function of the complex variable $k$ in the strip $\tau_- < \operatorname{Im} k < \tau_+$. Then the functions $f(x)$ and $F(k)$ are related by the inverse Fourier transformation:

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty+i\tau}^{\infty+i\tau} F(k)\, e^{-ikx}\, dk \tag{II-36}$$

where the integration is performed along any straight line, parallel to the real axis of the complex $k$-plane, lying in the strip $\tau_- < \operatorname{Im} k = \tau < \tau_+$. In particular, for $\tau_- < 0$ and $\tau_+ > 0$, the function $F(k)$ is analytic in the strip containing the real axis of the complex $k$-plane.

Thus, the function $V(x) = e^{-\alpha |x|}$ for $\alpha > 0$ has the Fourier transform

$$V(k) = \frac{1}{\sqrt{2\pi}} \cdot \frac{2\alpha}{\alpha^2 + k^2} \tag{II-37}$$

which is an analytic function of the complex variable $k$ in the strip $-\alpha < \operatorname{Im} k < \alpha$ containing the real axis.

Let us now examine the basic idea of the Wiener-Hopf method. We will first demonstrate it in solving a special type of integral equation.

## II.3. Integral Equations with a Difference Kernel

Let us begin with a homogeneous integral equation of the form

$$u(x) = \lambda \int_0^\infty v(x-s)\, u(s)\, ds \tag{II-38}$$

whose kernel, the function $v(x-s)$, depends on the difference $x - s = \xi$ and is defined for all values of its argument $-\infty < \xi < \infty$. The solution of this equation is obviously found to within an arbitrary factor; it may be found from the supplementary conditions of the problem, for instance the normalization conditions. We assume that equation (II-38) defines the function $u(x)$ for all values of the variable $x$, whether positive or negative. We introduce the functions $u_-$ and $u_+$:

$$u_-(x) = \begin{cases} u(x), & x<0, \\ 0 & x>0, \end{cases} \qquad u_+(x) = \begin{cases} 0, & x<0 \\ u(x), & x>0 \end{cases} \tag{II-39}$$

Clearly, $u(x) = u_+(x) + u_-(x)$, and equation (II-38) may be rewritten in the form

$$u_+(x) = \lambda \int_0^\infty v(x-s)\, u_+(s)\, ds, \quad x>0 \tag{II-40}$$

$$u_-(x) = \lambda \int_0^\infty v(x-s)\, u_+(s)\, ds, \quad x<0 \tag{II-41}$$

That is, the function $u_+(x)$ is determined from the solution of the integral equation (II-40), and the function $u_-(x)$ is expressed in terms of $u_+(x)$ and $v(x)$ with the aid of the quadrature formula (II-41). Here, the relation

$$u_+(x) + u_-(x) = \lambda \int_0^\infty v(x-s)\, u_+(s)\, ds \tag{II-42}$$

which is equivalent to the original equation (II-38), holds.

Let the function $v(\xi)$ satisfy the conditions

$$|v(\xi)| < Me^{\tau_- \xi} \text{ as } \xi \to \infty$$

and

$$|v(\xi)| < Me^{\tau_+ \xi} \text{ as } \xi \to -\infty \tag{II-43}$$

where $\tau_- < 0$, $\tau_+ > 0$. Then the function

$$V(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(\xi)\, e^{ik\xi}\, d\xi \tag{II-44}$$

is analytic in the strip $\tau_- < \operatorname{Im} k < \tau_+$.

We seek a solution of equation (II-38) that satisfies the condition*

$$|u_+(x)| < M_1 e^{\mu x} \text{ as } x \to \infty \tag{II-45}$$

where $\mu < \tau_+$. And, as can easily be verified, the integrals on the right sides of relations (II-40) and (II-41) are convergent, for the function $u_-(x)$ we have the evaluation

$$|u_-(x)| < M_2 e^{\tau_+ x} \text{ as } x \to -\infty \tag{II-46}$$

From the conditions (II-45) and (II-46) it follows that the Fourier transforms $U_+(k)$ and $U_-(k)$ of the functions $u_+(x)$ and $u_-(x)$ are

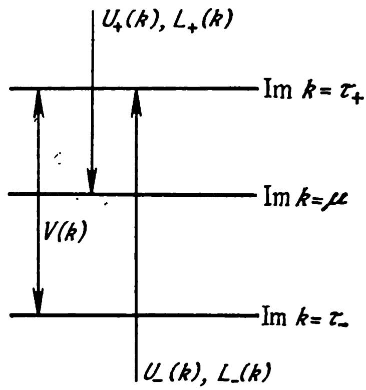


Fig. II.1

analytic functions of the complex variable $k$ for $\text{Im}\, k > \mu$ and $\text{Im}\, k < \tau_+$ respectively (for the sake of definiteness, we put $\mu > \tau_-$ in Fig. II.1).

Let us now solve the integral equation (II-38), or the equivalent equation (II-42). We take advantage of the Fourier transform. Using formula (II-9) of the convolution transformation, the truth of which in the given case of a semi-infinite interval is evident almost immediately, we get, from (II-42),

$$U_+(k) + U_-(k) = \sqrt{2\pi}\, \lambda V(k)\, U_+(k)$$

or

$$L(k)\, U_+(k) + U_-(k) = 0 \tag{II-47}$$

where

$$L(k) = 1 - \sqrt{2\pi}\, \lambda V(k) \tag{II-48}$$

Thus, with the aid of the Fourier transform we have again passed from the original integral equation to an algebraic equation for transforms. However, equation (II-47) now has two unknown func-

* We do not dwell on the proof of the existence of a solution of equation (II-40) having this property. For details, see for example [5].

tions. Generally speaking, two unknown functions cannot be determined uniquely from one algebraic equation. The Wiener-Hopf method permits solving this problem for a definite class of functions. It is primarily associated with a study of the domains of analyticity of the functions entering into the equation and with a special representation of this equation. The basic idea of the Wiener-Hopf technique consists in the following.

Suppose it has been possible to represent equation (II-47) in the form

$$L_+(k)\,U_+(k) = -L_-(k)\,U_-(k) \qquad \text{(II-49)}$$

where the left-hand side is analytic in the upper half-plane $\operatorname{Im} k > \mu$, and the right-hand side is analytic in the lower half-plane $\operatorname{Im} k < \tau_+$; note that $\mu < \tau_+$ so that there is a common strip of analyticity of these functions $\mu < \operatorname{Im} k < \tau_+$. Then, by virtue of the uniqueness of analytic continuation it may be asserted that there exists a unique entire function of the complex variable which coincides with the left-hand side of (II-49) in the upper half-plane and with the right-hand side of (II-49) in the lower half-plane, respectively. If it is also known that the functions entering into (II-49) do not increase at infinity faster than a finite power $k^n$, then by Liouville's theorem the given entire function is determined to within multiplicative constants. In particular, in the case of a function bounded at infinity we get

$$L_+(k)\,U_+(k) = -L_-(k)\,U_-(k) = \text{constant} \qquad \text{(II-50)}$$

Whence the functions $U_+(k)$ and $U_-(k)$ are determined uniquely.

Let us now apply the given scheme to solving equation (II-47). From the reasoning given above it follows that the domains of analyticity of $U_+(k)$, $U_-(k)$ and $L(k) = 1 - \sqrt{2\pi}\,\lambda V(k)$ are, respectively, the upper half-plane $\operatorname{Im} k > \mu$, the lower half-plane $\operatorname{Im} k < \tau_+$ and the strip $\tau_- < \operatorname{Im} k < \tau_+$. This equation thus holds true in the strip* $\mu < \operatorname{Im} k < \tau_+$, which is the common domain of analyticity of all the functions entering into the equation. In order to transform equation (II-47) to the form (II-49), assume that the following decomposition of the function $L(k)$ is possible:

$$L(k) = \frac{L_+(k)}{L_-(k)} \qquad \text{(II-51)}$$

where the functions $L_+(k)$ and $L_-(k)$ are analytic for $\operatorname{Im} k > \mu$ and $\operatorname{Im} k < \tau_+$, respectively. Besides, suppose that in the domains of their analyticity these functions do not increase at infinity faster than $k^n$, where $n$ is some positive integer. The splitting (II-51) of

* For definiteness, put $\mu > \tau_-$, otherwise the strip $\tau_- < \operatorname{Im} k < \tau_+$ will be the common domain of analyticity.

the analytic function $L(k)$ is ordinarily called *factorization.* The possibility of factorization of a given analytic function of a complex variable will be substantiated below (see Lemma 1 and Lemma 2 on pages 293-294).

Thus, as a result of factorization the original equation is reduced to the form

$$L_+(k)\,U_+(k) = -L_-(k)\,U_-(k) \tag{II-49}$$

From earlier arguments it follows that this equation defines some entire function of the complex variable $k$.

Since $U_\pm(k) \to 0$ as $|k| \to \infty$, and $L_\pm(k)$ increase at infinity as a finite power $k^n$, the given entire function may only be a polynomial $P_{n-1}(k)$ of degree no higher than $n-1$.

If the functions $L_\pm(k)$ increase at infinity only as the first power of the variable $k$, then from the relations (II-50), by Liouville's theorem, it follows that the corresponding entire function is the constant $C$. We then get the following expressions for the unknowns $U_+(k)$ and $U_-(k)$:

$$U_+(k) = \frac{C}{L_+(k)}, \quad U_-(k) = -\frac{C}{L_-(k)} \tag{II-52}$$

which define the Fourier transforms of the desired solution to within a constant factor; this factor may be found from the normalization conditions. In the general case, the expressions

$$U_+(k) = \frac{P_n(k)}{L_+(k)}, \quad U_-(k) = -\frac{P_n(k)}{L_-(k)} \tag{II-53}$$

define Fourier transforms of the sought-for solution of the integral equation (II-38) to within undetermined constants that may be found from the supplementary conditions of the problem. The solution itself is determined with the help of the inverse Fourier transform (II-31) and (II-35).

Let us illustrate the use of this method with an example.

*Example 2.* Consider the equation

$$u(x) = \lambda \int_0^\infty e^{-|x-s|} u(s)\, ds \tag{II-54}$$

the kernel of which is of the form $v(\xi) = e^{-|\xi|}$.

We find the Fourier transform of the function $v(\xi)$:

$$V(k) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} v(\xi)\, e^{ik\xi}\, d\xi = \frac{2}{\sqrt{2\pi}\,(k^2+1)} \tag{II-55}$$

The function $V(k)$ of (II-55) is an analytic function of the complex variable $k$ in the strip $-1 < \operatorname{Im} k < 1$. Represent the expression

$$L(k) = 1 - \sqrt{2\pi}\,\lambda V(k) = \frac{k^2-(2\lambda-1)}{k^2+1} \tag{II-56}$$

in the form of (II-51), where

$$L_+(k) = \frac{k^2 - (2\lambda - 1)}{k + i}, \quad L_-(k) = k - i \tag{II-57}$$

The function $L_+(k)$ in (II-57) is an analytic function of $k$ and is different from zero in the domain $\operatorname{Im} k > \operatorname{Im} \sqrt{2\lambda - 1}$. For $0 < \lambda < \frac{1}{2}$, this domain is determined by the condition $\operatorname{Im} k > \sqrt{1 - 2\lambda}$, and $\sqrt{1 - 2\lambda} \leqslant \mu < 1$. For $\lambda > \frac{1}{2}$ the function $L_+(k)$ is analytic and nonzero in the domain $\operatorname{Im} k > 0$. The function $L_-(k)$ is obviously a nonzero analytic function in the domain $\operatorname{Im} k < 1$. Therefore, for $0 < \lambda < \frac{1}{2}$ both functions satisfy the required conditions in the strip $\mu < \operatorname{Im} k < 1$.

For $\frac{1}{2} < \lambda$, the strip $0 < \operatorname{Im} k < 1$ is the common domain of analyticity of the functions $L_+(k)$ and $L_-(k)$. Thus, the necessary factoring of the function (II-56) has been performed.

Consider the expressions $U_\pm(k) L_\pm(k)$. Since $U_\pm(k) \to 0$ as $|k| \to \infty$, and $L_\pm(k)$, according to (II-57), increase at infinity as a first power of $k$, the entire function $P_n(k)$, which coincides with $U_+(k) L_+(k)$ for $\operatorname{Im} k > \mu$ and with $U_-(k) L_-(k)$ for $\operatorname{Im} k < 1$, can only be a polynomial of degree zero. Therefore,

$$U_+(k) L_+(k) = C \tag{II-58}$$

Whence

$$U_+(k) = C \frac{k + i}{k^2 - 2\lambda + 1} \tag{II-59}$$

and, by (II-31),

$$u_+(x) = \frac{C}{\sqrt{2\pi}} \int_{-\infty + i\tau}^{\infty + i\tau} \frac{k + i}{k^2 - (2\lambda - 1)} e^{-ikx}\, dk \tag{II-60}$$

where $\mu < \tau < 1$.

The integral (II-60) may be evaluated by the methods of Chapter 5. Closing the contour of integration for $x > 0$ by a semi-circular arc in the lower half-plane and evaluating the integral along this arc with the aid of the Jordan lemma, we get, after elementary computations,

$$u_+(x) = D \left\{ \cos \sqrt{2\lambda - 1}\, x + \frac{\sin \sqrt{2\lambda - 1}\, x}{\sqrt{2\lambda - 1}} \right\} \tag{II-61}$$

where $D$ is a new constant. For $0 < \lambda < \frac{1}{2}$ this solution grows exponentially with the growth of $x$; for $\frac{1}{2} < \lambda < \infty$, the solution is bounded at infinity.

Thus, the example of solving the homogeneous integral equation (II-38) has already demonstrated the basic idea of the Wiener-Hopf method, which consists in representing (by factoring) the original functional equation (II-47) in the form of the entire function (II-49). Let us now justify the factoring process of an analytic function of a complex variable. We will proceed from a somewhat more general functional equation than (II-47).

## II.4. General Scheme of the Wiener-Hopf Method

In the general case, a problem solvable by the Wiener-Hopf technique reduces to the following.

It is required to determine the functions $\Psi_+(k)$ and $\Psi_-(k)$ of a complex variable $k$, which are analytic respectively in the half-plane $\mathrm{Im}\, k > \tau_-$ and $\mathrm{Im}\, k < \tau_+$ ($\tau_- < \tau_+$) and tend to zero as $|k| \to \infty$ in both domains of analyticity and satisfy in the strip $\tau_- < \mathrm{Im}\, k < \tau_+$ the functional equation

$$A(k)\Psi_+(k) + B(k)\Psi_-(k) + C(k) = 0 \qquad \text{(II-62)}$$

Here, $A(k)$, $B(k)$, $C(k)$ are the given functions of the complex variable $k$, analytic in the strip $\tau_- < \mathrm{Im}\, k < \tau_+$; $A(k)$ and $B(k)$ are nonzero in this strip.

The main idea for the solution of this problem is based on the possibility of factoring the expression $A(k)/B(k)$, i.e. the possibility of representing it in the form

$$\frac{A(k)}{B(k)} = \frac{L_+(k)}{L_-(k)} \qquad \text{(II-63)}$$

where the functions $L_+(k)$ and $L_-(k)$ are analytic and different from zero, respectively, in the half-planes $\mathrm{Im}\, k > \tau'_-$ and $\mathrm{Im}\, k < \tau'_+$, the strips $\tau_- < \mathrm{Im}\, k < \tau_+$ and $\tau'_- < \mathrm{Im}\, k < \tau'_+$ having a common portion. Then, using (II-63), equation (II-62) may be rewritten as

$$L_+(k)\Psi_+(k) + L_-(k)\Psi_-(k) + L_-(k)\frac{C(k)}{B(k)} = 0 \qquad \text{(II-64)}$$

If the last term in (II-64) may be rewritten as

$$L_-(k)\frac{C(k)}{B(k)} = D_+(k) + D_-(k) \qquad \text{(II-65)}$$

where the functions $D_+(k)$ and $D_-(k)$ are analytic in the half-planes $\mathrm{Im}\, k > \tau''_-$ and $\mathrm{Im}\, k < \tau''_+$, respectively, and all three strips $\tau_- < \mathrm{Im}\, k < \tau_+$, $\tau'_- < \mathrm{Im}\, k < \tau'_+$ and $\tau''_- < \mathrm{Im}\, k < \tau''_+$ have a common portion—the strip $\tau^0_- < \mathrm{Im}\, k < \tau^0_+$—then in this strip the following functional equation holds true:

$$L_+(k)\Psi_+(k) + D_+(k) = -L_-(k)\Psi_-(k) - D_-(k) \qquad \text{(II-66)}$$

The left-hand side of (II-66) is a function analytic in the half-plane $\tau_-^0 < \operatorname{Im} k$, the right-hand side is a function analytic in the domain $\operatorname{Im} k < \tau_+^0$. From the equality of these functions in the strip $\tau_-^0 < \operatorname{Im} k < \tau_+^0$ it follows that there exists a unique entire function $P(k)$ coinciding, respectively, with the left and right sides of (II-66) in the domains of their analyticity. If all functions entering into the right sides of (II-63) and (II-65) increase at infinity in their domains of analyticity no faster than $k^{n+1}$, then from the condition $\Psi_\pm(k) \to 0$ as $|k| \to \infty$ it follows that $P(k)$ is a polynomial $P_n(k)$ of degree not higher than $n$. In this way, the equalities

$$\Psi_+(k) = \frac{P_n(k) - D_+(k)}{L_+(k)} \tag{II-67}$$

and

$$\Psi_-(k) = \frac{-P_n(k) - D_-(k)}{L_-(k)} \tag{II-68}$$

define the desired functions to within constants. The constants may be found from the supplementary conditions of the problem.

Thus, the use of the Wiener-Hopf method is based on the representations (II-63) and (II-65). The possibility of these representations is guaranteed by the following lemmas.

***Lemma 1.*** *Let a function $F(k)$ be analytic in the strip $\tau_- < \operatorname{Im} k < \tau_+$ and let $F(k)$, in this strip, tend uniformly to zero as $|k| \to \infty$. Then the following representation is possible in the given strip:*

$$F(k) = F_+(k) + F_-(k) \tag{II-69}$$

*where the function $F_+(k)$ is analytic in the half-plane $\operatorname{Im} k > \tau_-$ and the function $F_-(k)$ is analytic in the half-plane $\operatorname{Im} k < \tau_+$.*

*Proof.* Consider an arbitrary point $k_0$ lying in the given strip and construct a rectangle $abcd$ containing the point $k_0$ and bounded by the straight-line segments $\operatorname{Im} k = \tau'_-$, $\operatorname{Im} k = \tau'_+$, $\operatorname{Re} k = -A$, $\operatorname{Re} k = A$, where $\tau_- < \tau'_- < \tau'_+ < \tau_+$ (Fig. II.2). By the Cauchy formula

$$F(k_0) = \frac{1}{2\pi i}\int_{-A+i\tau'_-}^{A+i\tau'_-} \frac{F(\zeta)}{\zeta - k_0}\,d\zeta + \frac{1}{2\pi i}\int_{A+i\tau'_-}^{A+i\tau'_+} \frac{F(\zeta)}{\zeta - k_0}\,d\zeta + \frac{1}{2\pi i}\int_{A+i\tau'_+}^{-A+i\tau'_+} \frac{F(\zeta)}{\zeta - k_0}\,d\zeta + \frac{1}{2\pi i}\int_{-A+i\tau'_+}^{-A+i\tau'_-} \frac{F(\zeta)}{\zeta - k_0}\,d\zeta \tag{II-70}$$

In (II-70) proceed to the limit as $A \to \infty$. Since it is given that $F(k)$ tends uniformly to zero as $|k| \to \infty$, the limit of the second and fourth terms on the right of (II-70) is zero, and we obtain

$$F(k_0) = F_+(k_0) + F_-(k_0) \tag{II-71}$$

where

$$F_+(k_0) = \frac{1}{2\pi i} \int_{-\infty+i\tau'_-}^{\infty+i\tau'_-} \frac{F(\zeta)}{\zeta - k_0} d\zeta \qquad \text{(II-72)}$$

$$F_-(k_0) = -\frac{1}{2\pi i} \int_{-\infty+i\tau'_+}^{\infty+i\tau'_+} \frac{F(\zeta)}{\zeta - k_0} d\zeta \qquad \text{(II-73)}$$

As integrals dependent on a parameter,* the integrals (II-72) and (II-73) define analytic functions of the complex variable $k_0$, provided that the point $k_0$ does not lie on the contour of integration.

In particular, $F_+(k_0)$ is an analytic function in the half-plane $\text{Im}\, k_0 > \tau'_-$, and $F_-(k_0)$ is an analytic function in the half-plane

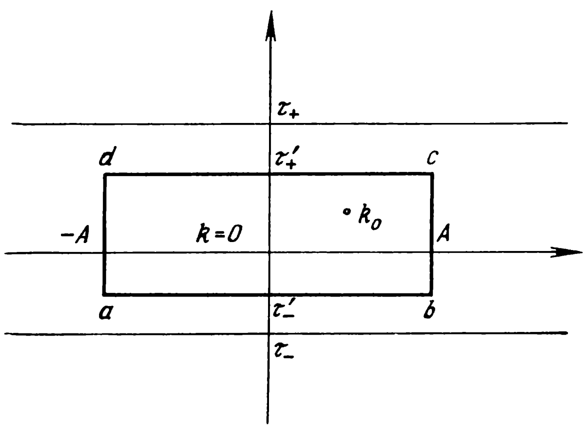


Fig. II.2

$\text{Im}\, k_0 < \tau'_+$. By virtue of the arbitrariness of choice of point $k_0$ and the straight lines $\tau'_-$ and $\tau'_+$, the relations (II-71) to (II-73) prove the lemma.

*Note 1.* Note that from the convergence of the integrals (II-72) and (II-73) it follows that the functions $F_+(k)$ and $F_-(k)$ thus constructed are bounded in the given strip as $|k| \to \infty$.

***Lemma 2.*** *Let a function* $\Phi(k)$ *be analytic and nonzero in a strip* $\tau_- < \text{Im}\, k < \tau_+$, *and let* $\Phi(k)$ *tend uniformly to unity in this strip as* $|k| \to \infty$. *Then in the given strip we will have the representation*

$$\Phi(k) = \Phi_+(k) \cdot \Phi_-(k) \qquad \text{(II-74)}$$

*where the functions* $\Phi_+(k)$ *and* $\Phi_-(k)$ *are analytic and different from zero in the half-planes* $\text{Im}\, k > \tau_-$ *and* $\text{Im}\, k < \tau_+$ *respectively.*

---

* See page 53.

*Proof.* Consider the function $F(k) = \ln \Phi(k)$ which clearly satisfies all the conditions of Lemma 1. Thus, for the function $F(k)$ we can have the representation (II-71)-(II-73). Putting

$$\Phi_+(k) = \exp\{F_+(k)\}, \quad \Phi_-(k) = \exp\{F_-(k)\} \tag{II-75}$$

where the functions $F_+(k)$ and $F_-(k)$ are defined by formulas (II-72), (II-73), we get

$$\ln \Phi_+(k) = F_+(k), \; \ln \Phi_-(k) = F_-(k) \tag{II-76}$$

Then formula (II-71) yields

$$\ln \Phi(k) = \ln \Phi_+(k) + \ln \Phi_-(k) \tag{II-77}$$

whence follows relation (II-74). Since, by Lemma 1 the functions $F_+(k)$ and $F_-(k)$ are analytic in the half-planes $\operatorname{Im} k > \tau_-$ and $\operatorname{Im} k < \tau_+$, respectively, so also the functions $\Phi_+(k)$ and $\Phi_-(k)$ defined by formulas (II-75) will have the required properties. That proves the lemma.

*Note 2.* The possibility of factoring (II-74) holds true when the function $\Phi(k)$ has a finite number of zeros $k_i$ in the strip $\tau_- < \operatorname{Im} k < \tau_+$.

To prove Lemma 2, in this case it suffices to introduce the auxiliary function

$$F(k) = \ln\left[\frac{(k^2+b^2)^{N/2}}{\prod\limits_i (k-k_i)^{\alpha_i}} \Phi(k)\right] \tag{II-78}$$

where $\alpha_i$ are the multiplicities of the zeros $k_i$; $N$ is the total number of zeros counting multiplicities; the positive constant $b > |\tau_-|$, $|\tau_+|$ is chosen from the condition that the function under the sign of the logarithm should not have additional zeros in the strip $\tau_- < \operatorname{Im} k < \tau_+$. This function clearly tends to unity at infinity. The function $F(k)$ thus constructed continues to satisfy all the conditions of Lemma 1.

The lemmas just proved determine the possibility of the representations (II-63), (II-65), which form the basis of the Wiener-Hopf method.

We considered the Wiener-Hopf method used for solving the functional equation (II-62). It is easy to see that the following *nonhomogeneous integral equation* with a difference kernel on a semi-infinite interval reduces to this equation:

$$u(x) = \lambda \int\limits_0^\infty v(x-s)\, u(s)\, ds + f(x) \tag{II-79}$$

We assume that the kernel of (II-79) and the function $f(x)$ satisfy the conditions (II-43) and we will seek a solution of (II-79) that satisfies the condition*

$$| u_+ (x) | < M_1 e^{\mu x} \text{ as } x \to \infty \quad (\mu < \tau_+) \tag{II-80}$$

Then with arguments similar to those involved in the derivation of the functional equation (II-47) for a homogeneous integral equation, we find that in the case of (II-79) the following functional equation should be satisfied in the strip $\mu < \text{Im } k < \tau_+$:

$$U_+ (k) + U_- (k) = \lambda \sqrt{2\pi} V (k) U_+ (k) + F_+ (k) + F_- (k) \tag{II-81}$$

or

$$L (k) U_+ (k) + U_- (k) - F (k) = 0 \tag{II-82}$$

where

$$L (k) = 1 - \sqrt{2\pi} \lambda V (k) \tag{II-83}$$

Equation (II-82) is a special case of equation (II-62). In the strip $\tau_- < \text{Im } k < \tau_+$ the function $L (k)$ is analytic and uniformly tends to unity as $| k | \to \infty$, since $| V (k) | \to 0$ as $| k | \to \infty$. If, besides, the function $V (k)$ has a finite number of zeros in this strip, then all the conditions of Lemma 2 are fulfilled and the function $L (k)$ may be represented in the form

$$L (k) = \frac{L_+ (k)}{L_- (k)} \tag{II-84}$$

where $L_+ (k)$ is an analytic function in the upper half-plane $\text{Im } k > \tau_-$, and $L_- (k)$ is an analytic function in the lower half-plane $\text{Im } k < \tau_+$. Then equation (II-82) takes the form

$$L_+ (k) U_+ (k) + L_- (k) U_- (k) - L_- (k) F_- (k) - F_+ (k) L_- (k) = 0 \tag{II-85}$$

To reduce this equation to the form of (II-66), it is sufficient to split up the last term:

$$F_+ (k) L_- (k) = D_+ (k) + D_- (k) \tag{II-86}$$

into the sum of the functions $D_+ (k)$ and $D_- (k)$ which are analytic in the half-planes $\text{Im } k > \mu$ and $\text{Im } k < \tau_+$, respectively.

To justify the possibility of such a representation, note that by condition (II-43) the function $F_+ (k)$ is analytic in the upper half-plane $\text{Im } k > \tau_-$ and tends uniformly to zero as $| k | \to \infty$. The function $L_- (k)$ is analytic in the lower half-plane $\text{Im } k < \tau_+$ and, by the mode of its construction, by virtue of Lemma 2 and the note referring to Lemma 1, it is possible to factor (II-84) in such a manner that $L_- (k)$ will remain bounded in the strip $\tau_- < \text{Im } k < \tau_+$ as $| k | \to \infty$. Whence it follows that all the conditions of Lemma 1

* Again, we dispense with justifying the existence of a solution of equation (II-79) that satisfies the condition (II-80).

are fulfilled for the function $F_+(k)\,L_-(k)$ in the strip $\tau_- < \operatorname{Im} k < \tau_+$, which is sufficient to substantiate the representation (II-86).

The foregoing considerations enable one, given the supplementary conditions that the functions $L_\pm(k)$ do not increase at infinity faster than $k^n$, to represent the Fourier transforms of the solution of the nonhomogeneous integral equation (II-79) in the form

$$U_+(k) = \frac{P_n(k) + D_+(k)}{L_+(k)},$$
$$U_-(k) = \frac{-P_n(k) + L_-(k)\,F_-(k) + D_-(k)}{L_-(k)}. \tag{II-87}$$

The solution itself may be obtained from (II-87) by means of formulas (II-31) and (II-35) of the inverse Fourier transform.

## II.5. Problems Which Reduce to Integral Equations with a Difference Kernel

### *a. Derivation of Milne's equation*

A large number of physical problems reduce to integral equations with a difference kernel. As a first instance, we take the classical Milne problem which describes the process of neutron (or radiative) diffusion (transport) through a substance.

Let there be a flux of neutrons in the half-space $x > 0$ filled with a homogeneous substance whose density is defined by the number $n_0$ of particles per unit volume. We consider the particles to be heavy atoms that scatter neutrons so that the absolute magnitude of neutron velocity remains constant and only the direction varies. We consider a steady-state process and assume that all neutrons have the same absolute magnitude of velocity $v_0 = 1$ and their distribution density depends solely on the coordinate $x$. We introduce a function $f(x, \mu)$ that characterizes the neutron density in the cross section $x$, the velocity of neutrons forming with the positive direction of the $x$-axis an angle $\theta$ where $\mu = \cos\theta$.* The number of neutrons in unit volume in a given cross section, the direction of velocity of which lies within the limits $(\mu, \mu + d\mu)$, is determined by the quantity $f(x, \mu)\,d\mu$.

The total neutron density $\rho(x)$ in a given section is

$$\rho(x) = \int_{-1}^{1} f(x, \mu)\,d\mu \tag{II-88}$$

Our immediate aim is to derive an equation for the distribution function $f(x, \mu)$. To do this, form the relation of the total balance

* It is obvious that for $0 \leqslant \theta \leqslant \pi$ we have $-1 \leqslant \mu \leqslant 1$.

of the number of neutrons having direction of velocity in the interval $(\mu, \mu + d\mu)$ and lying in the layer between the sections $x$ and $x + dx$. Due to the steady-state nature of the process, the flux of neutrons emerging from the given layer

$$\mu f(x + dx, \mu)\, d\mu - \mu f(x, \mu)\, d\mu \tag{II-89}$$

is determined by the difference between the number of neutrons that have acquired velocity in the given direction $(\mu, \mu + d\mu)$ as a result of scattering on particles of the substance in the given layer and the number of neutrons that had velocity in the given direction and changed this direction after scattering. We take it that the scattering of neutrons on particles of substance is isotropic (equally probable in all directions) and the probability of the scattering of a neutron on one particle is described by the effective scattering cross section $Q$. Then it is clear that the number of neutrons that had a given direction of velocity $(\mu, \mu + d\mu)$ and were scattered in a given layer is equal to

$$f(x, \mu)\, d\mu \cdot Q n_0\, dx \tag{II-90}$$

while the number of neutrons that acquired velocity in the required direction as a result of scattering is

$$\frac{1}{2}\, d\mu Q \cdot n_0\, dx \int_{-1}^{1} f(x, \mu')\, d\mu' \tag{II-91}$$

On the basis of (II-89), (II-90) and (II-91), the equilibrium equation is then written in the form

$$\mu f(x + dx, \mu)\, d\mu - \mu f(x, \mu)\, d\mu = -Q \cdot n_0 f(x, \mu)\, d\mu\, dx + \frac{Q n_0}{2}\, d\mu\, dx \int_{-1}^{1} f(x, \mu')\, d\mu' \tag{II-92}$$

Divide both sides by $d\mu\, dx$ and proceed to the limit as $dx \to 0$. Taking into account (II-88), we get an equation for the function of neutron distribution in the form

$$\mu \frac{\partial f}{\partial x} = -Q n_0 f(x, \mu) + \frac{Q n_0}{2} \rho(x) \tag{II-93}$$

This equation is frequently called the transport equation. It holds true not only in the case of the above-considered specific physical problem, but also for many other physical processes associated with the transport of matter or radiation.*

For what follows it will be more convenient to rewrite equation (II-93) in a somewhat different form, introducing a dimensionless

* For a detailed derivation of the transport equation for more general cases see, for example, [11].

spatial coordinate $\xi$ connected with $x$ by the relation $x = \lambda\xi$, where $\lambda = \frac{1}{nQ_0}$ is the mean free path. Then the transport equation takes the form*

$$\mu \frac{\partial f}{\partial \xi} = -f(\xi, \mu) + \frac{1}{2}\rho(\xi) \tag{II-94}$$

The function $f(\xi, \mu)$ must be subject to the boundary conditions that follow from the physical statement of the problem. We will assume that the flux of neutrons from the exterior half-space $\xi < 0$ is zero and as $\xi \to \infty$ there is a constant neutron flux of unit intensity in the negative direction of the $\xi$-axis (i.e. as $\xi \to \infty$ there are no neutrons whose direction of velocity forms an acute nonzero angle with the negative $\xi$-axis). Then the boundary conditions for the function $f(\xi, \mu)$ will be written in the form

$$\begin{aligned} f(0, \mu) &= 0, \qquad \mu \geqslant 0 \\ f(\infty, \mu) &= 0, \qquad -1 < \mu < 0 \end{aligned} \tag{II-95}$$

Let us establish important consequences of equation (II-94) and conditions (II-95). To do this, first integrate (II-94) with respect to $\mu$:

$$\frac{\partial}{\partial \xi}\int_{-1}^{1} f(\xi, \mu)\,\mu\,d\mu = -\int_{-1}^{1} f(\xi, \mu)\,d\mu + \frac{1}{2}\rho(\xi)\int_{-1}^{1} d\mu$$

$$= -\rho(\xi) + \rho(\xi) = 0 \tag{II-96}$$

Since the integral $j(\xi) = \int_{-1}^{1} f(\xi, \mu)\,\mu\,d\mu$ is equal to the neutron flux through a given cross section, equation (II-96) yields

$$\frac{\partial j}{\partial \xi} = 0 \quad \text{or} \quad j(\xi) \equiv \text{constant} \tag{II-97}$$

Due to the normalization conditions (as $\xi \to \infty$) we get $j(\xi) = -1$ (unit flux as $\xi \to +\infty$ is in the negative direction of the $\xi$-axis).

Now multiply (II-94) by $\mu$ and again integrate from $-1$ to $1$. Introducing the notation $K(\xi) = \int_{-1}^{1} f(\xi, \mu)\,\mu^2\,d\mu$ we have

$$\frac{\partial K}{\partial \xi} = 1 \quad \text{or} \quad K(\xi) = K(0) + \xi \tag{II-98}$$

where, by (II-95),

$$K(0) = \int_{-1}^{0} f(0, \mu)\,\mu^2\,d\mu \tag{II-99}$$

* For the function $f(\xi, \mu)$ we retained the old designation.

Equation (II-94) is an integro-differential equation, since the unknown functions $\rho(\xi)$ and $f(\xi, \mu)$ are connected by the integral relation (II-88). However, it is easy to get an integral equation for the function $\rho(\xi)$. Solving the ordinary differential equation (II-94) for the function $f(\xi, \mu)$, we get, by (II-95),

$$f(\xi, \mu) = \frac{1}{2\mu} \begin{cases} \int\limits_0^{\xi} e^{\frac{\eta-\xi}{\mu}} \rho(\eta)\, d\eta, & \mu > 0 \\ -\int\limits_{\xi}^{\infty} e^{\frac{\eta-\xi}{\mu}} \rho(\eta)\, d\eta, & \mu < 0 \end{cases} \tag{II-100}$$

Integrating (II-100) with respect to $\mu$ from $-1$ to $1$, we get an integral equation for the function $\rho(\xi)$:

$$\rho(\xi) = \frac{1}{2} \int\limits_0^1 \int\limits_0^{\infty} \rho(\eta)\, e^{-\frac{|\xi-\eta|}{\mu}}\, d\eta \frac{d\mu}{\mu} \tag{II-101}$$

Changing the order of integration in (II-101), we get the final equation for the neutron density in the cross section $\xi$:

$$\rho(\xi) = \int\limits_0^{\infty} v(\xi-\eta)\, \rho(\eta)\, d\eta \tag{II-102}$$

This will be seen to be an integral equation with a difference kernel in a semi-infinite interval:

$$v(\xi-\eta) = \frac{1}{2} \int\limits_0^1 e^{-\frac{|\xi-\eta|}{\mu}} \frac{d\mu}{\mu} \tag{II-103}$$

Equation (II-102) is ordinarily called Milne's equation, after E. Milne who first derived it in studies of the processes of radiative transport in stellar atmospheres.

Observe that in many cases it is convenient to give a somewhat different representation of the kernel that results from the change of integration variable $\mu = \frac{1}{\nu}$ in the integral $X(t) = \int\limits_0^1 e^{-\frac{|t|}{\mu}} \frac{d\mu}{\mu}$. Then

$$X(t) = \int\limits_1^{\infty} e^{-|t|\nu} \frac{d\nu}{\nu} \tag{II-104}$$

The integral (II-104) is often called the Hopf function. Integration by parts readily yields its asymptotic expansion for large positive values of $t$:

$$X(t) = \frac{e^{-t}}{t} \left\{ 1 - \frac{1}{t} + \frac{2!}{t^2} - \frac{3!}{t^3} + \dots \right\} \tag{II-105}$$

### b. Investigating the solution of Milne's equation

Equation (II-102) belongs to the type of equations considered in Section II.3 and it can be solved by the general algorithm of the Wiener-Hopf method. We will not go into the detailed solution of this equation and an investigation of its physical meaning,* but will confine ourselves to a few remarks.

In many problems of a practical nature the main interest lies only in determining the distribution function of neutrons emanating from a given medium, i.e. the function $f(0, \mu)$ for $\mu < 0$. According to (II-100), this function is defined by the expression

$$f(0, \mu) = -\frac{1}{2\mu}\int_0^\infty e^{\frac{\eta}{\mu}}\rho(\eta)\,d\eta = \frac{1}{2|\mu|}\int_0^\infty e^{\frac{-\eta}{|\mu|}}\rho(\eta)\,d\eta, \quad \mu < 0 \quad \text{(II-106)}$$

As is readily evident, by virtue of (II-29), the last integral is nothing other than a one-sided Fourier transform of the function $\rho(\eta)$ for $k = \frac{i}{|\mu|}$, i.e.

$$f(0, \mu) = \frac{1}{2|\mu|}\sqrt{2\pi}\,R_+\left(\frac{i}{|\mu|}\right) \quad \text{(II-107)}$$

Thus, in such problems it is enough to find the Fourier transform of the solution, not the solution itself, of the integral equation (II-102).

According to the general scheme of the Wiener-Hopf method, to solve this problem, one has to find the Fourier transform of the kernel of the integral equation, and then perform the factorization (II-51) of the function $L(k) = 1 - \sqrt{2\pi}\,\lambda V(k)$. In our case, $\lambda = 1$ and

$$V(k) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{ikx}v(x)\,dx = \frac{1}{2\sqrt{2\pi}}\left\{\int_{-\infty}^{0} e^{ikx}dx\int_0^1 e^{\frac{x}{\mu}}\frac{d\mu}{\mu}\right.$$

$$\left. + \int_0^\infty e^{ikx}dx\int_0^1 e^{-\frac{x}{\mu}}\frac{d\mu}{\mu}\right\} = \frac{1}{\sqrt{2\pi}}\int_0^1 \frac{1}{k^2+\frac{1}{\mu^2}}\cdot\frac{d\mu}{\mu^2} = \frac{1}{\sqrt{2\pi}}\cdot\frac{\arctan k}{k}$$

$$= \frac{1}{\sqrt{2\pi}}\cdot\frac{1}{2ik}\ln\frac{1+ik}{1-ik} \quad \text{(II-108)}$$

Therefore

$$L(k) = 1 - \sqrt{2\pi}\,\lambda V(k) = \frac{k - \arctan k}{k} \quad \text{(II-109)}$$

The function $L(k)$ is clearly analytic in the strip $-1 < \operatorname{Im} k < 1$ tending to zero in this strip as $|k| \to \infty$. The point $k = 0$ is a second-order zero of this function. This latter circumstance somewhat complicates factoring the function $L(k)$.

* For a detailed discussion, see [7].

In accordance with Note 2 (page 295), construct the auxiliary function $\frac{k^2+1}{k^2} L(k)$ to satisfy all the conditions of Lemma 2, and consider the function

$$\Phi(k) = \ln\left[\frac{k^2+1}{k^2} L(k)\right] = \ln\left[\frac{k^2+1}{k^2}\left(1 - \frac{\arctan k}{k}\right)\right] \qquad \text{(II-110)}$$

which can be readily represented in the form $\Phi(k) = \Phi_-(k) + \Phi_+(k)$, where the functions $\Phi_-(k)$ and $\Phi_+(k)$ are analytic in the lower $\operatorname{Im} k < \tau_+ < 1$ and upper $\operatorname{Im} k > \tau_- > -1$ half-planes respectively. Then

$$\Phi_+(k) = \frac{1}{2\pi i} \int_{-\infty + i\tau_-}^{\infty + i\tau_-} \Phi(\zeta) \frac{d\zeta}{\zeta - k} \qquad \text{(II-110')}$$

and the function $L_+(k)$, which is the numerator in the factorization formula (II-51) of the function $L(k)$:

$$L(k) = \frac{L_+(k)}{L_-(k)}$$

can be chosen in the form

$$L_+(k) = \frac{k^2}{k+i} e^{\Phi_+(k)} \qquad \text{(II-111)}$$

The function $L_+(k)$ is analytic in the upper half-plane $\operatorname{Im} k > \tau_-$ and, as $|k| \to \infty$, increases as the first power of $k$, since, due to the convergence of integral (II-110'), $\Phi_+(k)$ is bounded as $|k| \to \infty$. Therefore, the function $R_+(k)$ is determined from formula (II-52):

$$R_+(k) = \frac{A}{L_+(k)} = A \frac{k+i}{k^2} e^{-\Phi_+(k)} \qquad \text{(II-112)}$$

From this it follows that when determining the distribution function of neutrons emanating from the half-space $x > 0$, it is necessary to find $\Phi_+(k)$. This can be done with the aid of formula (II-110'). To compute this integral, put $\tau_- = 0$ and reduce it to the following form:

$$\Phi_+(k) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \Phi(\zeta) \frac{d\zeta}{\zeta - k}$$

$$= \frac{1}{2\pi i} \left\{ \int_{-\infty}^{0} \Phi(\zeta') \frac{d\zeta'}{\zeta' - k} + \int_{0}^{\infty} \Phi(\zeta) \frac{d\zeta}{\zeta - k} \right\} \qquad \text{(II-113)}$$

Taking advantage of the evenness of the function $\Phi(\zeta)$ and changing the integration variable $\zeta' = -\zeta$ in the first integral, we finally get

$$\Phi_+(k) = \frac{k}{\pi i}\int_0^\infty \Phi(\zeta)\,\frac{d\zeta}{\zeta^2 - k^2} \tag{II-114}$$

The last integral can easily be tabulated, and this enables us to find $f(0, \mu)$ for $\mu < 0$ to within the constant factor $A$. To determine $A$, take advantage of the normalization condition (II-97) and the following reasoning. Multiply equation (II-94), which holds for $\xi > 0$, by $\frac{1}{\sqrt{2\pi}}e^{ik\xi}$ and integrate it with respect to $\xi$ from 0 to $\infty$. Let $k$ be a complex quantity with small positive imaginary part. Then, using the formula of integration by parts

$$\frac{1}{\sqrt{2\pi}}\int_0^\infty e^{ik\xi}\frac{\partial f}{\partial \xi}\,d\xi = -\frac{1}{\sqrt{2\pi}}f(0, \mu) - ikF_+(k, \mu) \tag{II-115}$$

we get

$$-ik\mu F_+(k, \mu) - \frac{\mu}{\sqrt{2\pi}}f(0, \mu) = -F_+(k, \mu) + \frac{1}{2}R_+(k) \tag{II-116}$$

or

$$F_+(k, \mu) = \left[\frac{\mu}{\sqrt{2\pi}}f(0, \mu) + \frac{1}{2}R_+(k)\right]\frac{1}{1 - ik\mu} \tag{II-117}$$

The integration of (II-117) from $-1$ to 1 with respect to $\mu$ due to the obvious relation $R_+(k) = \int_{-1}^{1} F_+(k, \mu)\,d\mu$ and to condition (II-95), yields

$$R_+(k) = \left(1 - \frac{1}{2}\int_{-1}^{1}\frac{d\mu}{1 - ik\mu}\right)^{-1}\frac{1}{\sqrt{2\pi}}\int_{-1}^{0}\frac{f(0, \mu)}{1 - ik\mu}\,\mu\,d\mu \tag{II-118}$$

Since

$$\frac{1}{2}\int_{-1}^{1}\frac{d\mu}{1 - ik\mu} = \frac{1}{2ik}\ln\frac{1 + ik}{1 - ik} = \frac{\arctan k}{k} \tag{II-119}$$

we finally get

$$R_+(k) = \frac{1}{\sqrt{2\pi}}\left(1 - \frac{\arctan k}{k}\right)^{-1}\int_{-1}^{0}\frac{f(0, \mu)}{1 - ik\mu}\mu\,d\mu \tag{II-120}$$

Expand the right-hand side of (120) in a Laurent series in the neighbourhood of the point $k = 0$. Taking advantage of the equality*

$$\int_{-1}^{0} f(0, \mu)\, \mu\, d\mu = \int_{-1}^{1} f(0, \mu)\, \mu\, d\mu = j(0) = -1 \qquad \text{(II-121)}$$

and the earlier introduced notation (II-99), we obtain

$$R_{+}(k) = -\frac{3}{\sqrt{2\pi}} \frac{1}{k^2} \{1 + iK(0) \cdot k + \ldots\} \qquad \text{(II-122)}$$

On the other hand, it is possible to find the first terms of the Laurent-series expansion, about the point $k = 0$, of the function on the right side of formula (II-112).

First compute $\Phi_{+}(0)$. Take formula (II-110′) and choose for the path of integration the real axis with traversal of the point $\zeta = 0$ along a semicircular arc in the lower half-plane. Allowing the radius of this semicircle to approach zero and taking into account that by virtue of the oddness of the integrand function the integral over portions of the real axis is zero, we obtain

$$\Phi_{+}(0) = \frac{1}{2} \lim_{\zeta \to 0} \ln \left[ \frac{\zeta^2 + 1}{\zeta^2} \left( 1 - \frac{\arctan \zeta}{\zeta} \right) \right] = \ln \frac{1}{\sqrt{3}} \qquad \text{(II-123)}$$

Utilizing (II-123) we find that the expansion of the function $R_{+}(k)$ about the point $k = 0$ is of the form

$$R_{+}(k) = Ai\sqrt{3} \frac{1}{k^2} \{1 + iCk + \ldots\} \qquad \text{(II-124)}$$

Comparing (II-122) and (II-124) we determine the value of the constant $A$:

$$A = \frac{\sqrt{3}}{\sqrt{2\pi}}\, i \qquad \text{(II-125)}$$

Putting the results obtained into the formulas (II-107), (II-112), (II-114), we finally get, for $\mu < 0$

$$f(0, \mu) = \frac{\sqrt{3}}{2} (1 + |\mu|) \exp \frac{\mu}{\pi} \int_{0}^{\infty} \ln \left[ \frac{\zeta^2 + 1}{\zeta^2} \left( 1 - \frac{\arctan \zeta}{\zeta} \right) \right] \frac{d\zeta}{\mu^2 \zeta^2 + 1} \qquad \text{(II-126)}$$

This yields the function of angular distribution of neutrons emanating from the half-space $x > 0$.

---

* Equality (II-121) is valid by virtue of (II-95) and the normalization condition (II-97).

### *c. Diffraction on a flat screen*

The integral equations considered so far are Fredholm equations of the second kind. However, a number of physical problems naturally lead to integral equations of the first kind with a difference kernel in a semi-infinite interval. As an example, let us consider the problem of diffraction of electromagnetic waves on a flat screen. Let there be placed in a homogeneous space a flat, perfectly conducting screen coinciding with the half-plane $x > 0$, $y = 0$, $-\infty < z < \infty$. Outside this screen let there be located local sources of an electromagnetic field that generate periodic electromagnetic oscillations of frequency $\omega$ polarized so that the vector of electric-field intensity **E** is parallel to the $z$-axis and is independent of the coordinate $z$. Then for the amplitude $u(x, y)$ of the vector **E** we get the scalar problem

$$\begin{aligned} \Delta u + k^2 u &= -f(x, y) \\ u(x, 0) &= 0, \quad x > 0 \end{aligned} \qquad \text{(II-127)}$$

Besides, the function $u(x, y)$ must satisfy the conditions of radiation at infinity; these conditions determine the absence of waves arriving from infinity.* Here, $k = \frac{\omega}{c}$ is the wave number ($c$ is the velocity of light in the medium exterior to the screen), $f(x, y)$ is a given function defining the density of the sources. We will seek a solution of the problem (II-127) in the form $u(x, y) = u_0(x, y) + v(x, y)$, where the function $u_0(x, y)$ is the field generated by the given sources in the absence of a screen; this field is expressed in terms of the function $f(x, y)$ in the form of the wave potential**

$$u_0(x, y) = \frac{i}{4} \iint_S H_0^{(1)}(kr) f(\xi, \eta)\, d\xi\, d\eta \qquad \text{(II-128)}$$

where $H_0^{(1)}(z)$ is Hankel's function of the first kind, $r = [(x - \xi)^2 + (y - \eta^2]^{1/2}$ and the integration is carried out over the entire domain $S$ in which the sources are located. For the function $v(x, y)$ we get the problem

$$\Delta v + k^2 v = 0 \qquad \text{(II-129)}$$

$$v(x, 0) = -u_0(x, 0) \qquad x > 0$$

Besides, $v(x, y)$ must satisfy the radiation conditions at infinity. We seek the solution of the problem (II-129) in the form of the wave potential of a simple layer

$$v(x, y) = \int_0^\infty H_0^{(1)}(kr') \mu(\xi)\, d\xi \qquad \text{(II-130)}$$

---

* For details on the statement of diffraction problems see [17].

** Ibid., for the definition and properties of wave potentials.

where $r' = [(x - \xi)^2 + y^2]^{1/2}$ and $\mu(\xi)$ is the unknown density, for the determination of which, using the boundary condition $x > 0$ for $y = 0$, we get an integral equation of the first kind:

$$\int_0^\infty H_0^{(1)}(k|x-\xi|)\,\mu(\xi)\,d\xi = -u_0(x, 0), \quad x > 0 \qquad \text{(II-131)}$$

Again we have a nonhomogeneous integral equation with a difference kernel in a semi-infinite interval. However, unlike (II-79), this is now an equation of the first kind. This equation can also be solved by means of the Wiener-Hopf method, but we will not go into the details of this investigation.

## II.6. Solving Boundary-Value Problems for Partial Differential Equations by the Wiener-Hopf Method

The Wiener-Hopf method may be used effectively not only for solving integral equations but also for solving boundary-value problems for partial differential equations. The specific form of employing this method may differ somewhat from the foregoing, although the general idea involved in factoring expressions of the type (II-63), (II-65) always forms the basis of the method. A typical example is the following boundary-value problem for the Laplace equation.

*Example 3.* In the upper half-plane $y > 0$, find a harmonic function that satisfies, for $y = 0$, the mixed boundary conditions

$$u(x, 0) = e^{-ax}, \ a > 0, \ x > 0 \qquad \text{(II-132)}$$

$$\frac{\partial u}{\partial y}(x, 0) = 0, \ x < 0 \qquad \text{(II-133)}$$

and tends to zero as $y \to \infty$.

To solve this problem we employ a device that is frequently used in mathematical physics. First we solve the boundary-value problem (II-132), (II-133) for the equation

$$\Delta u - \varkappa^2 u = 0 \qquad \text{(II-134)}$$

where $\varkappa^2 = i\nu_0$, $\nu_0 > 0$, and then we proceed to the limit, as $\varkappa \to 0$, in the formulas obtained. Using the method of separation of variables (see [17]) it is easy to obtain the integral representation of the general solution of equation (II-134), which solution decreases as $y \to \infty$, in the form

$$u(x, y) = \int_{-\infty}^{\infty} f(k)\,e^{-\mu y}e^{ikx}\,dk \qquad \text{(II-135)}$$

where $f(k)$ is an arbitrary function of the parameter $k$, and $\mu = \sqrt{k^2 + \varkappa^2}$ is that branch of the root being taken which is an immediate analytic continuation of the arithmetic value of the root $\mu = |k|$ for $\varkappa = 0$. Note that then $\operatorname{Re} \mu > 0$ for $-\infty < k < \infty$ and this ensures the decay of the function (II-135) as $y \to +\infty$. The function (II-135) will satisfy the boundary conditions (II-132), (II-133) if the function $f(k)$ satisfies the functional equations

$$\int_{-\infty}^{\infty} f(k)\, e^{ikx}\, dk = e^{-ax}, \quad x > 0 \tag{II-136}$$

$$\int_{-\infty}^{\infty} f(k)\, L(k)\, e^{ikx}\, dk = 0, \quad x < 0 \tag{II-137}$$

where the notation $L(k) = \mu(k) = \sqrt{k^2 + \varkappa^2}$ is introduced. The solution of the problem (II-136), (II-137) can readily be constructed if the function $L(k)$ is an analytic function of the complex variable $k$ in the strip $\tau_- < \operatorname{Im} k < \tau_+$ $(\tau_- < 0,\ \tau_+ > 0)$ and if in that strip it may be represented in the form

$$L(k) = (k^2 + a^2)\, L_+(k) \cdot L_-(k) \tag{II-138}$$

where $L_+(k)$ is a function different from zero and analytic in the upper half-plane $\operatorname{Im} k > \tau_-$; for $|k| \to \infty$ $L_+(k)$ tends to zero more slowly than $k^{-2}$, and the function $L_-(k)$ is analytic in the lower half-plane $\operatorname{Im} k < \tau_+$ and uniformly tends to zero at infinity.

If these conditions are fulfilled, it is easy to see by direct verification that the equations (II-136), (II-137) are satisfied by the function

$$f(k) = \frac{C}{(k^2 + a^2)\, L_+(k)} = \frac{C L_-(k)}{L(k)} \tag{II-139}$$

where the constant $C$ is determined from the condition

$$C = \frac{a}{\pi} L_+(ia) \tag{II-140}$$

Indeed, substituting into integral (II-136) the first of the equalities (II-139), closing the contour of integration by a semicircular arc of infinitely large radius in the upper half-plane, the integral around which, by virtue of the Jordan lemma is zero, we find, on the basis of (II-140), that the integral (II-136) is equal to $e^{-ax}$ for $x > 0$. Similarly, using the Jordan lemma applied to the integral around the semicircular arc of infinitely large radius in the lower half-plane, it is easy to establish, for $x < 0$, the truth of (II-137) for the function $f(k)$ defined by the second formula in (II-139). And so the solution of the given problem is connected with the possibility of the representation (II-138). In this case, due to the above-indicated choice of branch of the root, the function $L(k) = \sqrt{k^2 + \varkappa^2}$ is a single-valued analytic function different from zero in the strip

$\mathrm{Im}\,(i\varkappa) < \mathrm{Im}\,k < -\mathrm{Im}\,(i\varkappa)$ $(\mathrm{Im}\,(i\varkappa) < 0)$. Let us consider the function

$$\widetilde{L}_k(k) = \frac{L(k)}{\sqrt{k^2+a^2}} = \frac{\sqrt{k^2+\varkappa^2}}{\sqrt{k^2+a^2}} \tag{II-141}$$

For $a > -\mathrm{Im}\,(i\varkappa)$ this function is also analytic and nonzero in the given strip; $\widetilde{L}(k) \to 1$ as $|k| \to \infty$. For this reason, by virtue of Lemma 2 the required factorization of the function $\widetilde{L}(k)$, and hence, $L(k)$, is possible. It is easy to see that the functions

$$L_+(k) = \frac{\sqrt{k+i\varkappa}}{k+ia}, \quad L_-(k) = \frac{\sqrt{k-i\varkappa}}{k-ia} \tag{II-142}$$

satisfy all the indicated requirements. Then, on the basis of formulas (II-135), (II-139), (II-142) we get the integral representation of the solution of equation (II-134), which satisfies the conditions (II-132) and (II-133) and decreases as $y \to +\infty$, in the form

$$u(x, y) = \int_{-\infty}^{\infty} \frac{C}{\sqrt{k+i\varkappa}\,(k-ia)} e^{-\mu y} e^{ikx}\,dk \tag{II-143}$$

where the constant $C$, on the basis of (II-140), (II-142), is equal to

$$C = \frac{\sqrt{ia+i\varkappa}}{2\pi i} \tag{II-144}$$

Proceeding to the limit in (II-143), (II-144) as $\varkappa \to 0$, we get the integral representation of the solution of the original problem

$$u(x, y) = \frac{\sqrt{a}}{2\pi} e^{-i\frac{\pi}{4}} \int_{-\infty}^{\infty} \frac{e^{-|k|y}}{\sqrt{k}\,(k-ia)} e^{ikx}\,dk$$

$$= \frac{\sqrt{a}}{2\pi} \left\{ e^{-i\frac{3\pi}{4}} \int_{-\infty}^{0} \frac{e^{k'y+ik'x}}{\sqrt{-k'(k'-ia)}}\,dk' + e^{-i\frac{\pi}{4}} \int_{0}^{\infty} \frac{e^{-ky+ikx}}{\sqrt{k}\,(k-ia)}\,dk \right\} \tag{II-145}$$

In the first integral (II-145) make a change of the integration variable $k' = -k$. Since

$$\int_{-\infty}^{0} \frac{e^{k'y+ik'x}}{\sqrt{-k'}\,(k'-ia)}\,dk' = -\int_{0}^{\infty} \frac{e^{-ky-ikx}}{\sqrt{k}\,(k+ia)}\,dk$$

$$= e^{i\pi} \int_{0}^{\infty} \frac{e^{-ky-ikx}}{\sqrt{k}\,(k+ia)}\,dk \tag{II-146}$$

it follows that (II-145) assumes the form

$$u(x, y)=\frac{\sqrt{a}}{2\pi}\left\{e^{i\frac{\pi}{4}}\int_0^\infty \frac{e^{-ky-ikx}}{\sqrt{k}(k+ia)}dk+e^{-i\frac{\pi}{4}}\int_0^\infty \frac{e^{-ky+ikx}}{\sqrt{k}(k-ia)}dk\right\}$$

$$=\frac{\sqrt{a}}{\pi}\,Re\left\{e^{-i\frac{\pi}{4}}\int_0^\infty \frac{e^{-ky+ikx}}{\sqrt{k}(k-ia)}dk\right\} \qquad \text{(II-147)}$$

To compute the integral (II-147), consider the integral

$$J(\alpha, \beta)=\int_0^\infty \frac{e^{-\alpha\xi}}{\sqrt{\xi}(\xi+\beta)}d\xi \qquad \text{(II-148)}$$

By the change of the variable of integration $\xi+\beta=\eta$, this integral may be reduced to the form

$$J(\alpha, \beta)=e^{\alpha\beta}I(\alpha, \beta) \qquad \text{(II-149)}$$

where

$$I(\alpha, \beta)=\int_\beta^\infty \frac{e^{-\alpha\eta}}{\eta\sqrt{\eta-\beta}}d\eta \qquad \text{(II-150)}$$

The integral (II-150) may be evaluated by differentiating with respect to the parameter:

$$\frac{\partial I}{\partial\alpha}=-\int_\beta^\infty \frac{e^{-\alpha\eta}}{\sqrt{\eta-\beta}}d\eta=-e^{-\alpha\beta}\sqrt{\frac{\pi}{\alpha}} \qquad \text{(II-151)}$$

Since

$$I(0, \beta)=\int_\beta^\infty \frac{d\eta}{\eta\sqrt{\eta-\beta}}=\frac{\pi}{\sqrt{\beta}} \qquad \text{(II-152)}$$

it follows that by integrating (II-151) we get

$$I(\alpha, \beta)=\frac{\pi}{\sqrt{\beta}}-\sqrt{\pi}\int_0^\alpha \frac{e^{-\alpha\beta}}{\sqrt{\alpha}}d\alpha=\frac{\pi}{\sqrt{\beta}}[1-\Phi(\sqrt{\alpha\beta})] \qquad \text{(II-153)}$$

where $\Phi(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-x^2}dx$ is the error function. Whence

$$J(\alpha, \beta)=\pi\frac{e^{\alpha\beta}}{\sqrt{\beta}}[1-\Phi(\sqrt{\alpha\beta})] \qquad \text{(II-154)}$$

Returning to (II-147), we obtain

$$u(x, y)=\mathrm{Re}\,\{e^{-az}[1-\Phi(\sqrt{-az})]\} \qquad \text{(II-155)}$$

where $z=x+iy$.

## APPENDIX III

# FUNCTIONS OF MANY COMPLEX VARIABLES

The theory of functions of many complex variables, which is a natural development of the theory of functions of one complex variable, has come to the fore due to effective applications of the methods of this theory in a variety of sciences, and in particular in quantum field theory. In this appendix we give a brief survey of the fundamentals of the theory of functions of many complex variables.

## III.1. Basic Definitions

We consider an $N$-dimensional complex space $C^N$ the points $z = (z_1, \ldots, z_N)$ of which constitute an ordered collection of complex variables $z_k = x_k + iy_k$. The complex space $C^N$ may be interpreted as an ordinary Euclidean space of the real variables $x_1, y_1, \ldots, x_N, y_N$ of dimension $2N$. For this reason, the notions of an open and a closed region, an interior, an exterior, and a boundary point, a $\delta$-neighbourhood, and so on are introduced just as they are in the theory of functions of many real variables. For example, the $\delta$-neighbourhood of a point $z^0$ will be regarded as a set $C(\delta, z^0)$ of points $z \in C^N$ that satisfy the condition

$$|z_k - z_k^0| < \delta_k \qquad k = 1, 2, \ldots, N$$

The symbol $\delta = (\delta_1, \ldots, \delta_N)$ stands for an ordered collection of real numbers $\delta_k > 0$. The set of points $z \in C^N$ that satisfy the condition $|z_k - z_k^0| < r_k$ $(r_k > 0)$ is called a polycircle $K(r, z^0)$ of radius $r = (r_1, \ldots, r_N)$ centred at the point $z^0 = (z_1^0, \ldots, z_N^0)$.

*A function $w = f(z) = f(z_1, \ldots, z_N)$ of many complex variables $z = (z_1, \ldots, z_N)$ specified on a set $E \subset C^N$ is defined by a law that associates with every value $z \in E$ a definite complex number $w \in C^1$.* Since the complex number $w$ consists of a pair of real numbers $u$ and $v$ $(w = u + iv)$, specification of the function $f(z)$ on the set $E \subset C^N$ is tantamount to a specification, on an appropriate set of a $2N$-dimensional Euclidean space, of the two real functions $u(x_1, y_1, \ldots, x_N, y_N)$, and $v(x_1, y_1, \ldots, x_N, y_N)$ of $2N$ real variables $x_1, y_1, \ldots, x_N, y_N$:

$$f(z) = u(x_1, \ldots, y_N) + iv(x_1, \ldots, y_N) \qquad \text{(III-1)}$$

The functions $u(x_1, \ldots, y_N)$ and $v(x_1, \ldots, y_N)$ are called the real and the imaginary part, respectively, of the function $f(z)$.

It is clear that a number of the concepts and properties of functions of many real variables can be carried over to functions of many complex variables. For example, the function $f(z)$ specified on the set $E \subset C^N$ is continuous, at a point $z^0 \in E$, with respect to the collection of variables $z_1, \ldots, z_N$ if for any $\varepsilon > 0$ a $\delta = (\delta_1, \ldots, \delta_N)$ can be found such that for all $z \in C(\delta, z^0)$ the following inequality holds:

$$|f(z) - f(z^0)| < \varepsilon \tag{III-2}$$

From now on we will use the term *continuous function* for the function $f(z)$ that is continuous with respect to the set of variables $z_1, \ldots, z_N$.

If the function $f(z)$ is continuous at every point $z \in E$, then it is said to be continuous on the set $E$. The following theorem holds true.

***Theorem III.1.*** *A necessary and sufficient condition for the continuity of the function* $f(z) = u(x_1, \ldots, y_N) + iv(x_1, \ldots, y_N)$ *on a set* $E \subset C^N$ *is the continuity with respect to the collection of variables of the real functions* $u(x_1, \ldots, y_N)$ *and* $v(x_1, \ldots, y_N)$ *of* $2N$ *real variables on the corresponding set of a* $2N$*-dimensional Euclidean space.*

The properties of continuous functions of a single complex variable carry over directly to the case of many complex variables. A series of continuous functions of many complex variables that is uniformly convergent in a domain $G$ converges to a continuous function.

## III.2. The Concept of an Analytic Function of Many Complex Variables

As in the case of a single complex variable, one of the basic concepts in the theory of functions of many complex variables is that of an analytic function.

Given in a domain $G \subset C^N$ a function $w = f(z)$ of many complex variables. Fixing the values of the variables $z_1^0, \ldots, z_{i-1}^0, z_{i+1}^0, \ldots, z_N^0$, we obtain the function

$$f_i(z_i) = f(z_1^0, \ldots, z_{i-1}^0, z_i, z_{i+1}^0, \ldots, z_N^0)$$

of a single complex variable $z_i$ specified in some domain $G_i$ of the complex plane $z_i$. Suppose for arbitrary fixed values $z_1^0, \ldots, z_{i-1}^0, z_{i+1}^0, \ldots, z_N^0$ each function $f_i(z_i)$ $(i = 1, 2, \ldots, N)$ is an analytic function of the complex variable $z_i \in G_i$. In this case we say that the function $f(z)$ is an *analytic function in each variable* in the domain $G$. The derivatives $f_i'(z_i)$ of the function $f_i(z_i)$ with respect to the variable $z_i$ will be called the partial derivatives of $f(z)$ and will be denoted as $\frac{\partial f}{\partial z_i}$. Clearly,

$$\frac{\partial f}{\partial z_i} = \lim_{\Delta z_i \to 0} \frac{f(z_1, \ldots, z_{i-1}, z_i + \Delta z_i, z_{i+1}, \ldots, z_N) - f(z_1, \ldots, z_N)}{\Delta z_i} \qquad \text{(III-3)}$$

The partial derivatives $\frac{\partial f}{\partial z_i}$ can be expressed in terms of the partial derivatives of the functions $u(x_1, \ldots, y_N)$ and $v(x_1, \ldots, y_N)$:

$$\frac{\partial f}{\partial z_i} = \frac{\partial u}{\partial x_i} + i\frac{\partial v}{\partial x_i} \qquad \text{(III-4)}$$

For them the Cauchy-Riemann conditions hold:

$$\frac{\partial u}{\partial x_i} = \frac{\partial v}{\partial y_i}, \quad \frac{\partial u}{\partial y_i} = -\frac{\partial v}{\partial x_i} \qquad \text{(III-5)}$$

We now introduce a basic definition:

*A function $f(z)$ of many complex variables $z = (z_1, \ldots, z_N)$ is said to be analytic* in a domain $G$ if in that domain the function $f(z)$ is analytic in each variable $z_i$ and all its partial derivatives $\frac{\partial f}{\partial z_i}$ are continuous.*

Analytic functions of many complex variables have a number of remarkable properties similar to those of an analytic function of one complex variable. Below we give a brief survey of these properties. For the sake of simplicity we consider the case of two independent variables, since the reasoning holds true for a larger number of variables.

## III.3. Cauchy's Formula

Suppose $f(z_1, z_2)$ is an analytic function in the domain $G = G_1 \times \times G_2$, the domains $G_1$ and $G_2$ being simply connected. In $G_1$ and $G_2$ take arbitrary closed contours $C_1$ and $C_2$, respectively, and consider the iterated integral

$$I = \int_{C_1} d\zeta_1 \int_{C_2} \frac{f(\zeta_1, \zeta_2)}{(z_1 - \zeta_1)(z_2 - \zeta_2)} d\zeta_2 \qquad \text{(III-6)}$$

where $z_1$ and $z_2$ are arbitrary points lying inside the contours $C_1$ and $C_2$, respectively.

The integrand in (III-6) is continuous with respect to the set of variables, which fact is a sufficient condition for the possibility of

* As in the case of a single complex variable, to simplify later proofs we included in the definition of an analytic function of many complex variables the extra condition of continuity of the partial derivatives, which, however, does not restrict the class of functions under consideration; this follows from the so-called Hartogs theorem (see, for example, [20]).

changing the order of integration in the iterated integral. Hence

$$I=\int\limits_{C_2} d\zeta_2 \int\limits_{C_1} \frac{f(\zeta_1, \zeta_2)}{(z_1-\zeta_1)(z_2-\zeta_2)}\, d\zeta_1 \qquad \text{(III-6}')$$

Since the function $f(\zeta_1, \zeta_2)$ is analytic in each variable, the inner integral in (III-6) is, by virtue of the Cauchy formula (1-59), equal to

$$\int\limits_{C_1} \frac{f(\zeta_1, \zeta_2)}{(z_1-\zeta_1)(z_2-\zeta_2)}\, d\zeta_2 = 2\pi i\, \frac{f(\zeta_1, \zeta_2)}{(z_1-\zeta_1)} \qquad \text{(III-7)}$$

Taking advantage once again of the Cauchy formula, we finally get

$$I=\int\limits_{C_1} d\zeta_1 \int\limits_{C_2} \frac{f(\zeta_1, \zeta_2)}{(z_1-\zeta_1)(z_2-\zeta_2)}\, d\zeta_2 = (-4\pi^2)\, f(z_1, z_2) \qquad \text{(III-8)}$$

which can be rewritten as

$$f(z_1, z_2) = -\frac{1}{4\pi^2}\int\limits_{C_1} d\zeta_1 \int\limits_{C_2} \frac{f(\zeta_1, \zeta_2)}{(z_1-\zeta_1)(z_2-\zeta_2)}\, d\zeta_2 \qquad \text{(III-9)}$$

Similarly, for the case of $N$ variables we have the formula

$$f(z)=f(z_1, \ldots, z_N)=\frac{1}{(2\pi i)\, N}\int\limits_{C_1} d\zeta_1 \ldots \int\limits_{C_N} \frac{f(\zeta)\, d\zeta_N}{\prod\limits_{k=1}^{N}(z_k-\zeta_k)} \qquad \text{(III-10)}$$

where the points $z_k$ lie inside the closed contours $C_k$ that belong to the simply connected domains $G_k$, and the function $f(z)$ is analytic in the domain $G = G_1 \times \ldots \times G_N$. Formulas (III-9) and (III-10) are generalizations of the Cauchy formula (1-59) to the case of many complex variables.

From these formulas we can obtain some important properties of an analytic function of many complex variables. In particular, as in the case of one complex variable, using formula (III-9) we can show that an analytic function of two complex variables has partial derivatives of all orders for which the following expressions hold true:

$$\frac{\partial^{n+m} f(z_1, z_2)}{\partial z_1^n\, \partial z_2^m} = -\frac{n!\, m!}{4\pi^2}\int\limits_{C_1} d\zeta_1 \int\limits_{C_2} \frac{f(\zeta_1, \zeta_2)\, d\zeta_2}{(z_1-\zeta_1)^{n+1}(z_2-\zeta_2)^{m+1}} \qquad \text{(III-11)}$$

The maximum modulus principle and other properties can similarly be established.

The appropriate results are obtained from formula (III-10) for an analytic function of many complex variables.

## III.4. Power Series

In the case of two independent variables, the following expression is called a power series:

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} (z_1 - a_1)^n (z_2 - a_2)^m \qquad \text{(III-12)}$$

where $C_{n,m}$, $a_1$, $a_2$ are specified complex numbers. An assertion similar to the Abel theorem (Theorem 2.5) holds true.

***Theorem III.2.*** *If the series* (III-12) *converges absolutely at the point* $z^0 = (z_1^0 \neq a_1, z_2^0 \neq a_2)$, *then it is absolutely convergent inside the polycircle* $K(r^0, a)$ *of radius* $r^0 = (|z_1^0 - a_1|, |z_2^0 - a_2|)$, *and in any polycircle of smaller radius * centred at the point a the series is uniformly convergent.*

*Proof.* By virtue of the absolute convergence of the series (III-12) at the point $z^0$, all terms of the series are uniformly bounded at this point. We therefore have the following estimate for the coefficients of series (III-12):

$$|C_{n,m}| \leqslant \frac{M}{|z_1^0 - a_1|^n \cdot |z_2^0 - a_2|^m} \qquad \text{(III-13)}$$

with the common constant $M$ for all coefficients. Take an arbitrary point $z = (z_1, z_2)$ inside the polycircle $K(r^0, a)$ and set

$$|z_1 - a_1| = q_1 |z_1^0 - a_1|, \quad |z_2 - a_2| = q_2 |z_2^0 - a_2|$$

where $0 < q_1 < 1$, $0 < q_2 < 1$. Then, using the estimate (13), we get, for the chosen point $z$,

$$\left| \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} C_{n,m} (z_1 - a_1)^n (z_2 - a_2)^m \right| \leqslant M \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} q_1^n q_2^m = \frac{M}{(1-q_1)(1-q_2)} \qquad \text{(III-14)}$$

which completes the proof of the convergence of the series (III-12) at the point $z$.

Since $z$ is an arbitrary point of the polycircle $K(r^0, a)$, the absolute convergence of the series (III-12) inside $K(r^0, a)$ follows. The uniform convergence of the series (III-12) in any polycircle $K(r^{(1)}, a)$ of smaller radius can be demonstrated with the aid of (III-14), just as in the case of one complex variable (Theorem 2.5).

The theorem just proved enables one to establish that the domain of convergence of the power series is the polycircle $K(R, a)$ of radius $R = (R_1, R_2)$. Inside $K(R, a)$ the series (III-12) is absolutely

* We will say that the radius $r^{(1)}$ of the polycircle $K(r^{(1)}, a)$ is less than the radius $r^{(2)}$ of the polycircle $K(r^{(2)}, a)$ if $r_1^{(1)} < r_1^{(2)}, \ldots, r_N^{(1)} < r_N^{(2)}$.

convergent, outside it the series diverges; the series (III-12) is uniformly convergent in any closed subdomain of $K(R, a)$. Note that the radii $R_1$ and $R_2$ are defined jointly and cannot, generally, be defined separately.

To illustrate let us consider the power series

$$f(z) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(n+m)!}{n!\,m!} z_1^n z_2^m \tag{III-15}$$

the coefficients of which are binomial coefficients. Since the series is absolutely convergent within its polycircle of convergence, the series with positive terms

$$\sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \frac{(n+m)!}{n!\,m!} |z_1|^n \cdot |z_2|^m \tag{III-16}$$

is convergent in the polycircle of convergence of the series (III-15). Collecting terms in (III-16) with the sum of the powers $n + m =$ $= s$, we get

$$\sum_{s=0}^{\infty} (|z_1| + |z_2|)^s \tag{III-16'}$$

whence it follows that the radii $R_1$ and $R_2$ of the polycircle of convergence of the series (III-15) are determined from the condition $R_1 + R_2 = 1$, i.e., when $R_1$ decreases, the value of $R_2$ increases and vice versa.

Consider the series (III-12) within its polycircle of convergence $K(R, a)$. Taking advantage of the absolute convergence of the series, collect those terms whose sum of the powers $m + n = s$:

$$f(z) = f(z_1, z_2) = \sum_{s=0}^{\infty} u_s(z_1, z_2) \tag{III-17}$$

The expression (III-17) is a representation of the original series in the form of a series of homogeneous polynomials in the variables $\hat{z}_1 = z_1 - a_1$, $\hat{z}_2 = z_2 - a_2$

$$u_s(z_1, z_2) = \sum_{k=0}^{\infty} C_{k,\, s-k} \hat{z}_1^k \hat{z}_2^{s-k} \tag{III-18}$$

Since the functions $u\ (z_1, z_2)$ are analytic in each variable and the series converges uniformly within the polycircle $K(R, a)$, it follows by the Weierstrass theorem (Theorem 2.3) that the function $f(z)$ is also analytic in each variable within $K(R, a)$, and its partial derivatives may be computed by means of termwise differentiation of the series (III-17). As can be readily seen, the radius of convergence of the resulting series is then equal to the radius of convergence of the series (III-12), and the partial derivatives $\frac{\partial f}{\partial z_1}$ and $\frac{\partial f}{\partial z_2}$ are continuous inside $K(R, a)$. From this it follows that *within the polycircle of convergence the power series* (III-12) *converges to an analytic function of many complex variables.*

As in the case of one complex variable, it is easy to establish that the coefficients of the power series (III-12) are expressed in terms of the values of the partial derivatives of its sum in the centre of the polycircle of convergence—at the point $a = (a_1, a_2)$—via the formulas

$$C_{n,m} = \frac{1}{n!\,m!}\,\frac{\partial^{n+m}}{\partial z_1^n\,\partial z_2^m} f(z)\,\Big|_{z=a} \tag{III-19}$$

## III.5. Taylor's Series

We now show that with a function analytic in some polycircle $K(R, a)$ there can be associated a power series that converges to the given function within $K(R, a)$. The following theorem holds.

***Theorem III.3.*** *A function $f(z)$ that is analytic inside a polycircle $K(R, a)$ is uniquely represented within $K(R, a)$ in the form of the sum of an absolutely convergent power series*:

$$f(z) = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} C_{n,m}(z_1 - a_1)^n (z_2 - a_2)^m$$

*Proof*. Take an arbitrary point $z \in K(R, a)$. By the formula (III-9) we have

$$f(z) = -\frac{1}{4\pi^2}\int\limits_{C_1'} d\zeta_1 \int\limits_{C_2'} \frac{f(\zeta_1, \zeta_2)}{(\zeta_1 - z_1)(\zeta_2 - z_2)}\,d\zeta_2 \tag{III-20}$$

where $C_1'$ and $C_2'$ are circles with centres at the points $a_1$ and $a_2$ and with radii $R_1'$ and $R_2'$ that satisfy the conditions $|z_1 - a_1| < R_1' < R_1$ and $|z_2 - a_2| < R_2' < R_2$. From earlier reasoning it follows that the rational fraction $\frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)}$ may be expanded into an absolutely and uniformly convergent series with respect to $\zeta_1$ and $\zeta_2$:

$$\frac{1}{(\zeta_1 - z_1)(\zeta_2 - z_2)} = \sum_{n=0}^{\infty}\sum_{m=0}^{\infty} \frac{(z_1 - a_1)^n (z_2 - a_2)^m}{(\zeta_1 - a_1)^{n+1}(\zeta_2 - a_2)^{m+1}} \tag{III-21}$$

Substituting the expansion (III-21) into (III-20) and again performing term-by-term integration of the corresponding uniformly convergent series, we obtain

$$f(z)=\sum_{n=0}^{\infty}\sum_{m=0}^{\infty}C_{n,m}(z_1-a_1)^n(z_2-a_2)^m \qquad \text{(III-22)}$$

where $C_{n,m}$ denotes the expressions

$$C_{n,m}=-\frac{1}{4\pi^2}\int\limits_{C_1'}d\zeta_1\int\limits_{C_2'}\frac{f(\zeta_1,\zeta_2)}{(\zeta_1-a_1)^{n+1}(\zeta_2-a_2)^{m+1}}\,d\zeta_2 \qquad \text{(III-23)}$$

By (III-11) this can be rewritten as

$$C_{n,m}=\frac{1}{n!\,m!}\left.\frac{\partial^{n+m}f(z)}{\partial z_1^n\,\partial z_2^m}\right|_{z=a} \qquad \text{(III-24)}$$

Since $z$ is an arbitrary point of $K(R, a)$, from (III-22) follows the expandability of the function, which is analytic in the polycircle $K(R, a)$, into a convergent power series.

From a comparison of formulas (III-24) and (III-19) we conclude that the expansion is unique, and this completes the proof of the theorem.

By analogy with the results obtained for a function of a single complex variable (see Theorem 2.6), it is natural to call the expansion (III-22) the Taylor series of the function $f(z)$.

To conclude, note that the radius $R^0$ of convergence of the series (III-22) may turn out to be greater than the radius $R$ of the polycircle $K(R, a)$. In that case the sum of that series will be a function that is analytic in the polycircle $K(R^0, a)$ and is coincident with the original analytic function $f(z)$ in a polycircle $K(R, a)$ of smaller radius.

The foregoing reasoning is readily carried over to the case of many complex variables.

## III.6. Analytic Continuation

As in the case of a single complex variable, representing an analytic function of many complex variables with the aid of a power series permits illuminating the question of the uniqueness of definition of an analytic function (see Theorem 2.8). For instance, if we have two analytic functions $f_1(z_1, z_2)$ and $f_2(z_1, z_2)$ in a domain $G$ that coincide in the subdomain $G'$ of $G$, then it can readily be demonstrated that $f_1(z_1, z_2) \equiv f_2(z_1, z_2)$ for $z=(z_1, z_2) \in G$. On this basis we can introduce the principle of analytic continuation in the following form.

***The principle of analytic continuation.*** Given, in domains $G^{(1)}$ and $G^{(2)}$ which have a common subdomain $G^{(1,2)}$, the analytic functions $f_1(z)$ and $f_2(z)$ that are coincident in $G^{(1,2)}$. These functions are then an analytic continuation of each other, which is to say that in the domain $G = G^{(1)} + G^{(2)}$ there is a unique analytic function $f(z)$ that coincides with $f_1(z)$ in $G^{(1)}$ and with $f_2(z)$ in $G^{(2)}$.

As in the case of one complex variable, it is possible to construct an analytic continuation of an analytic function $f_1(z)$, originally specified in some domain $G^{(1)}$, along all possible chains of domains emanating from $G^{(1)}$ and having pairwise common portions.

For example, such an analytic continuation can be obtained by expanding the function $f(z)$ in the Taylor power series (III-22) about various points $z^{(i)} \in G^{(1)}$. If the radius of the polycircle of convergence of any one of these expansions turns out to be greater than the distance of the point $z^{(i)}$ to the boundary of the domain $G^{(1)}$, then we obtain an analytic continuation of $f(z)$ into the greater domain $G$ that contains $G^{(1)}$.

In this manner we arrive at the concept of the total analytic function $F(z)$ and its natural domain of existence $G$ or, as it is common to say, the region of analyticity (also sometimes called the region of holomorphy). Generally, an analytic continuation can also lead to a multivalued function whose region of analyticity is a certain multisheeted manifold that results from the introduction of so-called domains of superposition.*

An essential point in applications of the theory of functions of many complex variables, in particular in the quantum field theory, is whether or not a given domain $G$ is a region of analyticity. In other words, whether there is a function $f(z)$ analytic in $G$ for which the domain $G$ is the natural domain of existence. If $G$ is not a region of analyticity, then any function $f(z)$ analytic in $G$ can be continued analytically into a greater domain $G^*$ containing $G$.

As we have seen (Example 4, Sec. 3.2), in the case of one complex variable the unit circle $|z| < 1$ is a region of analyticity. Making use of Riemann's theorem on the possibility of a conformal mapping of an arbitrary domain into the unit circle, it is easy to show that *in the case of one complex variable any domain is a region of analyticity*.

This assertion does not hold true in the case of many complex variables.

To prove this we will show that even in $C^2$ the domain

$$G:\ \{z = (z_1, z_2):\ 1 < |z| = (|z_1|^2 + |z_2|^2)^{1/2} < 5\}$$

is not a region of analyticity.** To do this it suffices to demonstrate that any function analytic in $G$ can be continued analytically into a greater domain $G^*$ that contains $G$, for instance, into the sphere $|z| < 5$.

---

* This is discussed in detail in [20].

** This example is a slight modification of the example considered in [2]. See also [20].

Now suppose $f(z)$ is an arbitrary function analytic in $G$. Consider the function

$$\varphi(z) = \varphi(z_1, z_2) = \frac{1}{2\pi i} \int\limits_{|\zeta_1|=4} \frac{f(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1 \qquad \text{(III-25)}$$

The function $\varphi(z)$ is an integral dependent on the variables $z_1$ and $z_2$ as parameters. The subdomain $\{|\zeta_1| = 4, |z_2| < 3\}$ belongs to

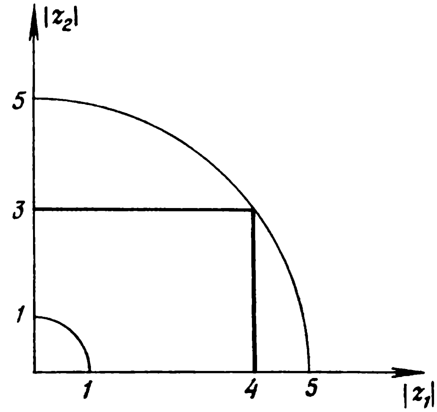


**Fig. III.1**

$G$ (see Fig. III.1). Therefore the function $\varphi(z)$ is analytic in each variable $z_1$ and $z_2$ in the polycircle $K$: $\{|z_1| < 4, |z_2| < 3\}$. It is easy to see that the partial derivatives of the function $\varphi(z)$ are then continuous. From this it follows that in the polycircle $K$: $\{|z_1| < 4, |z_2| < 3\}$ the function is an analytic function of two complex variables $z_1$ and $z_2$. In particular, $\varphi(z)$ is also analytic in the closed domain $\bar{G}'$: $\{|z_1| \leqslant 4, 1 \leqslant |z_2| \leqslant 3\}$ that simultaneously belongs to the polycircle $K$ and to the original domain $G$. By Cauchy's formula (1-59), we have, in $G'$,

$$\frac{1}{2\pi i} \int\limits_{|\zeta_1|=4} \frac{f(\zeta_1, z_2)}{\zeta_1 - z_1} d\zeta_1 = f(z_1, z_2) \qquad \text{(III-26)}$$

whence it follows that the analytic functions $f(z)$ and $\varphi(z)$ coincide in $G'$. Hence, in the extended domain $G^*$ (in the sphere $|z| < 5$) that contains the original domain $G$, there is defined an analytic function $F(z)$, equal to $f(z)$ in $G$ and $\varphi(z)$ in $K$, which is an analytic continuation of $f(z)$ in $G^*$. The proof is complete.

To summarize, then, *in the case of many complex variables not every region is a region of analyticity*. This fact markedly distinguishes the theory of functions of many complex variables from the theory of functions of one complex variable.

# APPENDIX IV

## WATSON'S METHOD

The Watson method is used chiefly in the summation and asymptotic analysis of series. This method was originally proposed by G. N. Watson in 1919 in a study of the problem of the diffraction of radio waves on a sphere. The method of separation of variables readily enables one to obtain an analytic representation of the solution of this problem in the form of a series in terms of eigenfunctions. However, when the length of the incident wave is much less than the radius of the sphere, which occurs for instance in problems of the diffraction of radio waves on the earth's surface, the resulting series converges very slowly. Watson was able to develop a method that permitted transforming this slowly convergent series into another series that converges quite rapidly. This method came to be known as Watson's method.

The principal idea behind Watson's method is unusually simple and is based on the fact that when computing the integral with respect to a complex variable with the aid of residue calculus, one can, by closing the contour of integration in various ways, obtain a representation of the original integral in the form of various series. However, despite the simplicity of the basic idea of the Watson method, its realization in many specific cases requires a great deal of skill.

We illustrate the basic propositions of the Watson method in a few simple instances.* Let it be required to sum the series

$$S=\sum_{n=-\infty}^{\infty}\frac{1}{n^2+a^2} \tag{IV-1}$$

where $a$ is a positive number.

Note that when $a \gg 1$ the numerical summation of the series (IV-1) to a high degree of accuracy is not exactly a trivial problem.

Consider the auxiliary integral

$$I=\frac{1}{2i}\int\limits_{\mathscr{L}^{+}+\mathscr{L}^{-}}\frac{1}{\nu^2+a^2}\cdot\frac{e^{i\pi\nu}}{\sin\pi\nu}\,d\nu \tag{IV-2}$$

---

* The examples given below in the use of the Watson method were suggested by S. Ya. Sekerzh-Zenkovich to whom the authors are indebted.

where the integration is performed on the complex $\nu$-plane along the straight lines $\mathscr{L}^+$ and $\mathscr{L}^-$ parallel to the real axis and at a distance $d$ from it in the upper and lower half-planes with $d < a$ (Fig. IV.1). The integration along the line $\mathscr{L}^+$ is from right to left,

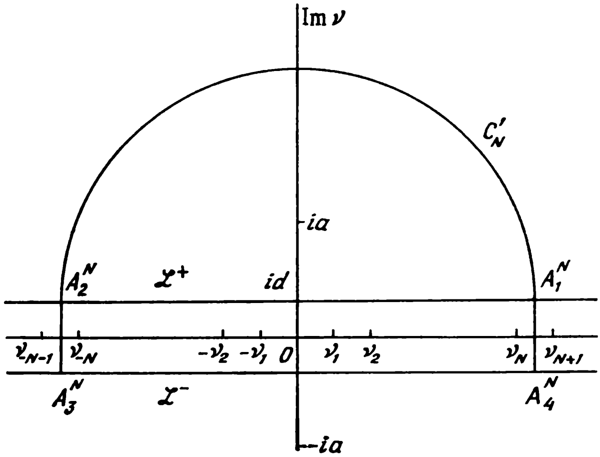


**Fig. IV.1**

along the line $\mathscr{L}^-$, in the reverse direction. The improper integral (IV-2) is absolutely convergent. Indeed, we have the obvious estimate

$$\left|\frac{e^{i\pi\nu}}{\sin\pi\nu}\right|\Bigg|_{\operatorname{Im}\nu=d}=\frac{2}{\left|1-e^{-2i\pi\nu}\right|\Big|_{\operatorname{Im}\nu=d}}\leqslant\frac{2}{e^{2\pi d}-1} \qquad \text{(IV-3)}$$

We have a similar estimate also for $\operatorname{Im}\nu = -d$. Thus the second factor in the integrand of (IV-2) is bounded, while the first tends to zero as $1/|\nu|^2$, and this ensures the absolute convergence of the integral (IV-2).

We will show that the integral (IV-2) is equal to the sum of the original series (IV-1). Construct the auxiliary integral

$$I_N=\frac{1}{2i}\int\limits_{\Gamma_N}\frac{1}{\nu^2+a^2}\cdot\frac{e^{i\pi\nu}}{\sin\pi\nu}\,d\nu \qquad \text{(IV-4)}$$

around the closed contour $\Gamma_N$ which is made up of the segments $\mathscr{L}_N$ and $\mathscr{L}_{\bar{N}}$ of the straight lines $\mathscr{L}^+$ and $\mathscr{L}^-$ between the points

$$A_1^N=\left(N+\frac{1}{2},\, d\right),\quad A_2^N=\left(-N-\frac{1}{2},\, d\right),\quad A_3^N=\left(-N-\frac{1}{2},\, -d\right),$$
$$A_4^N=\left(N+\frac{1}{2},\, -d\right)$$

respectively, and the vertical segments $\gamma_N$ $(A_4^NA_1^N)$ and $\tilde{\gamma}_N$ $(A_2^NA_3^N)$ connecting them (Fig. IV.1). Within the domain bounded by the contour $\Gamma_N$, the integrand in (IV-4) has poles of the first order at the points $\nu_k = k$ ($k = 0, \pm 1, \ldots, \pm N$). Therefore, when computing the integral (IV-4) by the calculus of residues, we get

$$I_N = \frac{2\pi i}{2i} \sum_{k=-N}^{N} \operatorname{Res}\left[\frac{1}{\nu^2+a^2}\cdot\frac{e^{i\pi\nu}}{\sin\pi\nu}, k\right] = \sum_{k=-N}^{N} \frac{1}{k^2+a^2} \quad \text{(IV-5)}$$

whence it follows that the sum $S$ of the series (IV-1) is equal to $\lim\limits_{N\to\infty} I_N$.

On the other hand, the limit of the integral $I_N$ as $N \to \infty$ is equal to the integral (IV-2). Indeed, by virtue of the absolute convergence of the improper integral (IV-2) we have

$$\lim_{N\to\infty} \frac{1}{2i} \int_{\mathscr{L}_N^+ + \mathscr{L}_N^-} \frac{1}{\nu^2+k^2}\cdot\frac{e^{i\pi\nu}}{\sin\pi\nu}\,d\nu = I \quad \text{(IV-6)}$$

and the integrals along the straight lines $\gamma_N$ and $\tilde{\gamma}_N$ tend to zero as $N \to \infty$, which fact can readily be established on the basis of the estimate

$$\left|\frac{e^{i\pi\nu}}{\sin\pi\nu}\right|\Bigg|_{\gamma_N} = \frac{e^{-\pi\,\mathrm{Im}\,\nu}}{|\sin\pi(\mathrm{Re}\,\nu + i\,\mathrm{Im}\,\nu)|}\Bigg|_{\gamma_N} = \frac{e^{-\pi\,\mathrm{Im}\,\nu}}{\cosh(\pi\,\mathrm{Im}\,\nu)}\Bigg|_{\gamma_N} \leqslant e^{\pi d} \quad \text{(IV-7)}$$

Thus

$$S = \sum_{n=-\infty}^{\infty} \frac{1}{n^2+a^2} = I \quad \text{(IV-8)}$$

and the original problem of summing the series (IV-1) reduces to computing the integral (IV-2). This problem can again be solved with the aid of residue calculus. Note that besides the singularities on the real axis the integrand in (IV-2) has two poles at the points $\nu = \pm ia$. To compute the integral along the straight line $\mathscr{L}^+$, we consider, in the upper half-plane, a closed contour $C_N$ consisting of the segment $\mathscr{L}_N^+$ and the closing arc $C'_N$ of the semicircle. It is easy to see that for $\mathrm{Im}\,\nu \geqslant d$ we have an estimate similar to (IV-3):

$$\left|\frac{e^{i\pi\nu}}{\sin\pi\nu}\right|_{\mathrm{Im}\,\nu\geqslant d} \leqslant \frac{2}{e^{2\pi d}-1} \quad \text{(IV-9)}$$

whence it follows that the integral around the arc $C'_N$ tends to zero as $N \to \infty$. Therefore, when computing the integral along the line $\mathscr{L}^+$ by means of residue calculus, we obtain

$$\frac{1}{2i}\int_{\mathscr{L}^+} \frac{1}{\nu^2+a^2}\cdot\frac{e^{i\pi\nu}}{\sin\pi\nu}\,d\nu = -\frac{2\pi i}{2i}\operatorname{Res}\left[\frac{1}{\nu^2+a^2}\cdot\frac{e^{i\pi\nu}}{\sin\pi\nu}, ia\right]$$

$$= -\pi\frac{1}{2ia}\frac{e^{-\pi a}}{\sin i\pi a} = \frac{\pi}{2a}\cdot\frac{e^{-\pi a}}{\sinh\pi a} \quad \text{(IV-10)}$$

Similarly

$$\frac{1}{2i}\int\limits_{\mathcal{L}^-}\frac{1}{\nu^2+a^2}\cdot\frac{e^{i\pi\nu}}{\sin\pi\nu}\,d\nu=\frac{\pi}{2a}\cdot\frac{e^{\pi a}}{\sinh\pi a}\qquad\text{(IV-11)}$$

From this we have

$$I=\frac{1}{2i}\int\limits_{\mathcal{L}^++\mathcal{L}^-}\frac{1}{\nu^2+a^2}\cdot\frac{e^{i\pi\nu}}{\sin\pi\nu}\,d\nu=\frac{\pi}{a}\coth\pi a\qquad\text{(IV-12)}$$

which completes the solution of the original problem of summing the series (IV-1).

Despite the simplicity of the foregoing example it contains all the basic elements of the Watson method. This method of an asymptotic analysis of series consists of several stages. In the first stage, it is necessary to construct an integral with respect to a complex variable that is equal to the sum of the original series. The integrand of this integral must contain, as a factor, the analytic continuation of the general term of the series into the complex plane of its number. The next stage consists of an independent calculation of the integral thus constructed. In many cases one is able to obtain an expression of the desired integral in terms of the sum of the residues of the integrand function at the singular points of the analytic continuation of the general term of the series. If the number of such singular points is finite, then we get an explicit expression for the sum of the original series; if the number of these singular points is infinite, then we transform the original series into a new series that may prove to be simpler for an asymptotic investigation.

For the next example we consider the problem of computing the series

$$F(\theta)=\sum_{n=1}^{\infty}(-1)^n\frac{\cos n\theta}{\cosh \alpha n}\qquad\text{(IV-13)}$$

where $0\leqslant\theta<\pi$ and $\alpha$ is a specified positive number that satisfies the condition $\alpha\ll 1$. The series (IV-13) is typical of many problems in mathematical physics, the solution of which is handled by the method of separation of variables. As is readily seen, by virtue of the condition $\alpha\ll 1$ a large number of the initial terms of the series are of the same order (for instance, when $\alpha=10^{-4}$ and $\theta=0$ the first 1000 terms of the series vary in absolute value from 1 to 0.995). For this reason it is extremely difficult to perform a direct numerical summation of the series (IV-13) for $\alpha\ll 1$. But if we apply the Watson method, it is possible to transform the series (IV-13) into a new series for which it is easy to find an asymptotic representation when $\alpha\ll 1$.

Let us consider the auxiliary integral

$$I\theta=\frac{1}{2i}\int\limits_{\Pi}\frac{\cos\nu\theta}{\cosh\alpha\nu\sin\nu\pi}\,d\nu \qquad \text{(IV-14)}$$

where the contour of integration $\Pi$ on the complex $\nu$-plane is an infinite loop enclosing the positive portion of the real axis (Fig. IV.2)

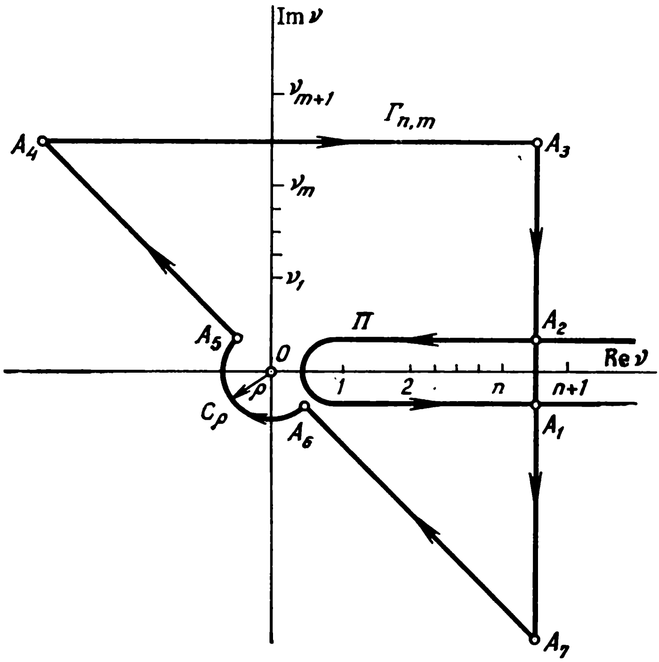


Fig. IV.2

and cutting the real axis at the point $\nu = 1/2$. Integration along the contour $\Pi$ is carried out in the positive direction so that the real axis remains to the left of the direction of motion. It is easy to see that the integral (IV-14) is equal to the original series (IV-13). Indeed, consider the integral

$$I_n(\theta)=\frac{1}{2i}\int\limits_{\Pi_n}\frac{\cos\nu\theta}{\cosh\alpha\nu\sin\nu\pi}\,d\nu \qquad \text{(IV-15)}$$

around the closed contour $\Pi_n$ which consists of a finite portion of the loop $\Pi$ and the vertical section $A_1A_2$ that closes it and cuts the real axis at the point $\nu = n+\frac{1}{2}$. The integrand $f(\nu)$ in (IV-15) is an analytic function of the complex variable $\nu$ inside the contour of integration, with the exception of a finite number of isolated singular points $\hat{\nu}_i = i$ $(i = 1, 2, \ldots, n)$ which are poles of the first order

Therefore, applying the residue theorem we obtain

$$I_n(\theta) = \sum_{k=1}^{n} (-1)^k \frac{\cos k\theta}{\cosh \alpha k} \tag{IV-16}$$

Let us estimate the value of the function $f(\nu)$ in (IV-15) on the section $A_1A_2$. Since here $\operatorname{Re}\nu = n+\frac{1}{2}$, by taking advantage of the relation

$$\sin \nu\pi \big|_{\operatorname{Re}\nu=n+1/2} = \sin(2n+1)\frac{\pi}{2}\cosh(\pi \operatorname{Im}\nu)$$
$$+ i \sinh(\pi \operatorname{Im}\nu)\cos(2n+1)\frac{\pi}{2} = (-1)^n \cosh(\pi \operatorname{Im}\nu) \tag{IV-17}$$

we obtain

$$|\sin \nu\pi|_{A_1A_2} = \cosh(\pi \operatorname{Im}\nu) \tag{IV-18}$$

We have the obvious estimate

$$|\cos \nu\theta|_{A_1A_2} \leqslant e^{\theta|\operatorname{Im}\nu|}\big|_{A_1A_2} \tag{IV-19}$$

whence

$$\left|\frac{\cos\nu\theta}{\sin\nu\pi}\right|_{A_1A_2} \leqslant \left.\frac{e^{\theta|\operatorname{Im}\nu|}}{\cosh(\pi\operatorname{Im}\nu)}\right|_{A_1A_2} < 2e^{-|\operatorname{Im}\nu|(\pi-\theta)|}\big|_{A_1A_2} < 2 \tag{IV-20}$$

On the other hand, it is clear that

$$|\cosh\alpha\nu|_{A_1A_2} = \frac{1}{2}e^{\alpha(n+1/2)}\,|1+e^{-2\alpha\nu}|_{A_1A_2}$$
$$> \frac{1}{2}|1-e^{-\alpha(2n+1)}|\,e^{\alpha(n+1/2)} \tag{IV-21}$$

By virtue of (IV-20) and (IV-21) the integrand in (IV-15) decreases exponentially on $A_1A_2$ as $n\to\infty$. Therefore, by passing to the limit in (IV-15) as $n\to\infty$, we get

$$I(\theta) = \sum_{k=1}^{\infty} (-1)^k \frac{\cos k\theta}{\cosh\alpha k} = F(\theta) \tag{IV-22}$$

which completes the proof that the original series (IV-13) is equal to the integral (IV-14).

Now let us evaluate the integral (IV-14). To do this, we analytically continue the integrand of (IV-14) into the entire complex $\nu$-plane and determine the singularities of the function $f(\nu) = \frac{\cos\nu\theta}{\cosh\alpha\nu\,\sin\nu\pi}$ outside the loop $\Pi$. These are obviously the points $\nu_n = n$ ($n = 0, -1, -2, \ldots$), $\nu_k = i\frac{1}{\alpha}(2k+1)\frac{\pi}{2}$ ($k = 0, \pm1, \pm2, \ldots$). All of these points are poles of the first order. Also note that the integrand $f(\nu)$ is an odd function of the complex variable $\nu$.

On the complex $\nu$-plane we construct a closed contour $\Gamma_{n\ m}$ consisting of a finite portion $\Pi_n$ of the loop $\Pi$ between the points $A_1$ and $A_2$ (see Fig. IV.2), the rectilinear segments $A_2A_3$, $\left\{A_3 = \left(n+\frac{1}{2}, \frac{(m+1)\pi}{\alpha}\right)\right\}$; $A_3A_4$, $\left\{A_4 = \left(-n-\frac{1}{2}, \frac{(m+1)\pi}{\alpha}\right)\right\}$; $A_7A_1$, $\left\{A_7 = \left(n+\frac{1}{2}, -\frac{(m+1)\pi}{\alpha}\right)\right\}$, and the contour $A_4A_5A_6A_7$, which consists of the straight-line segment $A_4A_7$ with the circuit around the point $\nu = 0$ along an arc of a semicircle of sufficiently small radius $\rho$.

Consider the integral

$$I_{n,m}(\theta) = \frac{1}{2i} \int_{\Gamma_{n,m}^{-}} \frac{\cos\nu\theta}{\cosh\alpha\nu\sin\nu\pi}\,d\nu \tag{IV-23}$$

where the integration is performed in the negative direction. Clearly

$$I_{n,m}(\theta) = -\pi\left\{\operatorname{Res}[f(\nu), 0] + \sum_{k=0}^{m}\operatorname{Res}[f(\nu), \nu_k]\right\} \tag{IV-24}$$

On the other hand (see Fig. IV.2),

$$I_{n,m}(\theta) = \frac{1}{2i}\left\{\int_{\Pi_n} f(\nu)\,d\nu + \int_{A_1}^{A_7} f(\nu)\,d\nu + \int_{A_7}^{A_6} f(\nu)\,d\nu \right.$$

$$\left. + \int_{C_\rho^-} f(\nu)\,d\nu + \int_{A_5}^{A_4} f(\nu)\,d\nu + \int_{A_4}^{A_3} f(\nu)\,d\nu + \int_{A_3}^{A_2} f(\nu)\,d\nu\right\} \tag{IV-25}$$

Since the function $f(\nu)$ is odd,

$$\int_{A_7}^{A_6} f(\nu)\,d\nu + \int_{A_5}^{A_4} f(\nu)\,d\nu = 0 \tag{IV-26}$$

Besides, it is clear that

$$\lim_{\rho\to 0}\int_{C_\rho^-} \frac{\cos\nu\theta}{\cosh\alpha\nu\sin\nu\pi}\,d\nu = -i \tag{IV-27}$$

Let us estimate the remaining integrals. By virtue of the estimates given above [see formulas (IV-20), (IV-21)] the function $f(\nu)$ tends exponentially to zero as $n \to \infty$ on the line segments $A_1A_7$ and $A_3A_2$. To estimate the function $f(\nu)$ on the line segment $A_3A_4$, note that, like (IV-17),

$$|\cosh\alpha\nu|_{\operatorname{Im}\nu = m\pi/\alpha} = \cosh(\alpha\operatorname{Re}\nu) \geqslant 1 \tag{IV-28}$$

It is also clear that

$$|\cos\nu\theta|_{\operatorname{Im}\nu = m\pi/\alpha} \leqslant e^{\frac{m\pi}{\alpha}\theta} \tag{IV-29}$$

and

$$|\sin \nu\pi|_{\operatorname{Im}\nu=m\pi/\alpha} > \frac{1}{2}\left|1-e^{-2m\pi^2/\alpha}\right| e^{m\pi^2/\alpha} > \frac{1}{4} e^{m\pi^2/\alpha} \quad \text{(IV-30)}$$

From (IV-29) and (IV-30) we get

$$\left|\frac{\cos \nu\theta}{\sin \nu\pi}\right|_{\operatorname{Im}\nu=m\pi/\alpha} < 4e^{-\frac{m\pi}{\alpha}(\pi-\theta)} \quad \text{(IV-31)}$$

By (IV-28) and (IV-31) we conclude that the function $f(\nu)$ tends exponentially to zero on the line segment $A_3A_4$ as $m \to \infty$ and $\theta < \pi$.

Passing to the limit in (IV-25) as $n, m \to \infty$ and $\rho \to 0$ and taking into account (IV-24) and (IV-15), we obtain, by virtue of the foregoing estimates,

$$I(\theta) - \frac{1}{2} = -\pi\left\{\operatorname{Res}[f(\nu), 0] + \sum_{k=0}^{\infty} \operatorname{Res}[f(\nu), \nu_k]\right\} \quad \text{(IV-32)}$$

Since

$$\operatorname{Res}\left[\frac{\cos \nu\theta}{\cosh \alpha\nu \sin \nu\pi}, 0\right] = \frac{1}{\pi} \quad \text{(IV-33)}$$

and

$$\operatorname{Res}\left[\frac{\cos \nu\theta}{\cosh \alpha\nu \sin \nu\pi}, i\frac{1}{\alpha}(2k+1)\frac{\pi}{2}\right] = -\frac{(-1)^k}{\alpha} \cdot \frac{\cosh\left[\frac{\theta\pi}{2\alpha}(2k+1)\right]}{\sinh\left[\frac{\pi^2}{2\alpha}(2k+1)\right]} \quad \text{(IV-34)}$$

we finally get

$$I(\theta) = F(\theta) = -\frac{1}{2} + \frac{\pi}{\alpha}\sum_{k=0}^{\infty}(-1)^k \frac{\cosh\left[\frac{\theta\pi}{2\alpha}(2k+1)\right]}{\sinh\left[\frac{\pi^2}{2\alpha}(2k+1)\right]} \quad \text{(IV-35)}$$

It is obvious that the terms of the series (IV-35) are of the asymptotic order $e^{-\frac{\pi}{\alpha}\left(k+\frac{1}{2}\right)(\pi-\theta)}$ as $\alpha \to 0$, and this ensures the rapid convergence of the series (IV-35) for $\theta < \pi$. If $\alpha$ is sufficiently small, one can confine himself, in practical calculations, to only a few of the terms of the series.

It must be stressed that the specific applications of Watson's method in each separate case may differ because there are different ways of constructing the integral equivalent to the original series, and there are different ways of evaluating it. The most effective realization of the method in each concrete case depends on the specific nature of the series being investigated.

# REFERENCES


1. Bitsadze, A. V. *Osnovy teorii analiticheskikh funktsii kompleksnogo peremennogo* (Fundamentals of the Theory of Analytic Functions of a Complex Variable), Moscow Nauka (1972).
2. Bochner, S., Martin, W. T. *Several Complex Variables*, Princeton (1948).
3. Courant, R., Hurwitz, A. *Functionentheorie*, Springer Verlag (1922).
4. Ditkin, V. A., Prudnikov, A. P. *Integral Transforms and Operational Calculus*, New York, Pergamon Press (1965).
5. Fock, V. A. On Certain Integral Equations of Mathematical Physics, *Matematich. sbornik*, 14, 1 (1944).
6. Goluzin, G. M. *Geometricheskaya teoriya funktsii kompleksnogo peremennogo* (The Geometric Theory of Functions of a Complex Variable), Moscow, Nauka (1966).
7. Hopf, E. *Mathematical Problems of Radiative Equilibrium*, Cambridge University Press (1934).
8. Lavrentyev, M. A., Shabat, B V. *Metody teorii funktsii kompleksnogo peremennogo* (Methods of the Theory of Functions of a Complex Variable), Moscow, Fizmatgiz (1972).
9. Lebedev, N. N. *Spetsialnye funktsii i ikh prilozheniya* (Special Functions and Their Applications), Moscow, Gostekhizdat (1953).
10. Markushevich, A. I. *Theory of Functions of a Complex Variable*, Vols. 1-3, Englewood Cliffs, N.J., Prentice-Hall (1965-1967).
11. Morse, P. M., Feshbach, H. *Methods of Theoretical Physics*, New York, McGraw-Hill (1953).
12. Noble, B. *Methods Based on the Wiener-Hopf Technique*, New York, Pergamon Press (1958).
13. Privalov, I. I. *Vvedenie v teoriyu funktsii kompleksnogo peremennogo* (Introduction to the Theory of Functions of a Complex Variable), 11th ed., Moscow, Nauka (1967).
14. Smirnov, V. I. *A Course of Higher Mathematics*, Vol. 3, Part 2: Complex Variables, Special Functions; Oxford, Pergamon Press (1964).
15. Stoilov, S. *Teoria functiilor de o variabilă complexă*. (Theory of Functions of a Complex Variable), Vols. I-II, Editura Academici Republicii Populare Romîne (1954, 1958).
16. Tamm, I. E. *Osnovy teorii elektrichestva* (Principles of the Theory of Electricity), Moscow, Nauka (1966).
17. Tikhonov, A. N., Samarsky, A. A *Partial Differential Equations of Mathematical Physics*, San Francisco, Holden-Day (1964, 1967).
18. Titchmarsh, E. C. *Introduction to the Theory of Fourier Integrals*, 2nd ed., New York, Oxford University Press (1948).
19 Van der Pol, B., Bremmer, H. *Operational Calculus Based on the Two-Sided Laplace Integral*, New York, Cambridge University Press (1955).
20. Vladimirov, V. S. *Metody teorii funktsii mnogikh kompleksnykh peremennykh* (Methods of the Theory of Functions of Many Complex Variables), Moscow, Nauka (1964).

# NAME INDEX

# SUBJECT INDEX

## TO THE READER

Mir Publishers would be grateful for your comments on the content, translation and design of this book. We would also be pleased to receive any other suggestions you may wish to make.

Our address is:
USSR, 129820, Moscow, I-110, GSP
Pervy Rizhsky Pereulok, 2
MIR PUBLISHERS

*Printed in the Union of Soviet Socialist Republics*

*Other Mir Titles*

**Problems in Mathematical Analysis**

under the editorship
of Professor B. Demidovich.

The book contains a total of over 3.000 problems. Emphasis is placed on the key sections of the course, which require well developed skills (such as limit determination, differentiation, graphical representation of functions, integration, application of some integrals, series and solution of differential equations). Each chapter is preceded by a short theoretical introduction, and includes basic definitions and formulas essential to the given section of the course. Examples of solutions are shown for the most important problems and answers are given for all the problems.

**A Course of Mathematical Analysis in 2 volumes**

by S. M. NIKOLSKY, Member, USSR Academy of Sciences.

A textbook for University students (physicists and mathematicians) with special supplementary material on mathematical physics. Based on the course read by the author at the Moscow Physico-technical Institute.

*Contents.* **Vol. I.** Introduction. Real numbers. Limit of sequence. Limit of function. Functions of one variable. $n$-dimensional space. Functions of several variables. Indefinite integral. Definite integral. Some applications of integrals. Series.

**Vol. II.** Multiple integrals. Scalar and vector fields. Differentiation and integration of integral with respect to parameter. Improper integrals. Normed linear spaces. Orthogonal systems. Fourier series. Approximation of functions with polynomials. Fourier integral. Generalized functions. Differential manifolds and differential forms Supplementary topics. Lebesgue integral.